Solution of the Black Scholes Equation Using the Green's Function of the Diffusion Equation

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Abstract

Without creating a new solution, we just show explicitly how to obtain the solution of the Black-Scholes equation for call option pricing using methods available to physics, mathematics or engineering students, namely, using the Green's function for the diffusion equation.

1 Introduction

We show one method of solving the Black-Scholes equation for the value of a call option using the Green's function approach to the diffusion equation. This is suitable to the background of many advanced physics, mathematics, and engineering students.

We consider a stock held at a variable market price x, and whose temporal evolution we treat as only determined by a random walk or a Gaussian probability distribution. This equity is hedged by selling call options at a price $w(x, t)$, which allow a call on the stock at the maturity date t^* at the strike price c. The conservative stock holder is then protecting themselves against some possible loss in the stock value by selling some possible profits from a large increase in the stock price. (Since I am not an economist, an arbitrager, a market analyst, a market bull, or even a day trader, I disown any responsibility for any errors or misunderstandings caused by this document.)

We refer the reader to the original Black-Scholes paper[1], to the website of the Nobel Prize in Economics explaining the origin and significance of the method and equation[2], and to a Scientific American article[3] which examines the limitations of the assumptions when applied to the real stock situation.

2 Boundary Condition

The temporal boundary condition on the option price is that at maturity at time t^*, if the stock has risen above c, the call option is worth $w(x, t^*) = x - c$, so that a caller could buy the option at time t^* at the strike price c, making a profit $x - c$, which would equal the cost or value of the call option then, $w(x, t^*)$. However, if the stock price x has fallen below the strike price c, then the call option is not exercised since it would result in a loss, and the value of the option is worthless, or $w(x, t^*) = 0$ if $x \leq c$. The boundary condition is continuous at $x = c$.

3 Derivation of the Black-Scholes equation

A neutral hedge equity is constructed by selling $\frac{1}{\partial w/\partial x} = \Delta x/\Delta w$ call options at price $w(x, t)$, so that the net equity invested is

$$x - w\frac{\Delta x}{\Delta w}. \tag{1}$$

A change in x by Δx accompanied by a change in w by Δw then gives no change in the equity

$$\Delta x - \Delta w\frac{\Delta x}{\Delta w} = 0. \tag{2}$$

In writing the Black-Scholes equation, we will find the value of the price of the call option $w(x, t)$ necessary to allow the hedge equity to grow at the same rate as investing the equity value in an interest account or instrument at the fixed interest rate r per day so that

$$\Delta x - \Delta w\frac{1}{\partial w/\partial x} = \left(x - w\frac{1}{\partial w/\partial x}\right) r\Delta t. \tag{3}$$

The Taylor expansion for the the change Δw is

$$\Delta w = w(x + \Delta x, t + \Delta t) - w(x, t) = \frac{\partial w}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 w}{\partial x^2}(\Delta x)^2 + \frac{\partial w}{\partial t}\Delta t. \tag{4}$$

It is now assumed that the variance $<(\Delta x)^2>$ comes from a random walk in the fractional price, and is therefore proportional to Δt, giving

$$(\Delta x)^2 = v^2 x^2 \Delta t. \tag{5}$$

v^2 is the variance per unit time, or the variance rate.

Putting Eq. 4 and Eq. 5 into Eq. 3, cancelling the Δx, dividing by Δt and multiplying by $\partial w/\partial x$ gives the Black-Scholes equation

$$\frac{\partial w}{\partial t} = rw - rx\frac{\partial w}{\partial x} - \frac{1}{2}v^2 x^2 \frac{\partial^2 w}{\partial x^2}. \tag{6}$$

4 Conversion of the Black-Scholes Equation to the Diffusion Equation

We first bring the equation into the standard form of the diffusion equation, and then solve it using the Green's function for the diffusion equation on the initial condition at $t = t^*$.

The first difference we notice from the canonical equation is that the coefficients depend on x. However, the equation is homogeneous or invariant under the scaling of $x \to ax$. The standard way to simplify this and eliminate the explicit coordinate dependence is to define a new variable $u = \ln(x/c)$, where we have scaled x by c to make it dimensionless. Then under $x \to ax$, $u \to u + \ln(a)$. Since the equation is invariant under this, it cannot have any explicit dependence on u in the coefficients. Changing variables to u using $\frac{\partial u}{\partial x} = \frac{1}{x}$, and defining $\tilde{w}(u,t) = w(x,t)$, the derivatives become

$$\begin{aligned} \frac{\partial \tilde{w}}{\partial x} &= \frac{\partial \tilde{w}}{\partial u}\frac{\partial u}{\partial x} = \frac{1}{x}\frac{\partial \tilde{w}}{\partial u}, && (7) \\ \frac{\partial^2 \tilde{w}}{\partial x^2} &= \frac{1}{x^2}\left(\frac{\partial^2 \tilde{w}}{\partial u^2} - \frac{\partial \tilde{w}}{\partial u}\right). && (8) \end{aligned}$$

The $1/x$ and $1/x^2$ factors cancel those in the Eq. 6 giving an equation with constant coefficients

$$\frac{\partial \tilde{w}}{\partial t} = r\tilde{w} - (r - v^2/2)\frac{\partial \tilde{w}}{\partial u} - \frac{1}{2}v^2 \frac{\partial^2 \tilde{w}}{\partial u^2}. \tag{9}$$

Now we observe that even if $\tilde{w}(u,t)$ is independent of u, it still grows as e^{rt} from the $r\tilde{w}$ term. Factoring this out at the start will remove the $r\tilde{w}$ term. We normalize this behavior where the boundary condition is at $t = t^*$ by writing the solution as

$$\tilde{w}(u,t) = e^{-r(t^*-t)}y(u,t). \tag{10}$$

Substituting this into Eq. 9 eliminates the $r\tilde{w}$ term giving

$$\frac{\partial y}{\partial t} = -(r - v^2/2)\frac{\partial y}{\partial u} - \frac{1}{2}v^2\frac{\partial^2 y}{\partial u^2}. \tag{11}$$

We next scale towards a canonical form. First we scale u to get a common coefficient for the u derivatives, and then absorb that coefficient into a rescaling for t. The new variables are

$$u' = u\frac{(r - v^2/2)}{v^2/2}, \tag{12}$$

and

$$t' = \frac{(r - v^2/2)^2}{v^2/2}(t^* - t). \tag{13}$$

With $\hat{y}(u', t') = y(u, t)$ the equation has become

$$\frac{\partial \hat{y}}{\partial t'} = \frac{\partial \hat{y}}{\partial u'} + \frac{\partial^2 \hat{y}}{\partial u'^2}. \tag{14}$$

Now, even with a constant gradient in u' there is an increase with time in $\hat{y}$. This is because the $\hat{y}$ is moving at constant velocity bringing a larger value to a fixed u' point. This term can then be eliminated by going to a comoving frame, or changing the u' spatial coordinate to $z = u' + t'$ where the velocity is 1. Thus with

$$\tilde{y}(z, t') = \tilde{y}(u' + t', t') = \hat{y}(u, t) \tag{15}$$

we finally get the canonical form of the diffusion equation with unit diffusion coefficient

$$\frac{\partial \tilde{y}}{\partial t'} = \frac{\partial^2 \tilde{y}}{\partial z^2}. \tag{16}$$

The boundary conditions at $t = t^*$ are now at $t' = 0$ where $z = u'$. $x \geq c$ translates into $u \geq 0$. We now use the case where $(r - v^2/2) \geq 0$ so the condition on u translates into $u' \geq 0$. The boundary conditions are then

$$\begin{aligned} \tilde{y}(z, 0) &= x - c = c(e^u - 1) \quad \text{for } z \geq 0, \text{ and} && (17) \\ &= 0 \quad \text{for } z \leq 0. && (18) \end{aligned}$$

5 Green's Function Solution

We now use the Green's function for the Diffusion or Heat equation[4], which is the solution to that equation for a point (or delta function) source at point z' at time $t' = 0$

$$G(z - z'; t) = \frac{1}{\sqrt{4\pi t'}}e^{-\frac{(z-z')^2}{4t'}}. \tag{19}$$

The verification of this Green's function solution is shown in Appendix A. The Green's function shows the Gaussian diffusion of the pointlike input with distance from the input $(z - z')$ increasing as the square root of the time t', as in a random walk.

We can use the Green's function to write the solution for $\tilde{y}(z, t')$ in terms of summing over its input values at points z' on the boundary at the initial time $t' = 0$

$$\tilde{y}(z,t') = \int_{-\infty}^{\infty} dz' \tilde{y}(z',0) \frac{1}{\sqrt{4\pi t'}} e^{-\frac{(z-z')^2}{4t'}}. \tag{20}$$

Putting in the initial condtions at $t' = 0$, where $\tilde{y}(z', 0)$ vanishes for negative z', gives

$$\tilde{y}(z,t') = \frac{1}{\sqrt{4\pi t'}} \int_{0}^{\infty} dz' c \left(e^{\frac{v^2/2}{(r-v^2/2)} z'} - 1 \right) \frac{1}{\sqrt{4\pi t'}} e^{-\frac{(z-z')^2}{4t'}}. \tag{21}$$

To do the integral we change the variable to

$$q = \frac{z' - z}{\sqrt{2t'}}, \quad \text{and} \tag{22}$$

$$dz' = \sqrt{2t'} dq. \tag{23}$$

The lower limit on the q integral is now

$$-\frac{z}{\sqrt{2t'}} = -d2, \tag{24}$$

where substitution gives the dimensionless

$$d2 = \frac{\ln x/c + (r - v^2/2)(t^* - t)}{v\sqrt{(t^* - t)}}. \tag{25}$$

In the first term we now complete the square to get a new variable

$$q' = q - \sqrt{2t'} \frac{v^2/2}{(r - v^2/2)}. \tag{26}$$

The new lower limit on q' in the first term is now $-d1$ where

$$d1 = \frac{\ln x/c + (r + v^2/2)(t^* - t)}{v\sqrt{(t^* - t)}}. \tag{27}$$

After completing the square on the first term, the exponent simplifies to

$$\ln x/c + r(t^* - t) - \frac{1}{2} q'^2. \tag{28}$$

Both integrals are now related to the Cumulative Distribution Function of the Normal Distribution

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt. \tag{29}$$

If we change t to $-t$ in the above integral and invert the limits we get the form of our integrals

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{\infty} e^{-\frac{1}{2}t^2} dt. \tag{30}$$

We now have our solution for the canonical $\tilde{y}$

$$\tilde{y}(z, t') = c(e^{\ln(x/c)+r(t^*-t)} N(d1) - N(d2)) \tag{31}$$

Finally, we use the facts that $\tilde{y} = \hat{y} = y$, and that $e^{\ln(x/c)} = x/c$, and the conversion

$$w(x, t) = \tilde{w}(u, t) = e^{-r(t^*-t)} y(u, t), \tag{32}$$

to get the Black-Scholes solution

$$w(x, t) = xN(d1) - ce^{-r(t^*-t)} N(d2). \tag{33}$$

6 Post-Analysis

We verify that the boundary conditions are satisfied. For $x > z$, $\log(x/c) > 0$, and as $t \to t^*$, $d1 \to \infty$ and $d2 \to \infty$. Then both $N(d1) \to 1$ and $N(d2) \to 1$, giving $w(x, t^*) = x - c$ as required. For $x < z$, $\log(x/c) < 0$, and as $t \to t^*$, $d1 \to -\infty$, and $d2 \to -\infty$, so both $N(d1)$ and $N(d2)$ vanish, and $w(x, t^*) = 0$.

To find the number of call options to hold at a given time $(1/(\partial w/\partial x))$, we calculate

$$\frac{\partial w}{\partial x} = N(d1) + \frac{1}{v\sqrt{2\pi(t^*-t)}} \left(e^{-d1^2/2} - \frac{x}{c} e^{-d2^2/2} e^{-r(t^*-t)} \right). \tag{34}$$

If $x < c$, as $t \to t^*$, $d1 \to \infty$ and $d2 \to \infty$, so $\partial w/\partial x \to 1$, and the number of call options to own at the maturity time t^* is 1. The value of the hedge equity at t^* is then $x - w/(\partial w/\partial x) = x - (x - c) * 1 = c$, as it should be.

A Green's function for the Diffusion equation

We show that the Green's function for the diffusion equation,

$$G(z - z'; t) = \frac{1}{\sqrt{4\pi t'}} e^{-\frac{(z-z')^2}{4t'}}, \tag{35}$$

satisfies the equation and behaves like a delta function at $t' = 0$.

Plugging the Green's function into the canonical diffusion equation, Eq. 16, gives on both sides

$$\frac{\partial G(z-z';t')}{\partial t'} = -\frac{1}{2t'}G(z-z';t') + \frac{(z-z')^2}{4t'^2}G(z-z';t') = \frac{\partial^2 G(z-z';t')}{\partial z^2}, \tag{36}$$

verifying that it is a solution to the equation.

As $t' \to 0$, for $z \neq z'$, the argument of the exponent goes to $-\infty$, and $G(z-z';t') \to 0$. For $z = z'$, it goes to infinity as $t' \to \infty$. The integral over z' can be found by substituting $q = (z-z')/\sqrt{2t'}$ and gives

$$\int G(z-z';t')dz' = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-q^2/2}dq = 1, \tag{37}$$

showing that it is correctly normalized to be the solution for a delta function source or point source at $z = z'$ when $t' \to 0$.

References

[1] Black, F. and Scholes, M., 1973, "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy*, Vol 81, pp. 637-654.

[2] "The Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel 1997" http://www.noble.se/announcement-97/ecoback97.html.

[3] Gary Stix, "A Calculus of Risk", Scientific American, May 1998, and http://www.sciam.com/1998/0598issue/0598stix.html.

[4] J. Mathews and R. L. Walker, Mathematical Methods of Physics, Second Ed., Addison-Wesley, 1970, pp. 242-244.