The Correct KK Calculation

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Abstract

In this note, we describe the descrepency between the naive KK- and the 5D-calculations. Then we show that the solution of this is to take the correct UV limit in the KK-calculation.

1 The naive KK-calculation

When matching the 5D and the 4D KK-calculation (using Lavora's result), we always get the second result to be $\mathcal{O}(10^{-3})$ smaller than the first one. To be more precise, when parameterizing the final amplitude (the p^μ part) in the form ($M \simeq 1/R'$)

$$\mathcal{M}_{p^\mu} = \frac{gv}{16\pi^2} f_\mu f_e \bar{u}_e u_\mu \times \left[\frac{a}{M^2} + \frac{b}{M^2}\left(\frac{v}{M}\right)^2 + \mathcal{O}\left(\frac{1}{M^2}\left(\frac{v}{M}\right)^3\right)\right], \tag{1.1}$$

the coefficient a in the KK-calculation vanishes, and the b-term is suppressed by $(v/M)^2$. Comparing to this, the 5D result gives $a \simeq 0.5$ and is much larger.

The Lavora-type calculation is performed in the full mass basis. i.e. we diagonalize the big mass matrix that includes both the Yukawa and the KK masses, then use the mass eigenvalues and the rotated Yukawa couplings to do the calculation. To compare with the 5D-calculation, we can also use the c-basis. In this basis, we do not diagonalize the mass matrix but use the kk-mass as the mass of the fields and do the v-insertion. When doing this, if we perform the loop-integral and the KK-sum in the following way

$$\lim_{N\to\infty} \sum_{n=1}^{N} \int_0^\infty dk\, \mathcal{M}^{(n)}(k), \tag{1.2}$$

the factor a in Eq. (1.1) is zero, and the result is also suppressed. This means the suppression does not come from the complicated flavor rotation of the mass matrix but really exists order by order in the v-insertion. The result is confusing. Since the 5D-propogator can be decomposed into the sum of the KK-propagators, the v-insertion calculation in the 5D should reproduce the 4D result. How can the two results be different? As we are going to show in this note, the two calculations do not match because the interval we use in Eq. (1.2) is wrong.

2 The decomposition of the 5D-propagator

The idea of the KK-decomposition is to reproduce the 5D result into a descrete but infinity sum of the 4D results. For the 5D-propagator $\Delta_{5D}(k, z, z')$, we can decompose it into the sum of the KK-propagators having the form (one can check the decomposition numerically with the 5D-propagators we derived)

$$\Delta_{5D}(k, z, z') = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{\not{k} + M_n}{k^2 - M_n^2} f^n(z) f^n(z'). \tag{2.1}$$

There is one subtlety for the decomposition. i.e., the expansion works only when the momentum k is smaller than the highest KK-mass M_N. When k gets larger, more KK-modes need to be included to reproduce the correct 5D propagator. This means we should add one more constraint in the above equation

$$\Delta_{5D}(k, z, z') = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{\not{k} + M_n}{k^2 - M_n^2} f^n(z) f^n(z')|_{k<M_N}. \tag{2.2}$$

This additional constraint is not important for most of the tree level processes, since the external energy scale we care is always smaller than the cutoff KK-masses. However, when doing the loop-integral as in Eq. (1.2), the correct interval in the integration should be

$$\lim_{N\to\infty}\sum_{n=1}^{N}\int_0^{rNM} dk\,\mathcal{M}^{(n)}(k), \tag{2.3}$$

where $r<1$ is a ratio between the momentum cutoff and the highest KK-mass we include. Naively thinking, Eq. (2.3) should be the same as Eq. (1.2). But as we are going to show in the next section, the way we deal with the UV in the momentum and the KK really effects the final result.

3 The correct KK integral

(I will put the full calculation later)

Define the two KK-masses when having one v-insertion as

$$M_1 = n_1 M, \qquad M_2 = n_2 M. \tag{3.1}$$

When using the correct KK-integral as in Eq. (2.3), the a-factor defined in Eq. (1.1) has the value

$$a = \lim_{N\to\infty}\sum_{n_1,n_2=1}^{N}\hat{a}(n_1,n_2) = \lim_{N\to\infty}\sum_{n_1,n_2=1}^{N}\frac{-(rN)^2\left[\,(n_1^2+n_2^2)\,(rN)^2+2n_1^2n_2^2\,\right]}{4\left[\,n_1^2+(rN)^2\,\right]^2\left[\,n_2^2+(rN)^2\,\right]^2}. \tag{3.2}$$

Before giving the numerical result directly, let us take some extreme values of the expression and try to understand the difference between Eq. (1.2) and (2.3). Assume rN (the momentum cutoff) is fixed and take $n_1=n_2=n$. For different regions of n w.r.t. rN, we have

$$\hat{a}(n) \sim -\frac{1}{(rN)^2}\left(\frac{n}{rN}\right)^2, \qquad n \ll rN, \tag{3.3}$$

$$\hat{a}(n) \sim -\frac{1}{(rN)^2}, \qquad n \sim rN, \tag{3.4}$$

$$\hat{a}(n) \sim -\frac{1}{(rN)^2}\left(\frac{rN}{n}\right)^4, \qquad n \gg rN. \tag{3.5}$$

When doing the n(KK)-sum of the function $\hat{a}(n)$, the main contribution comes from the peak around $n\sim rN$. Using the **correct** interval as in Eq. (2.3) corresponds to the case when $r<1$. The summation up to the Nth KK mode includes the peak around rN. The sum of $\hat{a}(n)$ then becomes non-zero. On the contrary, if we use the **wrong** interval as in Eq. (1.2), which corresponds to the case $r\to\infty$, we can never reach the peak around rN while doing the KK-sum. This makes the final a-factor goes to zero and is the origin of the suppression when using Lavora's result.

4 The numerical result

Now we know that when using the correct inverval in the momentum intergal, the a-factor in Eq. (1.1) is non-zero and the amplitude can be large. The ratio r between the momentum and the KK-mass cutoff should be smaller than one. When $r = 0.7$, the sum in Eq. (3.2) gives $a = -0.11$. It is of the same order as the 5D result. When $r = 0.1$, we have $a = -0.50$. This is very close to the 5D result. When $r = 0.05$, we have $a = -0.56$, which is a little bit larger than the 5D result. If r is very small, the result still converges to $a = -0.60$, since we have cover the whole peak around rN (as discussed in the previous section), and the result does not change with r.

One remaining problem of the KK-calculation is that, we do not know how to decide the ratio r precisely. We know it is smaller than one and cannot be too small (otherwise the UV in the loop momentum is not the true UV). It is good to see the 4D result being close to the 5D one. However, the because of the issue on r, the 4D calculation becomes more subtle.

5 The 4D calculation with one mass insertion

Here we derive the leading order p^μ coefficient in Eq. (3.2). As have discussed before, we use the mass insertion to compare the result with 5D. The diagrams are trivial: The first loop has a mass insertion before the photon vertex and the second one with the insertion after the vertex. The Yukawa mixes the flavor in the 4D KK mass basis[1]. Defining the mass after and before the insertion as M_1 and M_2, the loop-integral of the first diagram becomes (m is the higgs mass)

$$\int d\!\!\!{}^-{}^4k \frac{\left(\not{k} - \not{q} + M_1\right)\gamma^\mu\left(\not{k} + M_1\right)\not{k}}{\left(k^2 - M_1^2\right)^2\left(k^2 - M_2^2\right)\left(k^2 - m^2\right)}\left(1 + \frac{2k\cdot q}{k^2 - M_1^2}\right)\left(1 + \frac{2p\cdot k - m_\mu^2}{k^2 - m^2}\right). \tag{5.1}$$

The q- and p-expansions have been applied and only the leading order term is preserved. Evaluating the integral,

$$\int d\!\!\!{}^-{}^4k \left[\frac{\not{q}k^2\gamma^\mu k^2}{\Delta_1^3\Delta_2\Delta_H} + \frac{\not{p}k^2\gamma^\mu k^2}{\Delta_1^2\Delta_2\Delta_H^2} - \frac{\not{q}\gamma^\mu k^2}{\Delta_1^2\Delta_2\Delta_H}\right], \tag{5.2}$$

where

$$\Delta_1 \equiv k^2 - M_1^2, \qquad \Delta_2 \equiv k^2 - M_2^2, \qquad \Delta_H \equiv k^2 - m^2. \tag{5.3}$$

Extraqcting the p^μ term using the Clifford algebra, we have

$$p^\mu \sum_{M_1}\sum_{M_2}\int_0^\Lambda d\!\!\!{}^-{}^4k\left[\frac{\left(k^2\right)^2}{\Delta_1^3\Delta_2\Delta_H} + \frac{\left(k^2\right)^2}{\Delta_1^2\Delta_2\Delta_H^2} - \frac{2k^2}{\Delta_1^2\Delta_2\Delta_H}\right]. \tag{5.4}$$

As we can see, the terms inside the integral vanishes when all the masses are zero ($\Delta_1 = \Delta_2 = \Delta_H = k^2$). The leading order term then looks like

$$\sum_{M_1}\sum_{M_2}\int_0^\Lambda d\!\!\!{}^-{}^4k\frac{(M_1, M_2, m)^2}{(k^2 - (M_1, M_2, m)^2)^4}, \tag{5.5}$$

[1] Under this basis, the four component fermion propogator has KK-mass. We do the Yukawa mass insertion to include the Dirac mass.

which is finite no matter the order of the summations and the integral.

The calculation is the same for the second diagram, which gives the integral

$$\int đ^4k \frac{\left(\not{k}-\not{q}\right)\left(\not{k}-\not{q}+M_2 k\right)\gamma^\mu\left(\not{k}+M_2\right)}{\left(k^2-M_1^2\right)\left(k^2-M_2^2\right)^2\left(k^2-m^2\right)}\left(1+\frac{2k\cdot q}{k^2-M_1^2}+\frac{2k\cdot q}{k^2-M_2^2}\right)\left(1+\frac{2p\cdot k-m_\mu^2}{k^2-m^2}\right). \quad (5.6)$$

We again drop the higher order terms. Contracting the k's gives

$$\int đ^4k\left[\frac{\not{q}\gamma^\mu k^2}{2\Delta_1^2\Delta_2^2\Delta_H}+\frac{\not{q}\gamma^\mu k^2}{2\Delta_1\Delta_2^3\Delta_H}-\frac{\not{q}\gamma^\mu}{\Delta_1\Delta_2^2\Delta_H}+\frac{\not{p}\gamma^\mu k^2}{2\Delta_1\Delta_2^2\Delta_H^2}\right]M_2^2. \quad (5.7)$$

Extracting the p^μ term gives

$$p^\mu\sum_{M_1}\sum_{M_2}\int_0^\Lambda đ^4k\left[\frac{k^2}{\Delta_1^2\Delta_2^2\Delta_H}+\frac{k^2}{\Delta_1\Delta_2^3\Delta_H}-\frac{2}{\Delta_1\Delta_2^2\Delta_H}+\frac{k^2}{\Delta_1\Delta_2^2\Delta_H^2}\right]M_2^2. \quad (5.8)$$

Combining the Eqs. (5.4) and (5.8), doing the Wick rotation and substituting (M_1, M_2, Λ) into (n_1M, n_2M, rNM), the a-coefficient in Eq. (1.1) is Eq. (3.2).