Physics 213A

Problems marked with [M], [F] or [Q] appeared in previous midterms, finals or qualifying exams respectively.

Problem 1

Define

$$\delta_\sigma(\phi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\phi - |n|\sigma} ,$$

Show that for ϕ restricted to the interval $-\pi \leq \phi \leq \pi$

$$\lim_{\sigma \to 0} \delta_\sigma(\phi) = \delta(\phi) .$$

To do this show that

$$\begin{aligned} \int_{-\pi}^{\pi} d\phi \delta(\phi) &= 1 \\ \delta(\phi) &= 0, \text{ for } \phi \neq 0 ; \end{aligned}$$

for the latter, do the sum over n explicitly.

Problem 2

Use the results from Problem 1 to show that any function $f(\phi)$, for ϕ restricted to the interval $-\pi \leq \phi \leq \pi$ can be expanded as

$$f(\phi) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} f_n e^{in\phi}$$

and find the relation between $f(\phi)$ and f_n.

Problem 3 Jackson, 1.5

Problem 4 Jackson 1.10

Problem 5 Jackson, 1.15

Problems 6, 7 and 8 have a lot in common. For solutions to any problem, you may use results from a previous one.

Problem 6

Find the Green function for the space to the right of the grounded plane, $x = 0$. Do this by solving the differential equation for $G(\vec{r}; \vec{r}_0)$ and <u>not</u> by image methods.

Problem 7 [20 points]

Find the Green function for the space between two conducting planes, one at $x = 0$ and one at $x = a$. You may leave your answer as a two dimensional integral. Show that in the limit $a \to \infty$ your answer reduces to that of Problem 6.

Problem 8

For the geometry in Problem 7 write down the potential for a charge distribution $\rho(x, y, z)$ and the condition that the potential at the plane $x = 0$ is kept at $\phi(0, y, z) = 0$ while the one at $x = a$ is kept at a $\phi(a, y, z) = \phi_0$; ϕ_0 is a constant.

Problem 9 [20 points]

a. A charge distribution $\rho(\vec{r})$ is placed in a volume V bounded by a surface S. The surface is grounded and the Green function is $G(\vec{r}, \vec{r}_0)$. What is the energy of this charge distribution?

b. What is the energy required to bring a charge q from infinity to a point $\vec{r}$ in the volume V?

c. Two charges q_1 and q_2 are placed at $\vec{r}_1$ and $\vec{r}_2$ inside the volume described in part a. What is the force between the two charges?

Problem 10

For the the charge distributions described in Jackson, problem 4.1, find the electric dipole and quadrupole moments.

Problem 11

A charge distribution with a *cylindrically symmetric and diagonal* quadrupole moment with $Q_{zz} = eQ$ finds itself at the origin in electric field with a gradient $(\partial E_z/\partial z)_0$. What is the energy of this quadrupole interaction?

Problem 12 [20 points]

All of space is filled with two dielectrics. For $z < 0$ the dielectric constant is ϵ_L/ϵ_0 while for $z > 0$ it is ϵ_R/ϵ_0. Find the potential, $\phi(\vec{r})$ everywhere for a charge q placed at $(x, y, z) = (0, 0, d)$; $d > 0$. Do **not** use the method of images but solve the differentiable equation for ϕ.

Problem 13 [20 points]

A spherical shell is bounded by two concentric spheres with radii a and b with $b > a$.

The material in the shell has a dielectric constant that varies as

$$\epsilon(r) = \frac{C}{r^2};$$

r is the distance from the center of the spheres and C is a constant. A charge Q is placed at the center of the spheres.

(i) Find the electric field everywhere.

(ii) Find the volume charge density in the dielectric.

(iii) What are the surface charge densities at $r = a$ and at $r = b$?

Problem 14 [20 points]

A dielectric sphere of radius R and dielectric constant ϵ/ϵ_0 has a point charge q placed at its center and a charge $4\pi R^2\sigma$ distributed uniformly on its surface. The value of σ is chosen to make $\vec{E} = 0$ outside the sphere.

(a) What is $\vec{E}$ and $\vec{D}$ at a radius $r < R$?

(b) What is $\vec{D}$ outside the sphere?

(c) What is the value of σ?

(d) In addition to the surface charge σ there is a polarization charge density, σ_P. What is its value?

Problem 15 [20 points] [M]

A dielectric with variable dielectric constant $\epsilon(x)/\epsilon_0$ ($\epsilon(x)$ depends on x only and is continuous in x) is placed between two grounded plates at $x = 0$ and $x = a$. The Green function relates the potential $\Phi(\vec{r})$ to the free charge density $\rho(\vec{r})$

$$\Phi(\vec{r}) = \int d^3r_0\, G(\vec{r};\vec{r}_0)\rho(\vec{r}_0)$$

As in previous cases we can express the Green function for this problem as

$$G(\vec{r};\vec{r}_0) = \int d^2k\, g(x,\vec{k};\vec{r}_0)\exp(i\vec{k}\cdot\vec{r}_\perp);$$

with $\vec{r}_\perp = (y, z)$. Obtain (do not solve) the differential equation for $g(x,\vec{k};\vec{r}_0)$ and the boundary conditions at $x = 0$, $x = a$ and $x = x_0$.

Problem 16

A dielectric sphere of radius a and a dielectric constant ϵ/ϵ_0 is placed in an electric field which at large distances has the form

$$\vec{E} = E_0(2z\hat{z} - x\hat{x} - y\hat{y})\,.$$

Find the potential everywhere. What is the surface polarization?

Problem 17

Show that the magnetic induction corresponding to the vector potential

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \vec{r}}{r^3}\,,$$

with $\vec{m}$ a constant, is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left[\frac{3\hat{r}(\hat{r}\cdot\vec{m}) - \vec{m}}{r^3} + \frac{8\pi}{3}\vec{m}\delta(\vec{r})\right]\,.$$

[Don't just copy the discussion in Jackson, p. 188, but consider carefully what $\nabla_i\nabla_j r^{-1}$ is.]

Problem 18 [20 points]

Find the magnetic induction $\vec{B}(\vec{r})$ for a semi-infinite solenoid of radius a stretching along $z \le 0$ and $x = y = 0$. Take r to be much larger than a and $\vec{r}$ not along the negative z-axis.

Problem 19 [20 points]

Jackson, 5.1 on page 225. Hints:

- Take the dot product of both sided of the first equation with an arbitrary constant vector $\vec{n}$.
- Massage the equation for $\vec{n}\cdot\vec{B}$ to get it into a form where you can apply Green's Theorem: For any $\vec{V}(\vec{r})$
$$\int_S \vec{dS}\cdot(\vec{\nabla}\times\vec{V}) = \int_C d\vec{l}\cdot\vec{V}\,.$$
- You're on your own.

Problem 20

Using the results of Problem 16, find B_z along the z-axis for a loop of radius a, centered a the origin, carrying a current I and lying in the $x - y$ plane,

Problem 21

Jackson, 5.17 on page 229.

Problem 22

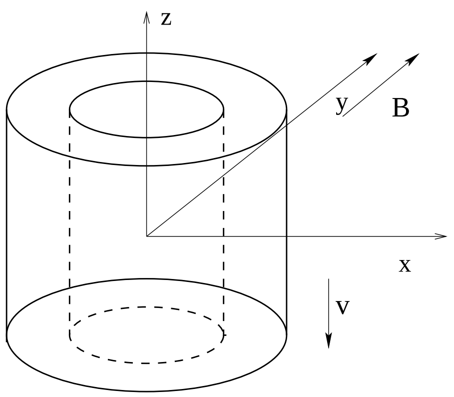
A current density o a cylinder of radius R has the form

$$\vec{j} = \frac{I \cos\phi}{2R} \delta(\rho - R) \hat{z} \,.$$

Find the magnetic induction everywhere.

Problem 23 [20 points] [F]

A long, hollow cylindrical conductor of inner radius a and outer radius b has its axis centered at the origin and along $\hat{z}$; the cylinder moves with velocity $-v\hat{z}$. There is a position dependent magnetic induction $\vec{B} = \frac{B_0}{a} x\hat{y}$



(i) What is the induced volume charge density?

For $\rho < a$ and $\rho > b$ the electrostatic potential has the form

$$\phi(\vec{r}) = \sum_{n=-\infty}^{\infty} \rho^n (A_n \cos n\phi + B_n \sin n\phi) \,,$$

with the components of $\vec{r}$ being (ρ, ϕ, z).

(ii) What are the electrical fields for $\rho < a$ and $\rho > b$?

(iii) What is the surface charge density at $\rho = a$ and $\rho = b$?

Problem 24

We have shown that the energy, E, due to an electric field $\vec{E}(\vec{r})$ and a magnetic induction $\vec{B}(\vec{r})$ in a volume V is

$$\mathrm{E} = \frac{1}{2}\int_V d\vec{r}\,[\epsilon_0 \vec{E}(\vec{r})\cdot\vec{E}(\vec{r}) + \frac{1}{\mu_0}\vec{B}(\vec{r})\cdot\vec{B}(\vec{r})]\,.$$

From considerations of how energy and momentum are related under Galilean transformations obtain the momentum carried by these fields.

Problem 25

Jackson, 6.1 on page 283.

Problem 26

Jackson, 6.4(a) on page 285.

Problem 27 [20 points]

Show that the solutions for the scalar and vector potentials given by

$$\begin{aligned}\phi(\vec{r},t) &= \frac{1}{4\pi\epsilon_0}\int d\vec{r}'dt'G(\vec{r},t:\ \vec{r}',t')\rho(\vec{r}',t')\,,\\ \vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi}\int d\vec{r}'dt'G(\vec{r},t:\ \vec{r}',t')\vec{j}(\vec{r}',t')\end{aligned}$$

satisfy the Lorenz condition

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0\,.$$

Find a $\Lambda(\vec{r},t)$ such that $\vec{A}'(\vec{r},t) = \vec{A}(\vec{r},t) + \vec{\nabla}\Lambda(\vec{r},t)$ satisfies the Coulomb gauge condition $\vec{\nabla}\cdot\vec{A}'(\vec{r},t) = 0$. Give your answer as a one space-time integral.

Problem 28

Jackson, 6.5(a) and (b) on page 286.

Problem 29 [20 points]

The torque, $\vec{K}$, on some currents and charges is

$$\vec{K} = \int d\vec{r}\ \vec{r}\times(\rho\vec{E} + \vec{j}\times\vec{B})\,.$$

The time derivative of the mechanical angular momentum $d\vec{L}_{\rm mech}dt = \vec{K}$. Find the electromagnetic field angular momentum, $\vec{L}_{\rm field}$ and show that

$$\frac{d\vec{L}_{\rm mech}}{dt} + \frac{d\vec{L}_{\rm field}}{dt} = \text{surface integral}\,.$$

Problem 30 [Q]

A stationary current distribution is established in a medium with a spatially varying dielectric constant $\epsilon(\vec{r})/\epsilon_0$ and a spatially varying conductivity $\sigma(\vec{r})$. As a result a free charge distribution $\rho(\vec{r})$ and an electric field $\vec{E}(\vec{r})$ are also established in the medium. Using the conservation of current show that

$$\rho(\vec{r}) = \left[\vec{\nabla}\epsilon(\vec{r}) - \frac{\epsilon(\vec{r})}{\sigma(\vec{r})}\vec{\nabla}\sigma(\vec{r})\right] \cdot \vec{E}(\vec{r})\,.$$

Problem 31

In a conductor the current density $\vec{j}(\vec{r},t)$ appearing as a source in Maxwell's equations may be replaced by $\sigma\vec{E}(\vec{r},t)$; where σ is the conductivity. The charge density $\rho(\vec{r},t) = 0$. The electric permittivity is ϵ and the magnetic permeability is μ.

(i) Obtain the equation satisfied by $\vec{E}(\vec{r},t)$ or $\vec{B}(\vec{r},t)$.

(ii) For

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{k},\omega)\exp(-i\omega t + i\vec{k}\cdot\vec{r})\,,$$

find the relation between k and ω.