’t Hooft and η’ail
Instantons and their applications

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Prompt

Explain the role of instantons in particle physics

1. As a warm-up explain how instantons describe tunneling amplitudes in quantum mechanics

2. Explain the vacuum structure of gauge theories. In particular explain what the winding number is and how the $|n\rangle$ vacua appear, and show how instantons describe tunneling between these vacua. Explain what the θ vacuum is.

3. As an application, explain the U(1) problem (“η problem”) of QCD and how instantons it. Do not simply say that $\mathrm{U}(1)_A$ is anomalous and broken by instantons; explain the Kogut-Susskind mechanism.

4. Explain the emergence of the ’t Hooft operator. Explain the relation between anomalies and instantons, focusing on the index theorem, and also show how the ’t Hooft operator encodes the breaking of the anomalous symmetries. As an example show how the ’t Hooft operator can lead to baryon and lepton number violation in the SM. As another example of the ’t Hooft operator consider $\mathcal{N} = 1$ SUSY QCD with$F = N - 1$ flavors (for and $\mathrm{SU}(N)$ gauge group). Write down the ’t Hooft operator for that theory, and explain how that could actually come from a term in the superpotential, and why $F = N - 1$ is special.

Contents

1 *Une dégustation*: Introduction

Instantons play a rather understated role in standard quantum field theory textbooks, tucked away as an additional topic or mentioned in passing. Despite being notoriously difficult to calculate, instanton effects play a very important role in our conceptual understanding of quantum field theory.

In this examination we provide a pedagogical introduction to the instanton and some of its manifestations in high energy physics. We begin with 'instanton' configurations in quantum mechanics to provide a controlled environment with few 'moving parts' that might distract us from the real physics. After we've used path integrals and imaginary time techniques to remind ourselves of several quantum mechanical facts that we already knew, we will make use of this foundation to understand the basis of Yang-Mills instantons and how tunneling between degenerate vacua can be [surprisingly] manifested in quantum field theory. We will discover that Yang-Mills theory has a surprisingly rich vacuum structure that can perform seemingly miraculous feats in the presence of fermions. In particular, we will see how Yang-Mills instantons (which *a priori* have nothing to do with fermions) can solve an apparent problem in the spectrum of low-lying spectra of hadrons. In doing this we will present a deep relationship between instantons and anomalous U(1) symmetries that can be used for baryon and lepton number violation in the early universe. We will end with an introduction to the moduli space of SUSY QCD where instanton effects end up providing a powerful handle to solve the ADS superpotential for any number of flavors and colors. By the end of this paper we hope that the pun in its title will be clarified to the reader.

There are many things that we will unfortunately not be able to cover, but hopefully this may serve as an introduction to a fascinating subject.

This work is a review of a broad subjects and we've made very little attempts to provide references to original literature. In the appendix we give a brief literature review of the sources that this paper drew most heavily upon. Of all the sections of this paper, the literature review is probably the most useful to other graduate students hoping to learn more about instantons. Nothing in this work represents original work, except any errors. Of course, at the time of compilation this document was perfect and flawless; any errors must have occurred during the printing process.

This examination was prepared over the course of a three week period along with two other similar exam questions. The reader will note that the tone in the document becomes increasingly colloquial and borderline sarcastic as the deadline approached. (The introduction was written last.)

That being said, *Allons-y!*

2 *Apéritif*: Quantum mechanics

Let's start with the simplest instanton configuration one could imagine: tunneling in quantum mechanics, i.e. 0+1 dimensional quantum field theory. This is a story that we are all familiar with from undergraduate courses, but we will highlight the aspects which will translate over to Yang Mills theory in 4D. We will summarize the presentation by Coleman [1]; for details see also [2].

2.1 The semiclassical approximation and path integrals

Exact solutions to quantum systems are intractably difficult to solve. To make progress, we proceed using perturbation theory about the analytically soluble (i.e. quadratic) classical path[1]. This corresponds to an expansion in $\hbar$ (in the language of Feynman diagrams, an expansion in loops).

In quantum mechanics we know how to take the semiclassical limit: the WKB approximation. Here we note that for a constant potential, the solution to the Schrödinger equation is just a plane wave,

$$\Psi(x) = \Psi(0)e^{\pm ipx/\hbar} \qquad\qquad p = \sqrt{2m(E-V)}. \tag{2.1}$$

For nontrivial $V \to V(x)$, we can promote $p \to p(x)$ and proceed to solve for $p(x)$. This plane-wave approximation is valid so long as the state is probing a 'sufficiently flat' region of the potential. This can be quantified by defining the [position-dependent] Compton wavelength of the quantum state

$$\lambda(x) = \frac{2\pi\hbar}{p(x)}. \tag{2.2}$$

Thus we can see that the $\hbar \to 0$ limit indeed allows the quantum state to probe smaller (and hence more slowly varying) regions of the potential.

From here one can proceed as usual from one's favorite introductory quantum mechanics textbook to derive nice results for scattering and even tunneling. Since we're grown ups, let us instead remind ourselves how this comes about in the path integral formalism. Feynman taught us that quantum ampilitudes can be written as a sum over paths

$$\langle x_f|e^{iHt/\hbar}|x_i\rangle = N\int [dx]e^{-iS/\hbar}. \tag{2.3}$$

The semiclassical limit $\hbar \to 0$ causes the exponential on the right-hand side to oscillate quickly so that nearby paths tend cancel one another. The [parametrically] dominant contribution to the path integral then comes from the path of stationary phase (steepest descent). This, of course, comes from the extremum of the action and corresponds to the classical path, x_{cl}. The usual game, then, is to expand about this path

$$x(t) = x_{\text{cl}}(t) + \sum_n c_n x_n(t) \tag{2.4}$$

using a convenient basis of functions $x_n(t)$ chosen so that the resulting functional determinant can be calculated easily. *Ho hum!* This is all familiar material since we're all grown up and already know all about fancy things like path integrals.

One thing that might cause us to pause, however, is to ask how this can possibly give us quantum tunneling when—by definition—for such processes there exists no classical path to perturb

[1]The identification of the quadratic part of the Lagrangian as classical is most easily seen in $\lambda\phi^4$ theory by scaling $\phi \to \phi' = \lambda^{1/4}\phi$ and noting that the partition function contains an exponential of $\mathcal{L}'/\hbar = (1/\lambda\hbar)[\frac{1}{2}(\partial\phi)^2 + \cdots]$. The semiclassical limit corresponds to small $(\lambda\hbar)$.

about. Conceptually this is a bit of a hum-dinger, but a hint can already be seen from (2.1): for $E < V$ the momentum becomes imaginary and we get the expected exponential behavior. Going back to our high-brow path integrals, we shall now see that we can get to this behavior by working in the imaginary time formalism via our old friend, the Wick rotation,

$$t = i\tau. \tag{2.5}$$

The validity of the Wick rotation may seem odd, but we shall simply treat this as a change of variables[2]. What does this buy us? Our amplitudes now take the form

$$\langle x_f|e^{-H\tau/\hbar}|x_i\rangle = N\int [dx]e^{-\frac{1}{\hbar}\int^\tau \mathcal{L}_{\rm E}d\tau'} \tag{2.6}$$

where

$$\mathcal{L}_{\rm E} = \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x). \tag{2.7}$$

We see that the potential has swapped signs relative to the kinetic term. This is easy to see from the equation of motion, $m\ddot{x} = -V'(x)$, where the left-hand side picks up an overall sign when $t \to \tau$. On the surface this seems like a completely trivial change (it is), but the point is that the minus sign flips the potential barrier upside down allows us to find a classical path between the two extrema. The imaginary time formalism provides a classical path about which we can sensibly make a semiclassical approximation for tunneling processes.

Let us make two remarks:

- The [imaginary] time evolution operator $\exp(-H\tau/\hbar)$ is not unitary, but Hermitian.
- An interesting consequence of this is that large time evolution leads to a projection to the ground state (if the state has any overlap with $|0\rangle$):

$$\begin{aligned}
\lim_{\tau\to\infty}\langle x\left|e^{-H\tau/\hbar}\right|x'\rangle &= \lim_{\tau\to\infty}\langle x|n\rangle\langle n|x'\rangle e^{-E_n\tau/\hbar} &(2.8)\\
&= \langle x|0\rangle\langle 0|x'\rangle e^{-E_0\tau/\hbar} &(2.9)\\
&= \psi_0(x)\psi_0^*(x')e^{-E_0\tau/\hbar}. &(2.10)
\end{aligned}$$

2.2 The harmonic oscillator

Before discussing any tunneling phenomena, let's review the harmonic oscillator in the imaginary time formalism. In all cases to follow we will consider events between an initial time $-T/2$ and a final time $T/2$ with $T \to \infty$. As before, we shall expand about the classical path

$$x(\tau) = x_{\rm cl}(\tau) + \sum_n c_n x_n(\tau), \tag{2.11}$$

[2] Analyticity is a rather deep idea in physics which the author is still trying to appreciate with the proper reverence. For example, non-analyticity in the effective potential signals the appearance of massless modes. More mundanely, in the Kramers-Kronig relation analyticity is related to causality. In the present case, the important feature is that the Hamilton-Jacobi equations still hold upon complexification. For more about this rather deep connection, see e.g. [3].

where we choose our basis functions x_n to satisfy the appropriate boundary conditions ($x_n(\pm T/2) = 0$) and to be eigenfunctions of $\delta^2 S/\delta x_{\rm cl}^2$,

$$-\frac{d^2 x_n}{x\tau^2} + V''(x_{\rm cl})x_n = \lambda_n x_n. \tag{2.12}$$

The path integral measure is converted to $[dx] \to \prod_n (2\pi\hbar)dc_n$, where we use Coleman's normalization [1]. In the semiclassical limit where $\hbar \to 0$, stationary phase tells us that the amplitude is dominated by

$$\langle x_f|e^{-H\tau/\hbar}|x_i\rangle = Ne^{-S(x_{\rm cl})/\hbar}\left[\det\left(-\partial_\tau^2 + V''(x_{\rm cl})\right)\right]^{-1/2} \tag{2.13}$$

up to higher order terms in $\hbar$. By our choice of basis functions the determinant over $\delta^2 S/\delta x_{\rm cl}^2$ can be written as $\prod_n \lambda^{-1/2}$. With suave arithmetic maneuvering one can show that the normalization and the determinant simplify to

$$\left[\det\left(-\partial_\tau^2 + V''(x_{\rm cl})\right)\right]^{-1/2} = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2}, \tag{2.14}$$

where $\omega \equiv V''(x_{\rm cl})$. Note that this agrees with our earlier remark that a long time evolution projects out the ground state. The derivation of this expression is not particularly enlightening, but (2.14) will be a useful result for future reference. A proper derivation can be found in [2].

Proof. **Derivation of (2.14).** We shall follow the steps in [2]. The boundary conditions on our basis functions allow us to solve for their eigenvalues,

$$\lambda_n = \frac{\pi n^2}{T^2} + \omega^2. \tag{2.15}$$

Let us then massage the factors in (2.14) as follows

$$\left[\det\left(-\partial_\tau^2 + V''(x_{\rm cl})\right)\right]^{-1/2} = N\prod_n \lambda_n^{-1/2} \tag{2.16}$$

$$= N\prod_n \left(\frac{\pi^2 n^2}{T^2}\right)^{-1/2} \prod_n \left(1 + \frac{\omega^2 T^2}{\pi^2 n^2}\right)^{1/2}. \tag{2.17}$$

We have simply pulled out the first factor from the expression for each λ_n. This term should look rather familiar. It is the only term to survive the limit $V \to V_0 =$ constant , i.e. it gives the 'classical' contribution for a plane wave solution. We have

$$N\prod_n \left(\frac{\pi^2 n^2}{T^2}\right)^{-1/2} = \int \frac{dp}{2\pi} e^{-p^2 T/2} = \frac{1}{\sqrt{2\pi T}}. \tag{2.18}$$

Meanwhile, for the other factor we can invoke an identity for the hyperbolic sine,

$$\sinh(\pi y) = \pi y \prod_n \left(1 + \frac{y^2}{n^2}\right). \tag{2.19}$$

This allows us to write

$$\left[\det\left(-\partial_\tau^2 + V''(x_{\rm cl})\right)\right]^{-1/2} = \frac{1}{\sqrt{2\pi T}}\left(\frac{\sinh(\omega T)}{\omega T}\right)^{-1/2} \tag{2.20}$$

$$= \sqrt{\frac{\omega}{\pi}}\left(2\sinh(\omega T)\right)^{-1/2} \tag{2.21}$$

$$= \sqrt{\frac{\omega}{\pi}}e^{-\omega T/2}\left(1 + \frac{1}{2}e^{-2\omega T} + \cdots\right). \tag{2.22}$$

□

2.3 The double well

Now that we're warmed up, let's move on to a quantum mechanical system with actual tunneling. As we discussed, the imaginary time formalism flips over the potential so that there now exists a classical path between the two extrema. This is shown heuristically in Fig. 1. The classical equation of motion tells us that if we start at $x = -a$, there is a conserved energy

$$E = 0 = \frac{1}{2}\left(\partial_\tau x\right)^2 - V(x). \tag{2.23}$$

From this we get $\partial_\tau x = \sqrt{2V(x)}$, which we can integrate.

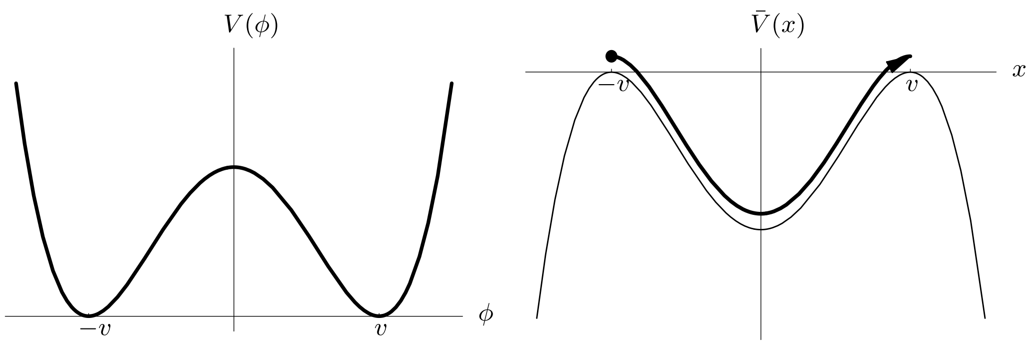


Figure 1: The double well potential and its flipped over euclidean version, from [4]. We use slightly different notation: the extrema will be labelled $x = \pm a$.

We start by writing out the action associated with this 'classical' tunneling path,

$$S_0 = \int d\tau \left[\frac{1}{2}(\partial_\tau)^2 + V(x)\right] = \int dt \left(\frac{dx}{dt}\right)^2 = \int_{-a}^{a} dx\sqrt{2V}. \tag{2.24}$$

Taylor expanding $\sqrt{2V}$ about $x = a$ at late times tells us that for $\tau \gg 1$

$$\partial_\tau x \approx \omega(a - x) \Rightarrow (a - x) \propto e^{-\omega\tau}. \tag{2.25}$$

As before, we've written $\omega = V''(a)$. This tells us that the instanton solution is a kink of some characteristic width $1/\omega$; these topological configurations are localized in time. This is the origin of the term **instanton**, or (as Polyakov suggested), "pseudo-particle."

One can solve for the structure of the instanton kink solution that interpolates between the two vacua. The answer is, as we would expect, a hyperbolic tangent. Instead of dwelling on this, let us move on to consider the tunneling amplitude. The usual formula with $S(x_{\text{cl}}) = S_0$ gives

$$\langle a|e^{-H\tau/\hbar}|-a\rangle = Ne^{-S_0/\hbar}\left[\det\left(-\partial_\tau^2 + V''(x_{\text{cl}})\right)\right]^{-1/2}. \tag{2.26}$$

In the case of the harmonic oscillator (single well) we already solved for the normalized determinant and found (2.14). Since the two minima in the double well locally behave like single wells and since the quantum state spends *most* of its time very close to one or the other well, our physics intuition tells us that (2.14) should hold up to some corrections to account for the instanton. Let us parameterize this correction as an overall factor K,

$$N\left[\det\left(-\partial_\tau^2 + V''(x_{\text{cl}})\right)\right]^{-1/2} = \left(\frac{\omega}{\pi\hbar}\right)^{-1/2} e^{-\omega T/2}K. \tag{2.27}$$

Thus we find that for a single instanton background,

$$\langle a|e^{-H\tau/\hbar}|-a\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{-1/2} e^{-\omega T/2}Ke^{-S_0/\hbar}. \tag{2.28}$$

2.3.1 The zero mode

This isn't the whole story. For large times (e.g. compared to the characteristic instanton lifetime $1/\omega$) we should properly account for the time translation invariance; i.e. we must include the effects of instantons occurring at any intermediate time. We can pitch this in fancier language. For example, we can say that the instanton has a **collective coordinate** which acts as a parameter to give different instanton solutions. In other words, the instanton has a **moduli space**.

More physically, we remark that the mode x_0 associated with this time translation can be thought of as a **Goldstone boson** for the spontaneously broken τ-symmetry. As one would expect, such a Goldstone mode has zero eigenvalue: $\lambda_0 = 0$. We can write out the form of this mode explicitly:

$$x(\tau) = x_{\text{cl}}(\tau + d\tau) = x_{\text{cl}}(\tau) + x_{\text{cl}}(\tau + d\tau) - x_{\text{cl}}(\tau) = x_{\text{cl}}(\tau) + \frac{dx_{\text{cl}}}{d\tau}d\tau + \cdots, \tag{2.29}$$

so that comparing to (2.11), we have

$$x_0 = S_0^{-1/2}\frac{dx_{\text{cl}}}{d\tau}, \tag{2.30}$$

where the normalization comes from our normalization of the classical path in (2.24) and the requirement that the modes x_n must be orthonormal.

We've made a bed of fancy words and now we have to lie in it. In particular, now that we've thrown around the idea of a **zero mode**, we need to stop and go all the way back to our evaluation of the functional integral. When we evaluated the functional integral into a determinant, we assumed that all of the eigenvalues λ_n were positive definite. This made all of our Gaussian integrals sensible. Now we have a zero eigenvalue, we have an apparent conundrum: $\exp(-c_0^2\lambda_0)dc_0 = dc_0$, so that we no longer have a Gaussian integral! This integral is formally infinite and we worry that our state is not normalizable.

Fortunately, we have nothing to fear. This apparently-divergent integral is simply corresponds to the integration over the instanton time that we mentioned above. In fact, in this light it is an *expected* 'divergence' for we need some kind of extensive dependence on the large time T.

To be more rigorous, we would like to convert the dc_0 integral into a $d\tau_c$ where τ_c is the instanton center. This is easy to work out since we know that the change in the instanton configuration $x(\tau) = x_{\text{cl}}(\tau) + \sum_n c_n x_n$ induced by a small change in τ_c is

$$dx = \frac{dx_{\text{cl}}}{d\tau}\, d\tau_c. \tag{2.31}$$

Meanwhile, the change in the instanton configuration from a change in c_0 is

$$dx = x_0\, dc_0. \tag{2.32}$$

Combining these results we find that

$$dc_0 = \sqrt{S_0}\, d\tau_c. \tag{2.33}$$

This wasn't particularly surprising, but we want to keep track of the constant of proportionality ($\sqrt{S_0}$) since it will feed into our determination of the factor K. This digression is prescient in another way: it is an example of how an instanton couples to zero modes of a system, and so is a crude prototype for the **'t Hooft operator** that we will explore in QCD.

2.3.2 Determining K

While the current treatise focuses primarily on the qualitative features of what instantons are and what they can do, it is nice to flesh out quantitative results when they don't require much work[3]. For our quantum mechanical example, it is nice to explicitly work out the K factor in (2.28). From the zero mode discussion we found that

$$\frac{1}{\sqrt{2\pi\hbar}}\, dc_0 = \sqrt{\frac{S_0}{2\pi\hbar}}\, d\tau_c. \tag{2.34}$$

Thus when we take into account the integration over the instanton position, our tunneling amplitude takes the form

$$\langle a|e^{-HT/\hbar}|-a\rangle = NT\sqrt{\frac{S_0}{2\pi\hbar}}e^{-S_0/\hbar}\left[\det\left(-\partial_\tau^2 + V''(x_{\text{cl}})\right)\right]^{-1/2}. \tag{2.35}$$

Comparing to (2.28) we find that

$$K = \sqrt{\frac{S_0}{2\pi\hbar}}\left|\frac{\det(-\partial_\tau^2+\omega^2)}{\text{Det}(-\partial_\tau^2+V''(x_{\text{cl}}))}\right|, \tag{2.36}$$

[3]Unfortunately, most of the 'honest' calculations in this field require a lot of work.

where we've introduced the notation Det to mean determinant *without* any zero modes. (Most review literature refers to this as det', but the author thinks this notation is silly because it is suggestive of some kind of derivative.)

2.3.3 Multi-instanton effects

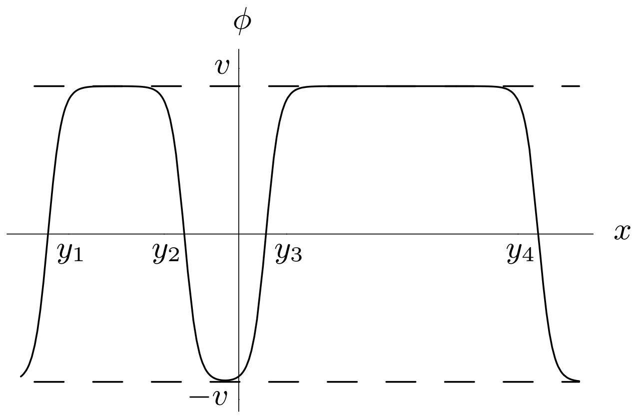


Figure 2: An example of a multi-instanton background from [4]. Note that we use different notation. The author is too pressed for time to draw his own diagrams properly.

We're not yet done with the double well. If we think about it a bit longer, we'll note that the one instanton solution is not the only possible classical background to perturb about. One could, in fact, have a chain of instantons and 'anti-instantons' tunneling back and forth between the vacua, as depicted in Fig. 2. The only restriction is that instantons from $|-a\rangle \to |a\rangle$ can only be followed by anti-instantons from $|a\rangle \to |-a\rangle$ and vice versa. Our previous logic leading up to the K factor should tell us that these n-instanton solutions should take the same form but with $K \to K^n$:

$$N\left[\det(-\partial_\tau^2 + V''(x_{\rm cl}))\right]^{-1/2} = \left(\frac{\omega}{\pi\hbar}\right)^{-1/2} e^{-\omega T/2} K^n e^{-S_0/\hbar}. \tag{2.37}$$

One should again integrate over these instanton positions (corresponding to their zero modes, as discussed above). Note that the alternating instanton/anti-instanton ordering gives us a slightly more non-trivial integral,

$$\int_{T/2}^{T/2} d\tau_1 \int_{-T/2}^{\tau_1} d\tau_2 \cdots \int_{-T/2}^{\tau_{n-1}} d\tau_n = \frac{T^n}{n!}. \tag{2.38}$$

This integral should look rather familiar from one's first introduction to perturbation theory and canonical quantization in quantum field theory.

The 'full' classical background for the semiclassical approximation must take into account these multi-instanton solutions. We thus find

$$\langle \pm a|e^{-H\tau/\hbar}|-a\rangle = \sqrt{\frac{\omega}{\pi\hbar}}e^{-\omega T/2}\sum_{n\text{ odd/even}}\frac{\left(Ke^{-S_0/\hbar}T\right)^n}{n!} \tag{2.39}$$

$$= \sqrt{\frac{\omega}{\pi\hbar}}e^{-\omega T/2}\frac{1}{2}\left[e^{Ke^{-S_0/\hbar}T} \mp e^{-Ke^{-S_0/\hbar}T}\right] \tag{2.40}$$

up to leading order in $\hbar$. Recalling that the time evolution operator has eigenstates of definite energy, e.g. (2.8), we find that the energy eigenstates and their eigenvalues are

$$|\pm\rangle \equiv |a\rangle \pm |-a\rangle \qquad\qquad E_\pm = \frac{1}{2}\hbar\omega \pm \hbar K e^{-S_0/\hbar}. \tag{2.41}$$

This is, of course, exactly as we would have expected since the action obeys a $\mathbb{Z}_2$ symmetry under which $x \to -x$. In such a case we know that the ground state of the system is degenerate with a small breaking coming from tunneling effects (exactly what we've re-derived). The energy eigenstates of the system are also eigenstates of the parity operator (since this commutes with the Hamiltonian) and the lowest states correspond to the symmetric and antisymmetric combinations. We note that the factor of $\exp(-S_0/\hbar)$ makes it clear that even though this term is small, it is clearly a *non-perturbative* effect that one would not have found doing naïve perturbation theory.

2.4 The periodic potential

There's one more obvious tunneling generalization in quantum mechanics. Since we've already made the leap form a single well harmonic oscillator to a double well potential, it's trivial to go to a triple or n-tuple well potential. The strategy is now completely analogous, though finding closed form solutions for the appropriate instanton sums become non-trivial; e.g. for a triple well one may have up to two instantons in a row but no more. One way to avoid this is to identify the sides of a well, so that we are led to the periodic (cosine) potential.

Remark: It should be 'obvious' that the functional form near the minima of each of these potentials is arbitrarily close to the harmonic oscillator for long times T so that our strategy of duplicating the single well result up to a K factor is justified in the semiclassical approximation.

For the periodic potential there is no constraint on how many instantons (n) or anti-instantons ($\bar{n}$) one might have nor is there any constraint on their ordering except that they must sum to give the appropriate shift in vacua. For example, consider the tunneling process $|j\rangle \to |k\rangle$. The appropriate generalization of our above techniques is

$$\langle k|e^{-HT/\hbar}|j\rangle = \sqrt{\frac{\omega}{\pi\hbar}}e^{-\omega T/2}\sum_{n=0}\sum_{\bar{n}=0}\frac{1}{n!\bar{n}!}\left(Ke^{-S_0/\hbar}T\right)^{n+\bar{n}}\delta_{(n-\bar{n}),(k-j)}. \tag{2.42}$$

We may now use one of our usual tricks and go to a Fourier series representation,

$$\delta_{ab} = \int_0^{2\pi}\frac{d\theta}{2\pi}\,e^{i\theta(a-b)}. \tag{2.43}$$

This allows us to simplify our amplitude to the form

$$\langle k|e^{-HT/\hbar}|j\rangle = \sqrt{\frac{\omega}{\pi\hbar}}e^{-\omega T/2}\int_0^{2\pi}\frac{d\theta}{2\pi}\,e^{i(j-k)\theta}\exp\left(2KTe^{-S_0/\hbar}\cos\theta\right). \tag{2.44}$$

Just as the double well potential contained a discrete permutation symmetry that forced the energy eigenstates to also be permutation eigenstates (whose energy is only split by tunneling effects), the periodic potential also contains a translation symmetry that forces eigenstates to be shift-invariant. This should again sound very familiar from toy models of solid state systems: the energy eigenstates are **Bloch waves**,

$$|\theta\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2\pi}}\sum_n e^{-in\theta}|n\rangle. \tag{2.45}$$

The $|\theta\rangle$ states are eigenstates of a shift operator, e.g. a shift to the next well to the right, T_1:

$$T_1|\theta\rangle = e^{i\theta}|\theta\rangle. \tag{2.46}$$

The energy eigenvalues for the Bloch waves are

$$E(\theta) = \frac{1}{2}\hbar\omega + 2\hbar Ke^{-S_0/\hbar}\cos\theta. \tag{2.47}$$

This example will turn out to be very handy when we discuss the θ-vacua of Yang-Mills theories.

2.5 The bounce

If you are doing everything well, you are not doing enough.
– Howard Georgi, personal motto [5]

One topic that is egregiously omitted in this document is the treatment of 'the bounce' and tunneling from metastable vacua. Such field theoretic calculations have been important in early universe cosmology and, more recently, the long-term stability of vacua in metastable SUSY-breaking models. This type of tunneling is also described very well by first considering a quantum mechanical archetype which demonstrates many subtle aspects of the semiclassical approximation. This is described in Section 2.4 of Coleman's lectures [1].

2.6 Polemics: tunneling in QM versus QFT

2.6.1 Summary of QM

Before wiping our hands of quantum mechanics and graduating to quantum field theory (straight to gauge theory, no less), let's pause to make a very important philosophical point about the transition from 0+1 dimensions to nontrivial spacetimes (i.e. QM $\to$ QFT). It is not a stretch to say that tunneling is one of the key results in quantum mechanics. The idea that a quantum state can pass through a classically-impenetrable barrier is the foundation for all of the manipulations we did above[4]. This is also related to the idea that energy eigenstates ought to also be eigenstates of any discrete symmetry relating theory's vacua, hence the appearance of symmetric and antisymmetric $|\pm\rangle$ states in the double well or the Bloch waves $|\theta\rangle$ in the periodic potential. The *cost* of barrier penetration was evident from our WKB formula (2.1): the oscillating plane wave picks picks up a factor of i in its argument from $\sqrt{E-V}$ and then becomes exponentially suppressed. The bigger the energy difference the more strongly suppressed the tunneling amplitude. This is all as we have grown to know and love: tunneling is one of the highlight calculations in any quantum mechanics course.

2.6.2 Zero (and one) spatial dimensions is special

In quantum field theory of *any* nonzero space dimension, however, we **never** talk about a physical field tunneling. Never (well, almost—certainly never in an introductory course). The reason is clear: quantum field theory is an infinite number of copies of quantum mechanics: there is a coupled QM oscillator at each point in spacetime so that any discussion about the vacuum is not a statement about a Hilbert space, but rather a *Fock* space. Instead of energy, QFT deals with energy *densities* that must be multiplied by the (infinite) volume of spacetime. More concretely: in order to tunnel from one vacuum to another, each of an *infinite number* of QM oscillators must tunnel. This picks up an infinite number of $e^{-\Delta E}$ suppression factors (where ΔE is the characteristic energy difference) which leads to *zero* tunneling probability between Fock space vacua. Thus in QFT there is never any tunneling phenomena between degenerate vacua with a potential barrier. The double well potential in 1+1 dimensional field theory leads to spontaneous symmetry breaking (as do its more famous 'Mexican hat' generalizations in higher dimensions), rather than any parity-symmetric vacuum states. (To be a bit more honest, we will remark below that the case of a single spatial dimension is also 'special' due to kink solutions and should be lumped together with the 0+1 dimensional case[5].)

From this point of view one ought to say that we've so far gone over a very nice review of undergraduate quantum mechanics, but the document should end right here. It's not clear why there should be anything else to be said about 'instantons'—certainly not in field theory where there is no tunneling phenomena to be considered. Thus you should be wondering why there are any more pages to this document at all.

2.6.3 Can we be clever?

The argument above is certainly heuristic. One could ask if we can be clever enough to find a loophole. A good first attempt is to imagine a situation in field theory with metastable vacua tunneling to a ground state via bubble nucleation. Here *finite* volumes of space tunnel and the difference in energy between the true ground state and the metastable state ($\delta E \sim$ volume) versus the surface tension coming from the energy barrier ($\sim$ surface area) can lead to either vacuum decay or bubble collapse depending on the characteristic size of the quantum fluctuation. Such tunneling events can indeed be calculated using fancy instanton methods (though they are unfortunately

[4]Note, for example, that a double well with an infinite finite-width barrier separating the minima will exactly have two degenerate spectra.

[5]We thank Zohar Komargodski for bringing this to our attention.

outside the scope of the present document). These cases, however, avoid the philosophical issue above because one transitions from a metastable vacuum to an energy-favorable vacuum. It should be clear that one can *never* have this sort of bubble nucleation for physical fields between *degenerate* vacua.

We can be a little more clever and consider the Sine-Gordon kink solution. We know that a scalar field in 1+1 dimensions and a double well potential can have kink solutions where the field only has a nonzero vacuum expectation value over a localized position in space. One typically looks for a time-independent solution from which one can reconstruct time-dependence from Lorentz invariance. One could then imagine Wick rotating to try to construct such a kink in the [imaginary] time direction rather than the space direction. Here, however, one still runs into the problem of a vanishing tunneling amplitude because an infinite number of QM oscillators must undergo barrier penetration. Fancier attempts involving just scalar fields will similarly fail for more general grounds: **Derrick's theorem** tells us that scalar fields cannot have solitonic solutions in dimension higher than one. (A short discussion of Derrick's theorem can be found in, e.g. [6].)

We're on the right track. If one wanted to naïvely generalize the Sine-Gordon kink into a vortex for a two dimensional scalar, it is a well-known result that Derrick's theorem manifests itself as a divergence in the energy of the static configuration. For the case of space-like solitons, however, we already know how to evade Derrick's theorem: we introduce gauge fields. Indeed, the vortex solution is given by the winding of a U(1) gauge field. Now we have a handle for how to proceed into QFT.

2.6.4 Gauge redundancy

What is so special about gauge theory that allows us to create solitons? In quantum mechanics where we spoke about the tunneling of a 'physical' state to another 'physical' state and we argued that in QFT it is impossible to pull an infinitely (spatially) extended field over a potential barrier of finite energy density. Gauge theory provides the loophole we wanted because it gives us lots of manifestly *unphysical* degrees of freedom to twist and wrap about. We will see that topological winding about gauge degrees of freedom will lead to classes of distinct gauge vacua. Surprisingly, tunneling can occur between these vacua (since such tunneling doesn't require a physical field being pulled over an energy potential) and this leads to the construction of Bloch waves (θ-vacua) and rather remarkable physical effects.

It is worth spouting some rhetoric about gauge symmetry that amounts to doctrine rather than physics. Gauge symmetry can be thought of in two ways:

1. A gauge symmetry is what one obtains by taking a global ('normal') symmetry and promote it to a local symmetry.

2. A gauge symmetry is a *redundancy* in the way one describes a physical system.

Both doctrines are compatible, but the latter point of view is particularly handy for the philosophical dilemma at hand. Physical states are defined modulo gauge orbits. In other words, gauge transformations form an equivalence class of physical states. The extra degrees of freedom afforded by this gauge redundancy is certainly convenient, but the actual *physical* system is described by modding out the gauge degree of freedom (fixing a gauge).

The Yang-Mills theories that we get when introducing a gauge redundancy[6] can have additional structure due to the gauge symmetry. In particular, we will find that this structure lends itself to multiple vacua that are gauge equivalent but distinguished topologically from one another. One will not be able to continuously deform one topological vacuum to another without pushing the *physical* field over a potential which, as we discussed, is forbidden in QFT. Further, each of these vacua appear to spontaneously break gauge invariance. The 'magic' now is that because gauge degrees of freedom are *redundant* (whether or not they are continuously connected along the vacuum manifold), one is free to construct the analog of our Bloch wave states *in gauge space*: i.e. we can construct gauge-invariant linear combinations of the topologically distinct vacua. These are called the θ vacua and will be the main topic in this paper.

3 *Entrée*: Vacua of gauge theories

In anticipation of non-trivial vacuum structure and tunneling phenomena, we now study Yang-Mills theory in Euclidean spacetime. We'll be a little bit loose with our conventions—we may miss a sign here or there—but the underlying physics will be transparent. There will be some relatively fancy ideas tossed around, but the physical intuition follows precisely the simple quantum mechanical examples above. This is why we invested so much into our quantum mechanical treatment, it gives us a little bit of wiggle room to play it fast and loose now that there are many more moving parts.

First we'll go over the relevant physics to get to the point. Then we'll close with a section that gives just the *slightest* flavor for the mathematical elegance that's running 'under the hood.'

3.1 Euclidean Yang-Mills

We would now like to consider classical solutions to the Euclidean equation of motion. We define our gauge field and generator normalization via

$$A_\mu = igA^a_\mu t^a \qquad\qquad \mathrm{tr}\left(t^a t^b\right) = \frac{1}{2}\delta^{ab}. \tag{3.1}$$

The Euclidean action takes the form

$$S_E = \frac{1}{4g^2}\int d^4x\, F_{\mu\nu}F^{\mu\nu}, \tag{3.2}$$

where we remind ourselves that we're working with a Euclidean metric so that raised and lowered indices are equivalent. The classical equation of motion is

$$D_\mu F^{\mu\nu} = 0. \tag{3.3}$$

[6]One might introduce such a gauge redundancy to describe vector particles in a handy way, or—more technically—to identify (via the R_ξ gauges) the extra degree of freedom that in a massive vector that disappears for a massless vector.

3.2 Finite action and the field at infinity

As we mentioned above, one of the principal differences between quantum mechanics and quantum field theory is the appearance of several extensive quantities—such as the action—that appear to diverge as the volume of spacetime go to infinity. This is not actually a problem since we are primarily concerned with densities. In the $\hbar \to \infty$ limit the path integral is dominated by configurations with finite action. Thus our approach in the semiclassical limit will be to perturb about solutions which give finite action. Perhaps more intuitively this is equivalent to the condition of finite energy since there is no potential in pure Yang-Mills theory.

Let us define a sphere of constant radius r in Euclidean spacetime, $S^3 \subset \mathbb{R}^4$. Finite action requires that the Lagrangian (density) falls of sufficiently quickly on this surface as $r \to \infty$. This means that the field strength $F_{\mu\nu}$ must must go like $F \sim r^{-3}$ for large distances (the power is because our boundary is a three-sphere), and so the gauge potential must go like $A_\mu \sim r^{-2}$. It is completely equivalent and—as we will see—more useful to think about this as boundary conditions for A at the three sphere boundary of spacetime S^3 with radius $R \to \infty$. Thus we are tempted to write

$$A_\mu(x)\big|_{|x|=R} = 0. \tag{3.4}$$

This, of course, is too restrictive and must be wrong: it is not gauge invariant. The correct statement is that the potential on the boundary must be in the same gauge orbit as $A_\mu(R) = 0$. Thus we are led to the general form that A must be **pure gauge**,

$$A_\mu(x)\big|_{|x|=R}\, g(x)\partial_\mu g^{-1}(x), \tag{3.5}$$

where $g(x)$ is an element of our gauge algebra (the Lie algebra of our gauge group). If you are already very fancy you might call this configuration a **Maurer-Cartan form**, but it won't earn you any friends. For our purposes it is sufficient to consider the gauge group SU(2). As physicists we are happy to accept that any more complicated non-Abelian unitary group will contain SU(2) as a subgroup so that our conclusions will hold rather generally.

The elements of SU(2) can be written as an exponential of its Lie algebra, e.g. expanded in terms of the Pauli matrices ($\tau = i\sigma/2$),

$$U = \exp(g_0 \mathbb{1} + \vec{g} \cdot \vec{\tau}) \tag{3.6}$$

with the restriction $g_0^2 + \vec{g}^2 = 1$. From this it is clear that SU(2) is isomorphic (as a Lie group and topologically) to the three sphere, SU(2)$\cong S^3$. The gauge transformations in (3.5) are then maps from the boundary of Euclidean spacetime to SU(2), $g : S^3 \to \mathrm{SU}(2) \cong S^3$.

3.3 Bastard topology

Even as physicists (e.g. from our experience with monopoles or other solitons) we know that maps between topologically equivalent spaces can be classified by the appropriate **fundamental group**, Π_n, with generalizes winding number. We will, in particular, be interested in $\Pi_3(\mathrm{SU}(2)) = \mathbb{Z}$ which tells us how many times the image of the map g wraps SU(2)$\cong S^3$. Hoity-toity people who like to smell their own farts might call this the **Pontryagin number**. We will say a little bit about the construction of this number in Section 4, but for now we'll pull weakly-motivated results out of the aether on a need-to-know basis in order to the physics.

Let us motivate this excursion with the simpler case of a map $f : S^1 \to \mathrm{U}(1) \cong S^1$. In this case the winding number has an analogous definition,

$$n = \frac{-i}{2\pi} \int_0^{2\pi} d\theta \left[\frac{1}{f(\theta)} \frac{df(\theta)}{d\theta} \right]. \tag{3.7}$$

One can check the validity of this equation by plugging in a simple form, e.g. $f(\theta) = e^{i(n\theta+a)}$.

Let us now return to the case of SU(2). One should expect that the winding number can be calculated by integrating some topologically-sensitive function of g over its domain S^3. It turns out that the correct integral is

$$n = \frac{-1}{24\pi^2} \int_{S^3} d\theta_1 d\theta_2 d\theta_3 \, \mathrm{tr}\left(\epsilon^{ijk} A_i A_j A_k\right), \tag{3.8}$$

with A a pure gauge configuration (3.5). The funny factor of $24\pi^2$ cancels terms from the angular integral so that n really is an integer. The trace runs over gauge representation indices. Gauge invariance is further manifested by the antisymmetric ϵ^{ijk} tensor which cancels the Jacobian when performing a gauge transformation. If you're particularly trusting you should thus accept that this quantity really is topological and is invariant under continuous deformations.

Proof. It is sufficient to show that n is invariant under small transformations. We will write this as $\delta g(x) = g(x)\delta T(x)$. The variation of of A is then

$$\delta(g\partial g^{-1}) = \delta g \cdot \partial g^{-1} + g\partial\delta g^{-1} = g\delta T \cdot \partial g^{-1} - g(\partial\delta T)g^{-1} - g\delta T \cdot \partial g^{-1} = -g(\partial\delta T)g^{-1}. \tag{3.9}$$

From this we can write the change in n (using auspicious integration by parts)

$$\delta n \propto \int d\theta_1 d\theta_2 d\theta_3 \, \epsilon^{ijk} \mathrm{tr}\left(g\partial_i g^{-1} \, g\partial_j g^{-1} \, g(\partial_k \delta T)g^{-1}\right) \tag{3.10}$$

$$\propto \int d\theta_1 d\theta_2 d\theta_3 \, \epsilon^{ijk} \mathrm{tr}\left[\left(\partial_i g^{-1}\right)\left(\partial_j g\right)\left(\partial_k \delta T\right)\right] = 0, \tag{3.11}$$

where we used the antisymmetry of ϵ in the final step.

To check that n is an integer we can do a quick calculation. At the north pole the angular directions can be treated as Cartesian directions, $d\theta \to dx$ and the Maurer-Cartan form is $g\partial_i g^{-1} = -i\sigma_1$ so that

$$\epsilon^{ijk} \mathrm{tr}\left(g\partial_i g^{-1} \, g\partial_j g^{-1} \, g\partial_k g^{-1}\right) = -12. \tag{3.12}$$

The hypersphere area is $2\pi^2$ so that we indeed get the desired result that $n \in \mathbb{Z}$. □

We can construct representative maps for each homotopy class in $\Pi_3(\mathrm{SU}(2))$. Writing $g^{(n)}$ for a map of winding number n,

$$g^{(0)} = 1 \qquad g^{(1)} = \frac{x^0 + i\vec{x} \cdot \vec{\sigma}}{r} \qquad g^{(n)} = \left[g^{(1)}\right]^n. \tag{3.13}$$

This generalizes the composition of $\Pi_1(\mathrm{U}(1))$ maps, $e^{in\theta} e^{im\theta} = e^{i(n+m)\theta}$.

3.4 Winding up an energy density

We now present what, at this cursory level, may appear to be a rather nice magic trick. For no particular reason, let us consider the **Chern-Simons current**,

$$K_\mu = 4\epsilon_{\mu\nu\lambda\sigma}\mathrm{tr}\left(A_\nu\partial_\lambda A_\sigma + \frac{2}{3}A_\nu A_\lambda A_\rho\right) = \frac{4}{3}\epsilon_{\mu\nu\lambda\sigma}\mathrm{tr}\left(g\partial_i g^{-1}\, g\partial_j g^{-1}\, g\partial_k g^{-1}\right). \tag{3.14}$$

One should recognize that in addition to the rather a striking resemblance to (3.7), this looks very similar to the abelian anomaly coefficient. We will see later that this is not a coincidence. Let us press on and consider taking the divergence of this current,

$$\partial^\mu K_\mu = 2\,\mathrm{tr}(F^{\mu\nu}\widetilde{F}_{\mu\nu}), \tag{3.15}$$

where $\widetilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma}$ is the dual field strength. Still with no particular motivation, let us integrate this quantity:

$$\int d^4x\, 2\,\mathrm{tr}(F^{\mu\nu}\widetilde{F}_{\mu\nu}) = \int d^3S\,\hat{r}_\mu K^\mu \overset{!}{=} 32\pi^2 n. \tag{3.16}$$

Here we've used (3.8) and (3.5). The punchline is that we've found a handy way to represent the winding number in terms of the field strength,

$$n = \frac{1}{16\pi^2}\int d^4x\,\mathrm{tr}\left(F\widetilde{F}\right). \tag{3.17}$$

3.5 The Bogomol'nyi bound

Continuing with our semiclassical approximation we would like to determine the action of the instanton gauge configurations that give tunneling between the vacua of different winding numbers. We know from our experience with spacelike solitons in field theory that a useful trick is to saturate the **Bobomol'nyi bound**. We note that the field strength and its dual trivially obey positivity,

$$\int d^4x\,\mathrm{tr}\left(F \pm \widetilde{F}\right)^2 \geq 0. \tag{3.18}$$

From this we obtain a bound that is written explicitly in terms of the winding

$$\int d^4x\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu} \geq \left|\int d^4x\,\mathrm{tr}\,F_{\mu\nu}\widetilde{F}^{\mu\nu}\right| = 16\pi^2 n. \tag{3.19}$$

This tells us that the Euclidean action must satisfy the Bogomol'nyi bound,

$$S_E[A] \geq \frac{8\pi^2 n}{g^2}. \tag{3.20}$$

The usual manipulations tell us that the bound is saturated for (anti-)self dual field strenghts, $F = \pm\widetilde{F}$. These correspond to **instantons** and **anti-instantons**.

Proof. A cute way to show this is to write out the action as

$$S_E = \frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} = \frac{1}{4}\int d^4x \left[F\widetilde{F} + \frac{1}{2}\left(F - \widetilde{F}\right)^2\right] = \frac{8\pi^2 n}{g^2} + \frac{1}{8}\int d^4x \left(F - \widetilde{F}\right)^2. \quad (3.21)$$

The anti-instanton case corresponds to $\vec{x} \to -\vec{x}$ which gives $F\widetilde{F} \to -F\widetilde{F}$ and so $n \to -n$. □

As usual, the real benefit of the Bogomol'nyi trick is that we have converted our problem of finding classical solutions from a second-order differential equation into one that is first order.

3.6 BPST instantons

In the present document we won't be too concerned with the explicit form of the instanton solution, but in the interests of completeness and passing this A-exam let's go over the construction anyway. We know that on the boundary the gauge potential must approach pure gauge. Because we know we want to generate a winding number, we will consider pure gauge configurations with winding n. Thus we know that as $r \to \infty$, the A must approach

$$A_\mu = f(r)g^{(n)}\partial_\mu \left[g^{(n)}\right]^{-1} + \mathcal{O}(1/r^2), \quad (3.22)$$

where $f(r) \to 1$ as $r \to \infty$ and the $g^{(n)}$s are given in (3.13). Since we know each tunneling buys us a factor of $\exp(-8\pi n/g^2)$ we'll focus on the one-instanton case, $(n) = (1)$.

Rotations and gauge transformations. One of the features of SU(2) that makes it particularly amenable to instanton solutions is its relation to the 'Euclidean Lorentz group' (i.e. the rotation group) O(4)≅SU(2)×SU(2). Under a rotation the $g^{(1)} \propto (x^0 + \vec{x}\cdot\vec{\tau})$ element rotates as $g^{(1)} \to h_L g^{(1)} h_R^{-1}$ for some h_L and h_R in SU(2). Now note that under a *constant* gauge transformation h_0,

$$A_\mu \to h_0 A_\mu h_0^{-1} + \mathcal{O}(1/r^2). \quad (3.23)$$

Thus we see that choosing $h_0 = h_L^{-1}$ allows us to undo any rotation. This is what motivated us to consider f a function of r in (3.22).

As an ansatz let's assume that the $\mathcal{O}(1/r^2)$ term can be dropped. Solving the self-duality condition from our Bogomol'nyi bound we find the BPST solution,

$$f(r) = \frac{r^2}{r^2 + \rho^2}, \quad (3.24)$$

where ρ is an arbitrary constant with dimensions of length. One can guess that it corresponds to the size of the instanton. We will not discuss the effects that cut off the instanton behavior at large and small values of ρ.

These instanton solutions are conformally invariant. Let us remark on the **collective coordinates** of our instanton configuration, that is, we can ask how can we generate new one-instanton solutions trivially from this solution. The instanton configuration can be scaled, special-conformal-transformed, translated, rotated, and gauge transformed. We have already shown that rotations and (at least a subset of) gauge transformations generate identical instantons. Further, it turns out that combined gauge transformations and translations can generate special conformal transformations. Thus we end up with an eight parameter set of collective coordinates corresponding to scaling (1), translations (4), and rotations (3).

3.7 Instantons enact vacuum tunneling

Let us explicitly see how instantons change the winding number of our vacuum. We will work in $A_0(x) = 0$ gauge. this affords us a leftover gauge freedom $g(\vec{x})$ with $\partial_0 g(\vec{x}) = 0$. It won't hurt anybody to put our system into a box[7] so long as the boundary has a definite winding number (we'll take $n = 1$) and respects our gauge choice. Let's heuristically draw our space as we have in Fig. 3. As we explained above, the vacuum states must be pure gauge on the boundary

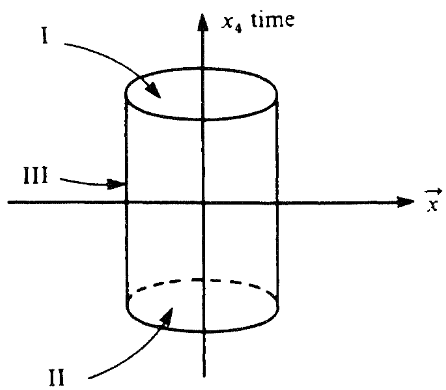


Figure 3: The instanton boundary over Euclidean spacetime. Image wrenched heartlessly from [7].

$$A_i|_{\text{boundary}} = g(\vec{x})\left[\partial_i g(\vec{x})\right]^{-1}. \tag{3.25}$$

For the spacelike boundary at $\tau = -T/2$ we can pick $A_i(-T/2, \vec{x}) = 0$ via $g(\vec{x}) = 1$. This corresponds to $g = g^{(0)}$ which, of course, is the trivial vacuum with zero winding. Now suppose that the volume of spacetime is filled with the one-instanton gauge configuration. Let's calculate the Pontryagin index (winding number),

$$n = \frac{1}{16\pi^2}\int d^4x \, \mathrm{tr}\left(F\widetilde{F}\right) \tag{3.26}$$

$$= \frac{1}{24\pi^2}\int_{\text{I-II}} d^3S \, \epsilon_{0ijk} \, \mathrm{tr}\left(A_i A_j A_k\right) + \int_{-T/2}^{T/2}\int_{\text{III}} d^2S_i \, \epsilon_{i\nu\lambda\sigma} \, \mathrm{tr}\left(A_\nu A_\lambda A_\sigma\right) \tag{3.27}$$

[7] `http://qdb.mit.edu/176`

Before plugging in the appropriate BPST one-instanton potential one should be careful to gauge transform to satisfy our gauge choice $A_0 = 0$. It turns out that one such transformation is $A_\mu \to V A_\mu V^{-1} - i(\partial\mu V)V^{-1}$ with

$$V = \exp\left[\frac{i\vec{x}\cdot\vec{\tau}}{\sqrt{r^2+\rho^2}}\tan^{-1}\left(\frac{x^0}{\sqrt{r^2+\rho^2}} - \frac{\pi}{2}\right)\right], \tag{3.28}$$

see [7, 8] for more details. This means that the integral over the cylinder (region III) vanishes and n is just the difference of the winding between the spacelike surfaces at $\tau = \pm T/2$. We stuck in the one-instanton potential so that $n = 1$ and we conclude that the gauge configuration has changed homotopy classes between the asymptotic past and future. With a little bit of elbow grease one can further see that the one-instanton potential in $A_0 = 0$ gauge indeed satisfies

$$A_i(-T/2, \vec{x}) \to ig^{(0)}\left[\partial_i g^{(0)}\right] \tag{3.29}$$

$$A_i(+T/2, \vec{x}) \to ig^{(1)}\left[\partial_i g^{(1)}\right]. \tag{3.30}$$

This suggests that we should have a term I in our effective Lagrangian encoding the instanton so that the tunneling between two asymptotic vacua:

$$\langle n|e^{-iH\tau}|m\rangle_J = \int [dA]_{(n-m)} e^{-i\int d^4x\,\mathcal{L}+JI(x)}, \tag{3.31}$$

where J is a source for the instanton $I(x)$. We'll identify what this $I(x)$ is in due course.

3.8 Fancy-pants notation and the special role of SU(2)

There is a very fancy way of writing the asymptotic behavior of the instanton solution in terms of the so-called 't Hooft symbols, $\eta_{a\mu\nu}$ and $\bar{\eta}_{a\mu\nu}$, which is popular in the literature. Let us recall the asymptotic form of the one-instanton solution,

$$A_\mu|_{r^2\to\infty} = i\left(\frac{ix^\mu\tau^+_\mu}{r}\right)\partial_\mu\left(\frac{ix^\mu\tau^+_\mu}{r}\right)^{-1}, \tag{3.32}$$

where we've used the notation $\tau^\pm \equiv (\mp i, \vec{\tau})$. The term in parenthesis is simply what we called $g^{(1)}$ earlier. To simplify our lives, let's define the **'t Hooft symbol**,

$$\eta_{a\mu\nu} = \begin{cases} \epsilon_{a\mu\nu} & \text{if } \mu,\nu = 1,2,3 \\ -\delta_{a\mu} & \text{if } \mu = 1,2,3 \quad \nu = 0 \\ +\delta_{a\mu} & \text{if } \nu = 1,2,3 \quad \mu = 0 \\ 0 & \text{if } \mu = \nu = 0 \quad . \end{cases} \tag{3.33}$$

The $\bar{\eta}_{a\mu\nu}$ symbol is the same with the explicitly written signs swapped. With this notation one can rewrite (3.32) as

$$A_\mu|_{r^2\to\infty} = 2\eta_{a\mu\nu}\frac{x_\nu}{x^2}\tau^a. \tag{3.34}$$

Anti-instantons can be written with $\bar{\eta}$. The properties of these operators can be found in Appendix B of [9]. Ordinarily the author would remark that this choice of notation is just a display of fancy-pants pomposity, but let us instead remark that there is some depth to this choice. The point is that these 't Hooft symbols mix SU(2) and 'Lorentz' (Euclidean) indices. We know that the SU(2) index a is an adjoint index. Further, it is clear that η and $\bar{\eta}$ are antisymmetric in the Lorentz indices. But we recall that $[\mu\nu]$ is an index of the adjoint representation of the rotation group SO(4) $\cong$SU(2)$\times$SU(2). In fact, if one stares at the definition long enough, one will recognize that $\eta_{a\mu\nu}$ projects onto the first SU(2) while the $\bar{\eta}_{a\mu\nu}$ projects onto the latter. This should ring true to our previous topological arguments about maps between spaces.

We can further appreciate the role of SU(2) by looking at the aforementioned relation between gauge transformations and rotations. Given an arbitrary gauge configuration $A_\mu(x)$, we can gauge fix to $\widetilde{A}_\mu(x)$ by some transformation

$$A_\mu(x) = S(x)\widetilde{A}_\mu(x)S^{-1}(x) + iS(x)\partial_\mu S_1(x). \tag{3.35}$$

This is not the end of the story. A_μ is still invariant under *global* transformations,

$$S(x) \to S(x)S_0^{-1} \qquad\qquad \widetilde{A}_\mu(x) \to S_0\widetilde{A}_\mu(x)S_0^{-1}. \tag{3.36}$$

Further, by the gauge invariance of any physical quantities, the theory is still invariant under a further global SU(2) symmetry

$$S(x) \to S_1 S(x) \qquad\qquad \widetilde{A}_\mu(x) \to \widetilde{A}_\mu(x). \tag{3.37}$$

Thus we have a leftover SU(2)$\times$SU(2) global symmetry.

Now we can clarify the relation between gauge symmetry and instanton rotations. The choice of $U(x)$ in (3.32) singled out a direction in the Lorentz SU(2)$\times$SU(2) space by virtue of x^μ that transforms under rotations. Thus the configuration naïvely appears to be an object with nonzero spin. Any such rotation, however, can be undone by invoking the residual SU(2)$\times$SU(2) global symmetry. For example, if we defined generators

$$T_1^a = \frac{1}{4}\eta_{a\mu\nu}M^{\mu\nu} + \tau_1 \tag{3.38}$$

$$T_2^a = \frac{1}{4}\bar{\eta}_{a\mu\nu}M^{\mu\nu} + \tau_2, \tag{3.39}$$

where the $\tau_{1,2}$ refer to the residual (global) gauge symmetry and $M_{\mu\nu}$ are the generators of Lorentz transformations. With respect to these SU(2) generators, the instanton has zero 'spin.'

3.9 θ vacua

Now we get to the question that should have been lingering since we introduced the distinct topoloical vacua of Yang-Mills: what are the true vacuum states? We have already shown that instanton gauge configurations are minima of our action (i.e. classical paths) that enact tunneling between these vacua. In fact, we should be very proud of coming this far: we've now side-stepped Derrick's theorem and found real tunneling effects in quantum field theory.

The next step is to recognize that because of instanton-mediated tunneling, the $|n\rangle$ vacua are not true vacua. Our quantum mechanical analogy serves us well, we would like to construct the Yang-Mills equivalent of the $|\theta\rangle$ states, the so-called θ **vacua**. In fact, if we wanted to play it fast and loose, we could just make a one-to-one correspondence between the $|n\rangle$ vacua of the sinusoidal potential in QM and the n-winding vacuum state of Yang-Mills theory. The analog of 'translation invariance' leads us to the 'Bloch waves'

$$|\theta\rangle = \sum_n e^{-in\theta}|n\rangle. \tag{3.40}$$

We can now calculate the vacuum-to-vacuum amplitudes using the effective action (3.31). We know that the $|\theta\rangle$ vacua should be orthogonal. Thus,

$$\langle\theta'|e^{-iH\tau}|\theta\rangle_J = \delta(\theta-\theta')\Delta_J(\theta). \tag{3.41}$$

Our task is to determine Δ, and in so doing learn something about the form of our effective Lagrangian in an instanton background.

$$\langle\theta'|e^{-iH\tau}|\theta\rangle_J = \sum_{m,n} e^{im\theta'}e^{-in\theta}\langle m|e^{-iH\tau}|n\rangle_J = \sum_{n,m} e^{-i(n-m)\theta}e^{im(\theta'-\theta)}\int [dA]_{(n-m)}e^{i\int d^4x\,\mathcal{L}+JI}. \tag{3.42}$$

This tells us that (writing $\nu = (n-m)$)

$$\Delta_J(\theta) = \sum_\nu e^{-i\nu\theta}\int [dA]_\nu e^{-i\int d^4x\,\mathcal{L}+JI} = \sum_\nu \int [dA]_\nu e^{-i\int d^4x\,\mathcal{L}_{\text{eff}}+JI}, \tag{3.43}$$

where the effective Lagrangian is

$$\mathcal{L}_{\text{eff}} = \mathcal{L} + \frac{\Theta_{\text{YM}}}{16\pi^2}\,\text{tr}\left(F_{\mu\nu}\widetilde{F}^{\mu\nu}\right), \tag{3.44}$$

where we've used (3.17) to write the index. Thus we find that instantons introduce a θ term. We could have written this in from the very beginning; it's dimension-4 and gauge invariant. We know, of course, that we can express the winding number as a derivative of the Chern-Simons form, so the effects of this term vanish in perturbation theory. Note that we will write $\theta \to \Theta_{\text{YM}}$ because (i) it looks cooler and (ii) this will help avoid any confusion with fermionic superspace directions when work with supersymmetric theories.

In case the above translation to our periodic potential in quantum mechanics was unsatisfying, let's consider the Θ vacua from a slightly different point of view. In analogy to our multi-instanton configurations in quantum mechanics (c.f. Fig. 2), we can consider integrating over gauge configurations over large boxes of Euclidean spacetime with some definite winding number, n. We will call the result of this integral

$$F(V,T,n) = N\int [dA]_n\, e^{-S_E}, \tag{3.45}$$

where we assume the measure is gauge-fixed. For large time separation the winding number simply adds,

$$F(V,T_1+T_2,n) = \sum_{n_1+n_2=n} F(V,T_1,n_1)F(V,T_1,n_1), \tag{3.46}$$

i.e. we sum over combined smaller-number instanton configurations subject to obtaining the total amount of winding number required. For sufficiently large volumes we can ignore surface effects when instantons approach one another. The above composition law is rather clunky and is representative of our working with a poor basis of states. The solution is to work with a Fourier transform,

$$F(V,T,\theta) = \sum_n e^{in\theta} F(V,T,n) = N \int [dA] e^{-S_E} e^{in\theta}, \tag{3.47}$$

so that we indeed get the composition

$$F(V, T_1 + T_2, \theta) = F(V, T_1, \theta) F(V, T_2, \theta). \tag{3.48}$$

The important point is that our θ vacua now properly take into account the tunneling between the gauge configurations of different winding. These semiclassical states are stable in that there is no way to tunnel from one θ to another.

There's one last question that comes up, especially for those who are familiar with the effects of the θ term: what is the physical significance of the particular θ vacuum that we inhabit? Recall that our Bloch waves had energies that depended on their 'momentum' θ. We can follow the exact same arguments for (2.47) to obtain

$$E(\theta)/V = -2Ke^{-S_0} \cos\theta, \tag{3.49}$$

where the K factor is now different and must be calculated in a slightly more honest manner. A heuristic derivation can be found in Section 3.6 [1], but a slightly more detailed form to leading order in coupling is

$$E(\theta)/V = -\frac{A}{g^2} e^{-\frac{8\pi^2}{g^2}} \cos\theta \int_0^\infty \frac{d\rho}{\rho^5} (\rho M)^{8\pi^2 \beta_1}. \tag{3.50}$$

Here we've determined the g dependence by following $\hbar$ dependence. We have to integrate over the characteristic scale ρ of the instanton; the ρ^{-5} dependence comes from dimensional analysis. M is an arbitrary renormalization scale coming from the running of the gauge coupling, which is the origin of the β_1 dependence. A is a factor that is difficult to calculate unless one was educated in the USSR.

3.10 Philosophy, gauge symmetry, and winding number

It is now timely to make more philosophical remarks about gauge invariance and the appearance of topologically distinct vacua. Let us remind ourselves that we started with Derreck's theorem as a 'no-go' theorem for tunneling amplitudes in $p+1$ dimensional field theory ($p > 1$). This boiled down to having to pull an infinite number of *physical* oscillators over a tunneling potential causing an infinite product of exponential suppressions to our tunneling amplitude. The way out was to put the potential not over physical field, but rather in the space of not-necessarily-physical gauge configurations.

One 'moral dilemma' that one should be concerned about is the apparent incompatibility of gauge invariance and the vacua of a given winding number n. We harped on and on in Section 2.6

about gauge invariance as a redundancy that must be modded out, i.e. physical states are equivalency classes. This seems to be in direct contradiction to the previous paragraph: the $|n\rangle$ vacua are distinct, yet gauge symmetry seems to say that they are physically equivalent. Meanwhile, we *know* that there's definitely *something* physical about this whole business because we found in (3.49) that our energy density is θ-dependent.

To resolve this conceptual issue it is important to distinguish between so-called **small gauge transformations** and **large gauge transformations**. Small transformations are homotopic to the identity while large transformations change winding number. One cannot continuously perform a sequence of small gauge transformations to produce a large gauge transformation since this pulls the field out of the pure gauge configuration at infinity—in other words, this requires changing the *physical* field over a *physical* potential barrier at an infinite number of points. (This is our usual argument about tunneling in QFT.)

We're thus concerned about large transformations. How can they take us to 'distinct' vacua when they're supposed to be modded out in a physical state? The answer is that they're not. Only the *small* gauge transformations form the 'physical' equivalency class. Topological arguments show that this subgroup is closed. For example, one can argue that at the origin a gauge transformation must have no winding or else it would be singular. A 'physical gauge transformation' (that's an oxymoron, but you know what I mean) thus cannot have winding at infinity. We will refine these ideas in Section 4.5 when we provide an alternate bundle-description for what the S^3 surface really is. Modding out by the large gauge transformations (those not homotopic to the identity) is certainly a valid procedure, but that defines a *different* theory with a *larger* gauge group. This would be analogous to 'gauging' out all nontrivial spin states so that all physical states are spin-0.

An orthogonal way of arguing that different $|n\rangle$ vacua are distinct (e.g. in the sense of gauge redundancy) is to invoke Gauss' law. For details see [10]. The general idea is that one may canonically quantize Yang-Mills theory in some nice gauge like $A = 0$ and look for Gauss' law,

$$D_\mu F^{\mu\nu} = 0. \tag{3.51}$$

Deriving the Hamiltonian equations, however, Gauss' law is nowhere to be found. In fact, the left-hand side of Gauss' law is an operator that does not commute with the canonical variables. If we *impose* Gauss' law as an operator equation, then it turns out that one finds that the only 'good' gauge transformations—those which can be built up from infinitesimal transformations—are those which have no winding. (Given our exhaustive discussion, this is actually a rather tautological statement.)

Note, however, that each $|n\rangle$ vacuum contains the same physics. This is just as in the case of the sinusoidal potential where it didn't matter which vacuum one started off in. We note that this is different from the $|\Theta_{\text{YM}}\rangle$ vacua, which *does* have physical significance, as we will discover later. In the sinusoidal potential this θ corresponded to a conserved 'pseudomomentum.' We can talk about Yang-Mills tunneling between $|n\rangle$ vacua, but left to its own devices instantons will not go between different $|\Theta_{\text{YM}}$ states. One might argue that the $|n\rangle$ vacua cannot contain the same physics since they have different winding. In particular, this means that highly wound states have more $F\widetilde{F}$ turned on somewhere in the spacetime. Since these are field strengths, wouldn't this have some physical effect? The answer is no: one should be careful to distinguish $FF \sim \vec{E}^2 + \vec{B}^2$ from $F\widetilde{F} \sim \vec{E} \cdot \vec{B}$. The highly wound states have, for example, the same vacuum energy.

Another question one might ask is whether or not instantons somehow break some gauge invariance. The answer is no since we must sum over instanton configurations in our generating functional. For a given correlation function, however, we will see that typically only one sector (e.g. one-instanton background) will contribute.

4 *Amuse-bouche*: a taste of mathematics

We've spent a lot of time dabbling half-heartedly with topological ideas. Let's now spend a bit of time to tip our heat properly to the mathematical structure that we have hitherto taken for granted.

Two spaces are topologically equivalent if they can be continuously deformed into one another. We say that the two spaces are **homeomorphic**. This would certainly be a useful idea, but it turns out that explicitly determining homeomorophisms between spaces is very difficult. We have to lower our expectations. Instead of directly determining topological equivalence, we will instead work with necessary conditions of such equivalence. For example, we can look for **topological invariants** which are equal for homeomorphic spaces. It turns out that even this is still very difficult. We can lower our expectations still by relaxing the condition of homeomorphism and instead consider spaces that are **homotopic**, i.e. one can be continuously deformable into one the but not necessarily in an invertible way (e.g. loops can shrink to a point). The **fundamental group** is the group of homotopy classes, and we've already met $\Pi_3(\mathrm{SU}(2))$ in our study of instantons. We've already developed an intuition for homotopic equivalency classes: they represent winding numbers that tell us how many times a map sends a covering of one space into the other.

4.1 Brouwer degree

Homotopies have served us rather well, though we were a bit lucky because the relevant fundamental group was very simple. Suppose we complained further and said that these are still very difficult to calculate—as they are in general. We can simplify even further by restricting to maps between spaces of the same dimension. This leads us to an even simpler object, the **Brouwer degree** of a map $\phi : M \to N$ with $\dim(M)$=$\dim(N)$,

$$\deg(\phi) = \int_M \phi^* \Omega, \tag{4.1}$$

where Ω is a normalized volume form on N, $\int_N \Omega = 1$. Since the difference between volume forms is exact this quantity doesn't depend on the choice of Ω. It is very straightforward to read off that $\deg(\phi)$ is counting how many 'volumes' of N we get back from integrating over a volume in M. From this explanation in words one can see that this is precisely the same thing as the winding number (or what we earlier called a Pontryagin number) that we've been making a big deal about.

4.2 The Maurer-Cartan form

For matrix Lie group manifolds there is a particularly handy volume form that one can construct from the Lie-algebra-valued **Maurer-Cartan form**, $g^{-1}dg$, where g is an element of the group.

Geometrically these are a basis of one-forms for a matrix Lie group. A left- and right-invariant volume form is then given conveniently by

$$\Omega = \frac{1}{24\pi^2} \operatorname{tr} \left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg \right) . \tag{4.2}$$

This should now look dramatically familiar, down to the factor of $24\pi^2$. This is just the integrand Pontryagin—ahem, sorry, the *winding* number in (3.8). Combining this with our Brouwer degree formula we now have a somewhat deeper appreciation for the nature of the topological quantities underlying our instanton derivation. One should note the mathematical game that we're playing: we are using the *geometric* structure of our gauge fibration of spacetime to determine *topological* information (which is independent of the particular geometry).

While I have your attention, let me remark that the Maurer-Cartan form is a surprisingly under-appreciated object in physics. Among its uses, it provides a natural way to understand the form of the supersymmetric covariant derivative, i.e. the *geometric* sense the SUSY covariant derivative is actually covariant.

4.3 Disco inferno (Chern, baby, Chern!)

The integral for winding number has another name, the **second Chern number**, c_2. Before trying to motivate what this means, let us remark on its significance. Thus far we've touched on topology and geometry. The second Chern number (and a slew of other objects named after Chern) connects our endeavor of finding topological invariants to cohomology. This will bring us to a mathematical connection to anomalies which will, in turn, lead to a remarkable connection between Yang-Mills instantons (which have been constructed in a purely gauge theory) and the axial anomaly (which is related to chiral fermions).

We'll drop a few fancy words and try to string coherent thoughts between them, but our treatment will be necessarily incomplete[8]. We will unfortunately begin with a sequence of definitions. Let us first introduce the idea of a **characteristic polynomial**. This is a polynomial $P(X_1, \cdots, X_n)$ of elements X_i in a Lie algebra that is invariant under transformations $X_i \to g^{-1}X_i g$ with g in the Lie group. This can be extended to a polynomial in Lie-algebra-valued forms, $\alpha = X\eta$ (where X is the Lie algebra element and η is a p-form),

$$P(\alpha_1, \cdots, \alpha_n) = P(X_1, \cdots, X_N)\eta_1 \wedge \cdots \wedge \eta_n. \tag{4.3}$$

We will be particularly interested in polynomials of the field strength. One can then define an **invariant polynomial** $P_n(F) = P(F, \cdots, F)$. These objects have the properties that (1) they are closed, $dP_n(F) = 0$, and (2) the difference of an invariant polynomial for different connections on the same bundle is exact, i.e. there exists a **transgression** $n-1$ form Q such that

$$P_n(F') - P_n(F) = dQ_{2n-1}(A', A) \tag{4.4}$$

[8]To the A-exam committee: I think all of the theory students (and a appreciable number of condensed matter, high energy experiment, and mathematics students) would be very interested in a course on differential geometry and physics. If there were a faculty member willing to teach such a 'special topics course,' there is a wealth of very interesting and relevant material to be covered.

where, in 'physics-speak,' F' is a gauge transformation of F. The quantity Q is called a **Chern-Simons form**. Because $P_n(F)$ is closed, it represents an element of a de Rahm cohomology class, $[P_n(F)] \in H^{2n}(M, \mathbb{R})$, called a **characteristic class**. Because the difference $P_n(F') - P_n(F)$ is exact, then their integral over a manifold without boundary vanishes. This means that the integrals ('periods') of $P_n(F)$ over the space are *independent* of the connection and are *topological* properties of the fibre bundle.

The particular invariant polynomial which will be of interest to us is the dterminant

$$c(F) = \det\left(1 + \frac{i}{2\pi}F\right), \tag{4.5}$$

called the **total Chern class**. (Mathematicians in this field aren't particularly creative with names.) This can be expanded into a series

$$c(F) = 1 + c_1(F) + c_2(F) + \cdots \tag{4.6}$$

where each $c_n(F)$ is a $2n$ form over the base space(time) and is called the $\mathbf{n^{th}}$ **Chern class**. These are precisely the Chern-Simons $2n-1$ forms Q_{2n-1}. These have the property that $c_n(F) = 0$ for $2n > \dim(M)$, where M is the base space(time) manifold. The first two Chern classes are

$$c_1(F) = \frac{i}{2\pi}\mathrm{tr}F \tag{4.7}$$

$$c_2(F) = \frac{1}{8\pi^2}\left[\mathrm{tr}F \wedge F - \mathrm{tr}F \wedge \mathrm{tr}F\right]. \tag{4.8}$$

For the particularly important case of an SU(2) fibration of a four-dimensional manifold we find that $c_1(F) = 0$ by the tracelessness of our generators and

$$c_2(F) = Q_3(A) = \frac{1}{8\pi^2}\mathrm{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right). \tag{4.9}$$

This should now look familiar once again: this is our celebrated Chern-Simons form that keeps popping up everywhere, most recently in (3.14). The integrals of Chern classes are, as we showed above, independent of geometry so that their integrals are topological quantities called **Chern numbers**. We already know the second Chern number for our instanton system as the winding number, Pontryagin index, and Brouwer degree. What will also be rather important is that we should *in addition* know the second Chern number as the abelian anomaly coefficient.

Just to show that we can keep constructing new phrases out of a very limited word bank, let us remark that one can also define a **Chern characters** ch_n and the **total Chern character** ch,

$$ch(F) = \sum_n ch_n(F) = \sum_n \frac{1}{n!}\mathrm{tr}\left(\frac{i}{2\pi}F\right)^n. \tag{4.10}$$

This is another characteristic class that is generally easier to compute than the Chern class and from which one can later compute Chern classes. Working things through, one finds

$$ch_2(F) = -c_2(F) + \frac{1}{2}c_1(F) \wedge c_1(F). \tag{4.11}$$

The second Chern number shows up somewhere else rather important, it is proportional to the index of the Dirac operator in the celebrated **Atiyah-Singer index theorem**. This will come back to us when we address the U(1) problem in Section 5.

4.4 Connection to fermions

We will soon connect instanton effects to massless (chiral) fermions. On the one hand you should be skeptical: our entire discussion thus far has been based on pure Yang-Mills theory with no mention of fermions at all. On the other hand, by now all of our discussion about the topological index should sound rather familiar. After all, we've seen all of this before: the $F\widetilde{F}$ term is just what we find when calculating the ABJ anomaly! Life is short and the author has other A-exam questions to address, so we'll leave it as homework for the reader to follow up on the mathematical nature of the second Chern class in the abelian anomaly. The key result is that one may write the anomaly in terms of differential forms,

$$d * j^5 = \frac{1}{4\pi}\mathrm{tr}FF. \tag{4.12}$$

4.5 Instantons for the mathematically inclined

Let us now describe the fibre-bundle set up for the BPST instanton. In grown up notation, our topological charge (winding number, second Chern class, Pontryagin number, whatever you want to call it) is

$$n = \frac{-1}{16\pi^2}\int d^4x\ \mathrm{tr}F_{\mu\nu} * F^{\mu\nu} = \frac{-1}{8\pi^2}\int_{S^4}\mathrm{tr}F^2 \tag{4.13}$$

where we have compactified our space via stereographic projection: $\mathbb{R}^4 \to S^4$. We may write the integrand—the **topological density**—as

$$Q = \frac{-1}{8\pi^2}\mathrm{tr}F^2, \tag{4.14}$$

which we already know is the transgression in (4.4)... so many names for the same thing. One can use the Bianchi identity $DF = 0$ to prove a result that we already know: the Chern-Simons form is closed, $dQ = 0$. (Use $D\mathrm{tr}F^2 = 2\mathrm{tr}DFF$.) We can divide S^4 into patches according to the upper and lower hemisphere as in Fig. 4. This construction should sound very familiar from the 't

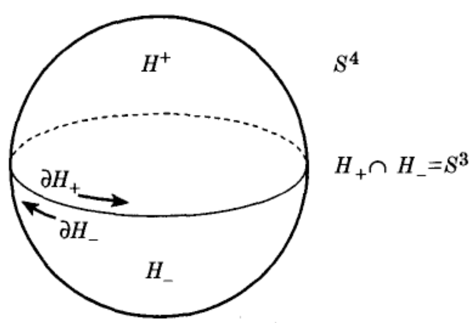


Figure 4: The compactified base manifold S^4 for the instanton bundle. Image from [11].

Hooft-Polyakov monopole. Poincaré's theorem then tells us that locally Q is exact. On the upper hemisphere, for example, we may write $Q = dK$ so that

$$n = \int_{H_+} dK = \int_{S^3} K. \tag{4.15}$$

K is, of course, the familiar Chern-Simons form and we see that it is indeed integrated over and S^3 as we were doing earlier. Now the nature of the S^3 is now clearer: it is the homotopic equivalent of the overlap region between the two patches we chose for S^4. Combining with the chiral anomaly equation (4.12),

$$d * j^5 = -2dK. \tag{4.16}$$

This should cause us to pause. The right-hand side of the anomaly appears to be a total divergence. Total divergences aren't so bad, especially over manifolds without boundary. One might wonder if such a total divergence can really have any effect on actual physics. To codify this, let's take the obvious step of combining the two exact forms into a new current,

$$*j' = *j^5 + 2K. \tag{4.17}$$

Then clearly $d * j' = 0$ and we've found a *new* axial current that appears to be conserved. No more anomalies, right? This will be the main point of Section 5. As a hint that there's much more to this than meets the eye, let us note that that the Chern-Simons form K is *not* gauge invariant.

Let's move on to constructing the instanton solution. The Bogomol'nyi bound takes the form

$$\int d^4x \, (F \pm *F)^2 \geq 0, \tag{4.18}$$

and we are led to the usual (anti) self-dual solutions $\pm * F$. This gives us a first-order system of differential equations for the gauge potential A. Let us make the ansatz,

$$A = f(r) \, \gamma^* \zeta = f(r) \, \gamma^{-1} d\gamma, \tag{4.19}$$

where r is the $\mathbb{R}^4$ radial direction, γ is an SU(2)-valued function, and ζ is the Maurer-Cartan one-form on SU(2). We already know that finite action leads us to functions f that tend to 1 as $r \to \infty$. We may write

$$\gamma(x) = \frac{1}{r} \left(x^0 + 2\vec{x} \cdot \vec{\tau} \right). \tag{4.20}$$

This map is singular at $r = 0$ but otherwise identifies the there-sphere of finite radius r with SU(2). Thus $\gamma^{-1} d\gamma$ differs from the Maurer-Cartan form ζ only by this identification. In other words, it is the pullback $\gamma^{-1} d\gamma = \gamma^* \zeta$.

We need $*F$, so let's define an orthonormal frame

$$e^j = \frac{1}{2} r \gamma^* \zeta \qquad \text{and} \qquad e^0 = dr. \tag{4.21}$$

Thus

$$F = \sum_{i=1}^{3} \tau_i \left[\frac{2}{r} \frac{df}{dr} e^0 \wedge e^i + \frac{2}{r^2} (f^2 - f) \sum_{j,k=1}^{3} \epsilon_{jki} e^j \wedge e^k \right]. \tag{4.22}$$

Now we can calculate the Hodge star,

$$*F = \sum_{i=1}^{3} \tau_i \left[\frac{-4}{r^2}(f^2 - f)e^0 \wedge e^i - \frac{1}{r}\frac{df}{dr} \sum_{j,k=1}^{3} \sum_{jki} e^j \wedge e^k \right]. \tag{4.23}$$

Imposing the self-duality condition we obtain

$$\frac{df}{dr} = -\frac{2}{r} f(f-1), \tag{4.24}$$

and similarly for the anti-self-dual case. We can solve this to get the instanton potential,

$$A_+ = \frac{r^2}{r^2 + c^2}\, \gamma^{-1} d\gamma, \tag{4.25}$$

where $\gamma : (\mathbb{R}^4 - \{0\}) \to$SU(2) and c is a constant. Our potential is regular at $x = 0$, but doesn't decay fast enough at $r \to \infty$ since it only goes as r^{-1} asymptotically. This is fine, of course: the whole point of the fibre bundle construction is that we can have different local trivailizations over different patches. For the physicists in the audience, this means that we can choose different gauge potentials over different coordinate patches. The only condition is that there exists a transition function (gauge transformation) that relates the different potentials where the patches overlap[9].

We're happy with (4.25) everywhere except at $r = \infty$, so let us take this as the potential over H_- in Fig. 4. Let us perform a gauge transformation by γ to get a potential that is valid at $r = \infty$

$$A_- = \gamma A \gamma^{-1} + \gamma d\gamma^{-1} = \frac{c^2}{r^2 + c^2}\, \gamma d\gamma^{-1}. \tag{4.26}$$

To fill in some details, let us define the stereographic projections

$$\alpha_\pm(x^0, \cdots, x^3) = \frac{1}{1 \pm x^3}(x^0, \cdots, x^3) \qquad\qquad x^4 \neq \mp 1. \tag{4.27}$$

We can consider $A_\pm$ as being defined over $U_\pm$ via the maps $\alpha_\pm$. Over $U_+ \cap U_- \cong S^3$ (homotopic to the 'hyper-equator') the two potentials are related by a gauge transformation. We say $A_\pm$ are each local representatives of an SU(2) **principal fibre bundle** (the instanton bundle) over S^4 with transition function $g_\pm = \gamma$. (A principal fibre bundle is a bundle whose fibre is identical to its structure group.) Let us remark that it is a fact that *any* finite action solution to the Euclidean Yang-Mills equation leads to a fibre bundle over S^4.

Let's say a few things about the resulting topology of the instanton bundle. As we've already remarked the base manifold is our compactified $\mathbb{R}^4 = S^4$ which we divide into two hemispheres $H_\pm$ which overlap on a band which is homotopic to the hyper-equator. This forms the boundary of the two hemispheres,

$$\partial H_\pm = \pm S^3, \tag{4.28}$$

[9] In fact, in case you were wondering earlier how integrals over differential structures (e.g. de Rahm cohomology classes) end up giving us topological data that is manifestly *insensitive* to geometry, this is your answer. The transition functions on the fibre bundle structure encode all of the information about the bundle topology.

where the sign comes from the orientation, as shown in Fig. 4. The structure group and the fibres are SU(2) and our local bundle patches are $H_\pm\times$ SU(2) with bundle coordinates $(x, f_\pm)$ where $f_\pm \in$ SU(2). These patches are stitched together with the **transition functions** h_{-+} so that

$$f_- = h_{-+} f_+ \quad \text{along the `equator' } H_+ \cap H_- = S^3. \tag{4.29}$$

Thus these represent the 'twist' in the fibration of the two patches. This 'twist' is, of course, just the winding number that we've made such a big deal about. These transition functions take value in SU(2) and we can see that as we go around the hyper-equator we wrap the group SU(2)$\cong S^3$. This is now a rather familiar story.

Let's reacquaint ourselves with terminology. The **connection one-form** associated with the instanton bundle are the gauge potential $A_\pm$ and must satisfy the compatibility condition

$$A_+ = h_{-+}^{-1} A_- h_{-+} + h_{+-}^{-1} dh_{-+} \tag{4.30}$$

on $H_+ \cap H_- = S^3$. This is of course just the gauge transformation law that we already know and love. The **curvature two-form** are the field strengths

$$F_\pm = dA + \pm + A_\pm^2 \qquad\qquad F_+ = h_{-+}^{-1} F_- h_{-+}. \tag{4.31}$$

Recall our discussion below (4.4) where we remarked that the periods of the invariant polynomials are are independent of connection and must thus be topological in character. Let us now see this for the Pontryagin index (or whatever you choose to call it),

$$\int_{S^4} \mathrm{tr} F^2 = \int_{H_+} d\,\mathrm{tr}\left(F_+ A + -\frac{1}{3}A_+^3\right) + \int_{H_-} d\,\mathrm{tr}\left(F_- A = -\frac{1}{3}A_-^3\right) \tag{4.32}$$

$$= \int_{S^3} \mathrm{tr}\left(F_+ A_+ - \frac{1}{3}A_+^3 - F_- A_- + \frac{1}{3}A_-^3\right) \tag{4.33}$$

$$= \int_{S^3} \left[-\frac{1}{3}\mathrm{tr}\left(h_{-+}^{-1} dh_{-+}\right)^3 + d\,\mathrm{tr} A_- dh_{-+} h_{-+}^{-1}\right] \tag{4.34}$$

$$= -\frac{1}{3}\int_{S^3} \mathrm{tr}\left(h_{-+}^{-1} dh_{-+}\right)^3, \tag{4.35}$$

where we've applied Stokes' theorem twice. What we find is that the Pontryagin index is manifestly independent of the connection A and only depends on the transition functions h_{+-}. As we suspected earlier, the transition functions encode the topological data of the fibre bundle.

We already know how to calculate the index, but let's remark upon how one would see this from our fancy-schmancy fibre bundle construction. We may simplify the S^3 integral by choosing a reference point at the north pole (of S^3, not S^4). We may do this since h_{-+} maps S^3 uniformly onto SU(2)$\cong S^3$. Then we may construct our Maurer-Cartan form

$$h_{-+}^{-1} dh_{-+} = i\sigma_k dx^k. \tag{4.36}$$

Then the Pontryagin index integrand may be written as

$$\mathrm{tr}(h_{-+}^{-1} dh_{-+})^3 = i^3\, \mathrm{tr}\sigma_i\sigma_j\sigma_k\, dx^i dx^j dx^k = 2\epsilon_{ijk} dx^i dx^j dx^k = 12\, dx^1 dx^2 dx^3. \tag{4.37}$$

This is now just a multiple of the area element at the north pole of S^3, so that we may integrate over the entire space using $\int_{S^3} dx^1dx^2dx^3 = 2\pi^2$. (Of course we write $dx^1dx^2dx^3 = dx^1\wedge dx^2\wedge dx^3$.) The result is the usual integer in (3.17),

$$n = \frac{1}{24\pi^2}\int_{S^3} \mathrm{tr}(h_{-+}^{-1}dh_{-+})^3 = 1, \tag{4.38}$$

for a one-instanton background. This brings us roughly up to the same point of the story that we'd developed on the physics side.

5 *Plat principal*: the U(1) problem

Now we get to the meat and gravy. We've made several oblique references to the instanton number and the chiral anomaly. Both depend on the Pontryagin number. In this section we clarify the nature of this relationship and demonstrate how instantons can lead to real phenomenological effects.

5.1 A fermion refresher

We would like to study fermions in an instanton background. Fermions are inherently quantum mechanical and never pick up a nontrivial semi-classical configuration so that all of our heavy lifting for pure Yang-Mills theory in the previous sections are unaffected by the fermionic path integral,

$$\int [d\Psi][d\bar{\Psi}]e^{i\int d^4x\, \bar{\Psi}i\not{D}\Psi}. \tag{5.1}$$

We see, however, that the quantum theory of our fermion fields will be affected by the nontrivial semi-classical gauge configuration.

Let us begin by establishing notation for fermions in Euclidean space. Clifford algebra takes the form,

$$\{\gamma^\mu,\gamma^\nu\} = 2\delta^{\mu\nu}. \tag{5.2}$$

The Euclidean γ matrices are

$$\gamma^0 = \begin{pmatrix} & 1\\ 1 & \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} & -i\sigma^j\\ i\sigma^j & \end{pmatrix} \qquad \Rightarrow \qquad \gamma^\mu \equiv \begin{pmatrix} & \sigma^\mu\\ \bar{\sigma}^\mu & \end{pmatrix}. \tag{5.3}$$

The chiral matrix is $\gamma^5 = \gamma^0\cdots\gamma^3$. We shall write out our Direct fermions Ψ in terms of Weyl fermions $\chi,\bar{\psi}$ using the 'standard' notation,

$$\Psi = \begin{pmatrix}\chi\\ \bar{\psi}\end{pmatrix} \qquad\qquad \bar{\Psi} = (\psi,\bar{\chi}). \tag{5.4}$$

The Dirac bilinear can then be decomposed into chiral components

$$\bar{\Psi}i\not{D}\Psi = \bar{\chi}i\bar{\sigma}\cdot D\chi + \psi i\sigma\cdot D\bar{\psi}. \tag{5.5}$$

Chiral Dirac operator. Let us make a parenthetical remark that the chiral Dirac operators $i\sigma \cdot D$ and $i\bar{\sigma} \cdot D$ are *not* self-adjoint. Instead, they are adjoints of one another. One can see this from looking at the operators on left-chiral fermions. First one can integrate by parts to obtain

$$\int d^4x\, \bar{\chi} i\bar{\sigma} \cdot D\chi = \int d^4x\, \bar{\chi} i(D_0 + i\vec{\sigma} \cdot \vec{D})\chi \tag{5.6}$$

$$= \int d^4x\, (-i) \left[\partial_0\bar{\chi} + ig\bar{\chi}A^a t^a + i(\vec{\partial}\bar{\chi}) \cdot \vec{\sigma} - g\bar{\chi}\vec{A}^a \cdot \vec{\sigma} t^a\right] \chi. \tag{5.7}$$

Next next one can compare this to the adjoint operator,

$$\left[(i\bar{\sigma} \cdot D)^\dagger \bar{\chi}\right]^T = i\left\{\left[\partial_0 - igA_0^a(-t_a^T)\right] \bar{\chi}^T + i\vec{\sigma}^T \cdot \left[\vec{\partial} - ig\vec{A}^a(-t_a^2)\right] \bar{\chi}^T\right\}. \tag{5.8}$$

Multiplying by σ^2 and using $\sigma^2\vec{\sigma}^T = -\vec{\sigma}\sigma^2$,

$$\sigma^2(i\vec{\sigma} \cdot D)^\dagger\sigma^2 = i\left[(\partial_0 - igA_0^a\bar{t}^a) - i\vec{\sigma} \cdot (\vec{\partial} - ig\vec{A}^a\bar{t}^a)\right], \tag{5.9}$$

where we've written the conjugate representation generators as $\bar{t}$. From this we finally obtain $(i\bar{\sigma} \cdot D)^\dagger \cong i\sigma D$ with the representation $r \to \bar{r}$. For the special case of SU(2) we know that $\mathbf{2} \cong \bar{\mathbf{2}}$.

The adjoint of spinor operators is something which is rarely done very carefully. This can become rather important in non-trivial cases, for example the Randall-Sundrum scenario where the dimensionality, orbifold compactification, and warping of the space make our fermion 'inner products' rather delicate. I only make a emphasize this now because none of the other students in my group seem to appreciate this subtlety.

Let us now consider the eigenspinors and eigenvalues of the chiral Dirac operators acting on the left and on the right. These take the form

$$(i\bar{\sigma} \cdot D)\chi_k = \bar{\lambda}_k\chi_k \qquad \chi_\ell(i\bar{\sigma} \cdot D) = \lambda_\ell\chi_\ell \tag{5.10}$$

$$(i\sigma \cdot D)\bar{\psi}_\ell = \lambda_\ell\bar{\psi}_\ell \qquad \bar{\psi}_k(i\sigma \cdot D) = \bar{\lambda}_k\bar{\psi}_k. \tag{5.11}$$

The spectrum of $i\bar{\sigma} \cdot D$ from the left and $i\sigma \cdot D$ from the right are identical and vice versa. It is thus tempting to say that $\lambda_\ell = \bar{\lambda}_k$,

$$\int d^4x\, \bar{\chi}_\ell(i\bar{\sigma} \cdot D)\chi_k = \bar{\lambda}_k \int d4x\, \bar{\chi}_\ell\chi_k = \lambda_\ell \int d^4x\, \bar{\chi}_\ell\chi_k. \tag{5.12}$$

Thus, indeed, while $\int d^4x\, \bar{\chi}_\ell\chi_k \neq 0$, we must have $\lambda_\ell = \bar{\lambda}_k$ as expected and all non-zero λ_ℓ and $\bar{\lambda}_k$ are paired. However, if $\lambda_\ell = 0$ (or, similarly $\bar{\lambda}_k$), then we can see that there's *no need* for such a zero mode to have a partner. Chiral symmetry alarm bells should be going off inside your head.

We can rephrase this statement by saying that the massive spectra of $i\sigma \cdot D$ and $i\bar{\sigma} \cdot D$ are matched, while the massless spectra—i.e the kernel of those operators—need not be. We can define an index[10],

$$\mathrm{ind}(i\bar{\sigma}\cdot D) = \dim[\ker(i\bar{\sigma}\cdot D)] - \dim[\ker(i\sigma\cdot D)], \tag{5.13}$$

which just counts the number of left-chiral zero spinors minus the number of right-chiral zero spinors.

The same statement using four-component spinors. For completeness, let us consider how an analogous argument would look using four-component Dirac spinors. The eigenvalue equation for the Dirac operator is

$$i\not{D}\Psi_j = \lambda_j \Psi_j. \tag{5.14}$$

We know that the λ_js are real by the usual Hermiticity of $i\not{D}$. Further, we know that γ^5 anticommutes with γ^μ so that we may write

$$i\not{D}\gamma^5\Psi_j = i\lambda_j\gamma^5\Psi_j. \tag{5.15}$$

This tells us that each non-zero λ_r eigenvalue must come in pairs according to chirality. However, if the $i\not{D}$ eigenvalue vanishes then one is free to choose Ψ_0 to also be an eigenvalue of γ^5,

$$\gamma^5\Psi_j = \pm\Psi_j, \tag{5.16}$$

where the eigenvalues are restricted to ± 1 via $(\gamma^5)^2 = \mathbb{1}$.

5.2 Friend or foe? Massless fermions and instantons

So far all that we have argued is that a zero mode does not necessarily need to have a partner. Now let us look for an explicit example where such a scenario is realized. We shall make the bold (though by now unsurprising) claim that this occurs precisely in the presence of a nontrivial instanton background. Let's consider the case $n = 1$. We would like to find a solution of

$$(D_0 + i\vec{\cdot}\vec{D})\chi = \bar{\lambda}\chi, \tag{5.17}$$

with $\bar{\lambda} = 0$. With a tremendous prescience, let us try and ansatz that the spinor takes the form

$$\chi_{\alpha i} = \frac{C_{\alpha i}}{(x^2+\rho^2)^a} \tag{5.18}$$

where α is our spinor Lorentz index, i is an SU(2) gauge index, and a is a power that we'd like to determine. $C_{\alpha i}$ is some constant matrix in $\mathrm{SU}(2)_{\mathrm{Lorentz}}\times\mathrm{SU}(2)_{\mathrm{gauge}}$ space. This ought to remind us

[10]...and by now you should already expect that there is only *one* index in this entire document and we keep referring to it in different contexts.

of our discussion about 'fancy pants' notation in Section 3.8, where we noted that the 'coincidence' that both the gauge and Lorentz group could be written in terms of SU(2)s was the reason that instantons are an SU(2) effect. (Of course the SU(2) may live inside larger groups.) Using the 't Hooft symbol defined in (3.8), we may write the covariant derivative as

$$D_\mu = \partial_\mu - igA^a_\mu \tau^a = \partial_\mu - i\frac{2}{x^2+\rho^2}\eta_{a\mu\nu}x^\nu \frac{\sigma^a}{2}. \tag{5.19}$$

The action on a left-chiral fermion thus takes the form

$$(D_0 + i\vec{\sigma}\cdot\vec{D})\chi = \frac{1}{(x^2+\rho^2)^{a+1}}\left[-2ax^0 - i\eta_{a0\nu}x^2\sigma^2 + i\sigma^j(-2ax^j) + \eta_{aj\nu}x^\nu\sigma^j\sigma^a\right]\cdot C \tag{5.20}$$

Let's consider the x^0 terms,

$$[\cdots]|_{x^0} = x^0(-2a + \eta_{aj0}\sigma^j\sigma^a)\cdot C = x^0(-2a - \vec{\sigma}\cdot\vec{\sigma})\cdot C. \tag{5.21}$$

By assumption these much vanish so that we want a to cancel $\vec{\sigma}\cdot\vec{\sigma}$. The evaluation of this latter quantity depends on the form of $C_{\alpha i}$. First, a sanity check. Note that there are two 'kinds' of Pauli σ matrix floating around here: one that acts on SU(2) spinor Lorentz indices and one that acts on SU(2) 'color' gauge indices. The quantity $\vec{\sigma}\cdot\vec{\sigma} = \sigma^i\sigma^j\delta_{ij}$ is contracted along the vectorial rotation indices but are *not* contracted along their matrix indices which instead act on the indices of $C_{\alpha i}$. We know what to do when we have a matrix $C_{\alpha i}$ with two SU(2) indices: we decompose it into irreducible representations and say words like 'Clebsch-Gordan.' In particular, the two cases are that C is symmetric or antisymmtric ($C_{\alpha i} \propto \epsilon_{\alpha i}$). In the antisymmetric case we find that C is an object with 'spin' 0 while in the symmetric ase it has 'spin' 1. The total 'spin' J^2 is

$$J^2 = \left(\frac{\vec{\sigma}_{\text{Lorentz}}}{2} + \frac{\vec{\sigma}_{\text{gauge}}}{2}\right)^2 = \frac{3}{4} + \frac{1}{2}\vec{\sigma}\cdot\vec{\sigma} + \frac{3}{4} = \frac{3}{2}\left(1 + \frac{\vec{\sigma}\cdot\vec{\sigma}}{3}\right). \tag{5.22}$$

Thus the two cases give

$$\text{spin-0} \quad\Rightarrow\quad \vec{\sigma}\cdot\vec{\sigma} = -3 \quad\Rightarrow\quad a = +3/2 \tag{5.23}$$

$$\text{spin-1} \quad\Rightarrow\quad \vec{\sigma}\cdot\vec{\sigma} = +1 \quad\Rightarrow\quad a = -1/2. \tag{5.24}$$

Looking back at our ansatz (5.18) we can see immediately that the $a = -1/2$ case is *not* normalizable and cannot exist in the spectrum. We note that we could do the analogous calculation for $\bar{\psi}$ to find values of a that are the same up to a sign flip. In this case, *both* modes are not normalizable. To be complete we should also check the terms with x^i (we only did the x^0 terms above); one will find that our results here are consistent.

The punchline is that we have indeed found a solution in an instanton background where there is *only* a zero mode for the left-chiral spinor χ and *not* the right-chiral spinor χ_R. We conclude that in the one-instanton background $\text{ind}(i\bar{\sigma}\cdot D) = 1$.

Motivating the ansatz. We've seen that (5.18) indeed gives us the structure that we would like. It is appropriate to insert a quick remark motivating that particular ansatz given its suggestive similarity to the instanton profile. See [12] for further details. The trick is to consider the commutator $[D_\mu, D_\nu] \sim F_{\mu\nu}$. This connects our Dirac operator to the structure of the instanton solution. This ends up giving us $D^2\psi = 0$ and $D^2\chi$ suggestive of our ansatz. The $D^2\psi = 0$ statement in fact tells us that the ψ field has no vanishing eigenvalues since $-D^2$ is a sum of the squares of Hermitian operators and is hence positive definite. We can see that indeed the zero mode picks up many characteristics of the instanton such as its localization and size.

5.3 The 't Hooft operator

Let's now consider what happens to the fermion path integral in an instanton background. Let us expand our fields in a basis of $i\not{D} = i\bar{\sigma} \cdot D \oplus i\sigma \cdot D$ eigenstates. For left-chiral spinors,

$$\chi = c_0\chi_0 + \sum_{k=1} c_k\chi_k \tag{5.25}$$

$$\bar{\chi} = \bar{c}_k\bar{\chi}_k \tag{5.26}$$

and similarly for right-chiral spinors

$$\bar{\psi} = \bar{d}_k\psi_k \tag{5.27}$$

$$\psi = d_0\psi_0 + \sum_{k=1} d_k\psi_k. \tag{5.28}$$

These obey the conditions that $\bar{\chi}_k = (\sigma^2\psi_k)^T$ and $\psi_{k,0} = (\sigma^2\chi_{k,0})^T$. We remind ourselves that the c and d coefficients now carry the anticommuting structure of the fermions. Using the orthonormality of the fields χ and ψ, we see that the fermionic action takes the form

$$\int d^4x\, \bar{\Psi} i\not{D}\Psi = \sum_{k=1} \lambda_k \left(\bar{c}_k c_k + \bar{d}_k d_k\right), \tag{5.29}$$

i.e. the zero modes c_0 and $\bar{d}_0$ do not appear because $\bar{c}_0 = d_0 = 0$. This fact manifests itself dramatically when we perform the path integral over the fermion field. The functional measure can be decomposed into a product of measures over the anticommuting coefficients c and d,

$$\int [d\Psi][d\bar{\Psi}] \cdots = \int dc_0 \prod_k dc_k d\bar{c}_k\, d\bar{d}_0 \prod_k d\bar{d}_k dd_k \cdots . \tag{5.30}$$

(In retrospect, the choice of writing d_k was probably not very wise.) Note the important point that the dc_0 and $d\bar{d}_0$ still appear in the functional integral even though they do not appear in the action. This means that there's no Gaussian over those modes in the generating functional,

$$\int [d\Psi][d\bar{\Psi}] e^{-\int d^4x\, \bar{\Psi} i\not{D}\Psi} = \prod_k (\lambda_k^2) \cdot \int dc_0\, d\bar{d}_0. \tag{5.31}$$

the first factor is the usual Gaussian integral for fermionic modes while the leftover integral runs over the zero modes. We know very well from the path integral formulation of fermions[11] that this integral vanishes. A constant integral over a Grassmanian measure vanishes.

We now reach an important lesson. In the presence of massless fermions, the contribution of the $n = 1$ instanton configuration to the path integral is zero. In other words, massless fermions suppress vacuum-to-vacuum instanton tunneling. This does not mean, however, that all of our pure Yang-Mills discussion is in vain: it should also now be very obvious how to calculate objects from the partition function that do *not* vanish.

The answer is the calculate an axial correlation function, $\langle\psi\chi\rangle$. This 'order parameter' violates axial current conservation since $Q^5 \neq 0$ so that we know any *perturbative* calculation should give us $\langle\psi\chi\rangle = 0$. (This still holds in the presence of the axial anomaly, at least in the perturbative picture.) Fortunately, we are *not* doing a purely perturbative calculation since we are expanding about a nontrivial semiclassical configuration. Let's see what this buys us.

$$\langle\chi\psi\rangle = \frac{\int [dA]_n [d\Psi][d\bar{\Psi}]\, e^{-\int d^4x\,\mathcal{L}} \psi(x)\chi(x)}{\int [dA]_n [d\Psi][d\bar{\Psi}]\, e^{-\int d^4x\,\mathcal{L}}}. \tag{5.32}$$

Note that one should, in fact, sum over winding sectors n appropriate to the Θ_{YM} vacuum of the theory. We will see (in case you're not already convinced) that only the $n = 1$ sector contributes. Evaluating the numerator about the $n = 1$ instanton background,

$$\int [dA]_1 [d\Psi][d\bar{\Psi}] e^{-\int d^4x\,\mathcal{L}} \psi\chi = \int [dA]_1 e^{-\int d^4x\, F^2} e^{i\Theta_{\text{YM}}} \int [d\Psi][d\bar{\Psi}] e^{-\sum_k \lambda_k (c_k \bar{c}_k + \bar{d}_k d_k)}\, \psi\chi \tag{5.33}$$

$$= \int [dA]_1 e^{-\int d^4x\, F^2} e^{i\Theta_{\text{YM}}} \det(i\not{D}) \int dc_0\, d\bar{d}_0\, \psi\chi \tag{5.34}$$

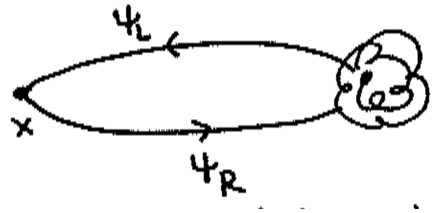
$$= \int [dA]_1 e^{-\int d^4x\, F^2} e^{i\Theta_{\text{YM}}} \det(i\not{D})\, \psi_0\chi_0. \tag{5.35}$$

Voilà! The determinant, which we have defined to run only over non-zero modes, is manifestly non-zero and so we end up with $\langle\psi\chi\rangle \neq 0$! Thus find that instantons violate axial charge conservation by two units. The realization of this mechanism is that the instanton configuration spits out zero modes of different chirality. (The anti-instantonc configuration just gives the Hermitian conjugate of the above operator.) More generally, for F flavors of fermions coupling to the SU(2) gauge field, one expects a violation

$$\Delta Q^5 = -2F(n_L - n_R), \tag{5.36}$$

where $(n_L - n_R)$ is the number of unpaired zero modes (the number of left zero modes minus right zero modes). Below is a schematic diagram of such an instanton effect from [13].

[11] Most of the review literature on instanton methods appear to have been written before fermionic path integrals were a staple of quantum field theory textbooks and so spend a lot of time belaboring this point. We will assume that the reader does not live in the 1970s.



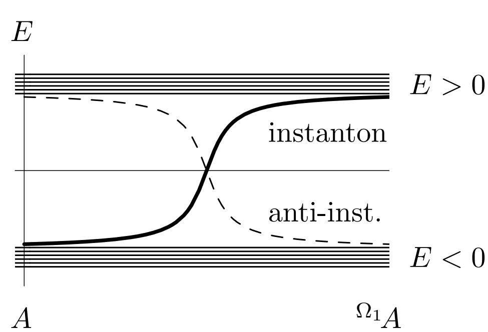
The author could have drawn a fancy computer-generated version, but there's something charming about Michael Peskin's drawings. One can see that the operator is promoted to

$$\mathcal{O} \sim \det(\psi_i \chi_j). \tag{5.37}$$

There is an overall prefactor which includes the integration over instanton collective coordinates and the exponential of the classical action. The actual *amplitude* is not our concern, rather we are more interested that these surprising events can happen at all. This operator was discovered by 't Hooft and is sometimes called the **'t Hooft operator** or the **'t Hooft vertex**. To really drive home the point, let's say it once more time: instantons, as represented by a 't Hooft operator, generate effective couplings between the zero modes of all fermions coupling to that instanton's SU(2) gauge field.

5.4 Instantons make zero modes

Where did this zero mode come from? Surely I can have a theory where all of my fermions have some non-zero mass (e.g in the Standard Model after electroweak symmetry breaking). What does it mean to talk about an electron ($m_e \neq 0$) zero mode in an instanton background? It turns out that such zero modes are created by the instanton field itself (e.g. by the 't Hooft operator). These are precisely the solutions we constructed in Section 5.2. A slightly more poetic statement is to consider particle and antiparticles relative to the 'Dirac sea.' The effect of an instanton is to create a left-handed particle by pulling up a right-handed antiparticle over the 'mass gap' around $-m < E < m$ for a particle of mass parameter m. By doing this the field configurations in the asymptotic past and future end up with different axial charge. The act of pulling the state across the 'mass gap' means that there exists some point in time where the fermion being lifted crosses $E = 0$, as shown in the following diagram from [14],



5.5 Physical nature of the $\Theta_{\rm YM}$ term

Of course, we should have expected something like this ought to happen. We know that axial symmetry is anomalous so *something* weird occurs. Even though—as we saw above—it seemed like we could still use the axial current perturbatively[12], we could have even guessed that there must be some non-perturbative effect since we could see that the anomaly funciton was related to the $\Theta_{\rm YM}$ term. Let us remark on the nature of the $\Theta_{\rm YM}$ term as a physical object.

Under a *redefinition* of our path integral functional argument $\Psi \to e^{i\alpha\gamma^5}\Psi$, Fujikawa taught us that the funtional measure transformas as

$$[d\Psi][d\bar{\Psi}] \to \exp\left(\frac{-i\alpha}{32\pi^2}\int d^4x\, F\widetilde{F}\right)[d\Psi][d\bar{\Psi}]. \tag{5.38}$$

This is completely equivalent to a shift in the $\Theta_{\rm YM}$ parameter,

$$\Theta_{\rm YM} \to \Theta_{\rm YM} + 2\alpha. \tag{5.39}$$

The transformation also changes our fermion mass parameters. Let us first write them in terms of 'chiral' masses

$$\mathcal{L}_m = -m\bar{\Psi}P_L\Psi - m^*\bar{\Psi}P_R\Psi. \tag{5.40}$$

The chiral redefinition of Ψ also forces us to redefine $m \to e^{2i\alpha}m$. This is a problem. A complex mass parameter violates P and CP symmetries. These are, of course, very well constrained by experiment. More to the point, such a rephasing appears to be a *physical* effect.

This does not seem to make sense; we know that *redefinitions* of our path integral functionals *cannot* lead to physical effects. The resolution is that observables cannot depends *separately* on $\Theta_{\rm YM}$ or m phases, but only on the combination

$$e^{-i\Theta_{\rm YM}}m, \tag{5.41}$$

where $m \to \prod_f m_f$ for many flavors (in which case the $\Theta_{\rm YM}$ shift is $\Theta \to \Theta + 2\sum_f \alpha_f$). This tells us that we can always define our fermion masses and effective Lagrangian relative to $\Theta_{\rm YM} = 0$. However, the cost is possible P and CP violation. Of course, if there exists a massless fermion f with $m_f = 0$, then the $\Theta_{\rm YM}$ angle has no effect and both P and CP are conserved.

(CP violation is an important question regarding the $\Theta_{\rm YM}$ term; why have we found ourselves in a vacuum where $\Theta_{\rm YM}$ is so small? This topic is unfortunately beyond the scope of this article but readers are encouraged to explore the axion solution to this so-called **strong CP problem**.)

5.6 The U(1) problem

Now that we've done a lot of heavy lifting, let's try to reap something useful: a solution to the U(1) problem. Stated succinctly, the U(1) problem asks: where is the would-be Goldstone boson associated with the $\mathrm{U}(1)_A$ axial symmetry of QCD?

As we know, we can describe low-energy QCD using chiral perturbation theory. The Lagrangian satisfies an $\mathrm{U}(3)_V\times\mathrm{U}(3)_A$ flavor symmetry for the light quarks (u, d, s). One assumes

[12] Old-timers will refer to the partially-conserved axial current (PCAC).

that the QCD vacuum forms a quark-antiquark condensate $\langle q\bar{q}\rangle \neq 0$ that breaks $\mathrm{U}(3)_A$ spontaneously. One can then construct the Goldstone modes associated with $\mathrm{U}(3)_A$ by acting upon the QCD ground state with the axial currents and calling the resulting modes pions (or otherwise light mesons). In the limit of nonzero quark masses the axial U(3) symmetry is preserved in the Lagrangian so that these pions must indeed be massless Goldstone modes. Turning on small quark masses lifts the modes and breaks their degeneracy.

Since the strange quark mass is much heavier than the other light quarks, let us ignore it for the moment and only consider the $\mathrm{U}(2)_V \times \mathrm{U}(2)_A$ subgroup which is not-so-badly broken by $m_{u,d}$. Further, us decompose this symmetry into

$$\mathrm{SU}(2)_V \times \mathrm{SU}(2)_\times \mathrm{U}(1)_V \times \mathrm{U}(1)_A. \tag{5.42}$$

We recognize the preserved $\mathrm{SU}(2)_V$ flavor symmetry and $\mathrm{U}(1)_V$ baryon number current. Since this is not a paper on chiral perturbation theory[13] we will only state the result that one can indeed construct the $\mathrm{SU}(2)_A$ goldstone modes corresponding to the pseudoscalar isotriplet (pions). We can turn on the light quark masses and they obtain masses at the expected scale and we indeed see these particles in our low-energy hadronic spectrum. Splendid.

The problem is the $\mathrm{U}(1)_A$ current. There is no pseudoscalar isosinglet would-be Goldstone boson (pseudo-Goldstone) anywhere near the pions in our hadronic spectrum. If we relax our search parameters and look at higher masses, we find that η has the correct quantum numbers, but is oddly heavy. Weinberg provided an upper bound for the expected would-be Goldstone boson [15, 6]

$$m_{\mathrm{U}(1)_A} \leq \sqrt{3} m_\pi, \tag{5.43}$$

which is violated by the η and any other candidate meson.

It turns out that the η is a red herring, anyway [1]. Even though the η happens to have the right quantum numbers of the $\mathrm{U}(1)_A$ pseudo-Goldstone, it is actually a member of the $\mathrm{SU}(3)_A$ pseudo-Goldstone octet. It's just another particle. There are a bunch of QCD resonances that happen to have the right quantum numbers. Typically the $\mathrm{U}(1)_A$ problem is associated with the next-lightest meson, the η'. We will see that it doesn't really matter (nor does it necessarily make sense to identify) which meson is the pseudo-Goldstone of $\mathrm{U}(1)_A$ precisely because there *isn't* a pseudo-Goldstone of $\mathrm{U}(1)_A$. The U(1) problem is sometimes called the η or η' problem, so that at this point one may sit back and reflect upon the title of this paper.

5.7 The U(1) solution

We could attempt to wave away the previous argument by saying that there is no problem because $\mathrm{U}(1)_A$ is anomalous and so is not a 'real' symmetry of the system. One might choose to say that $\mathrm{U}(1)_A$ is broken explicitly by these anomalies and that's that. However, we must still deal with the uneasy proposal made much earlier in (4.17): the anomalous term in the divergence of the axial current is itself a divergence so that we are free to define a modified axial current that really

[13] Chiral perturbation theory is an interesting topic in itself that, were it not for little Higgs models, would be something of a dying art. The author refers you to his personal notes on the subject.

is divergence-free:

$$j' = j^5 - 2K. \tag{5.44}$$

As far as perturbation theory about a trivial gauge background is concerned, this is perfectly good current and should lead to precisely the axial pseudo-Goldstone that we've been rambling on about. We will see that the punchline is, in fact, that the anomaly breaks $\mathrm{U}(1)_A$ and that there shouldn't be a Goldstone associated with it. The real 'problem,' then, is to understand how this actually happens given that simply saying 'anomaly' doesn't appear to make the Goldstone go away.

Kogut[14] and Susskind[15] proposed a way out of the U(1) problem [16]. The crux of their proposal was that the divergence-free modified axial current j'_μ could couple to a particle which remains massless in the $m_{u,d,s} = 0$ limit. This would allow a way out of the partially conserved axial current (PCAC) constraints that relate $m_{\mathrm{U}(1)_A}$ to m_π. (The story of PCAC is beyond the scope of this paper but is presented in older texts such as [8, 17].) Heuristically, Kogut and Susskind hoped that ghost fields (e.g. from covariant gauge fixing) could couple to j' in such a way. Their prototype was Schwinger's model of spinor electrodynamics in 1+1 dimensions where massless fields $\phi_\pm$ were, respectively, positive and negative norm fields so that their sum is free of any poles (in particular, from Goldstone modes). Gauge *invariant* quantities couple to a combination with zero propagator $\phi_+ + \phi_i$ while *non-invariant* quantities could couple to combinations whose propagators did not cancel, e.g. $\partial_\mu(\phi_+ - \phi_-)$. Coleman calls such a thing a '**Goldstone dipole**,' but this is a rather misleading (and altogether silly) name.

The moral of the story is really gauge invariance, which we've subtly been trying to promote over the course of this paper. The Chern-Simons current K_μ is *not* gauge invariant. The key to the proposed Kogut-Susskind mechanism is that the massless pole which the non-invariant current couples to should not also appear as poles in *physical* quantities. In other words, such poles are a kind of gauge artifact. 't Hooft found that this mechanism can be realized using instantons [18].

In fact, we've already done most of the relevant heavy-lifting. We've already seen how instantons cause a shift in the axial charge $\Delta Q^5 = 2F$. The Chern-Simons current is precisely the gauge non-invariant object which couples to axial charge. In Section 5.5 we saw that the effect of chiral rotations can be undone by a rotation on the Θ_{YM} angle. This tells us that the Goldstone boson for chiral rotations corresponds to oscillations of the Θ_{YM} parameter. This is the 'order parameter' of chiral symmetry breaking. Unlike the usual σ models for scalar fields with U(1) symmetry, however, the Θ_{YM} Goldstone is *not* physical. It is certainly no more physical than the unphysical 'pseudomomentum' θ of the periodic potential in quantum mechanics. In other words, angular rotations about a Mexican hat potential are really oscillations in physical field space that do not cost any energy, where as rotations in Θ_{YM} are '*unphysical.*'

This proposed solution may sound trivial. The point, however, was that the PCAC framework was developed before gauge theories were widely understood so that historically very intelligent people applied the usual arguments based on Ward identities to gauge non-invariant sectors of QCD [19].

Returning to the title of this paper, is it plausible that the η' is the missing $\mathrm{U}(1)_A$ pseudo-Goldstone which, by instanton effects (i.e. mass terms from the 't Hooft operator), have picked

[14] at around the same time he was developing the renormalization group

[15] at around the same time he was connecting the Veneziano amplitude to a theory of strings

up a much heavier mass than the pion octet? In some sense this question is ill-posed; one would like to say that the η' somehow becomes very light when 'instanton effects are turned off.' Unlike taking $m_{u,d} \to 0$, it is certainly not a well-prescribed continuous limit to 'turn off anomalies.' However, given that the question is ill-posed, we can act like theorists and push valiantly forward *anyway*. The relevant scales are

$$m_{\pi_0} \approx 135 \text{ MeV} \qquad\qquad m_\eta \approx 548 \text{ MeV} \qquad\qquad m_{\eta'} \approx 958 \text{ MeV}. \tag{5.45}$$

The pion masses are right where they ought to be, somewhere around Λ_{QCD}. The η, as we explained, is part of the pseudo-Goldstone octet and gets mass contributions from the strange quark which breaks chiral symmetry significantly more than the up and down. To get a feel for whether η has a reasonable mass, we can compare to the kaon (a $\bar{s}s$ bound state), which has $m_K \approx 498$ MeV. This is also in the right ball park. Now we get to the η'. We posit that this gets contributions form QCD instantons. In the context of the 't Hooft interaction, we say that there is no $\text{U}(1)_A$ symmetry at all and instead of a zero mass the η' should have masses on the order of the strong interaction scale. A reasonable comparison is the vector meson ϕ, which has $m_\phi = 1019$ MeV. This precisely the neighborhood of the observed η', so we can pat ourselves on the back.

6 *Fromage*: Baryon and lepton number violation

Let us make an incredibly brief remark about generalizations of these techniques for anomalous symmetries. Let us forget about SU(3) color and return to the $\text{SU}(2)_L$ of electroweak theory. We've seen how instantons provide a non-perturbative mechanism to realize the anomalous breaking of a symmetry. Now that we've developed a hammer, we might as well look around for some nails. Are there any anomalies in electroweak theory? Yes! Although all of the currents coupling to gauge symmetries are non-anomalous by algebraic miracles (which may come about from embedding into an anomaly-free gauge group), there are certainly two global symmetries that are broken by anomalies: baryon and lepton number. (Recall, however, that $(B - L)$ is conserved.)

$$j_L^\mu = \sum_f \bar{\ell}_f \gamma^\mu \ell_f + \bar{\nu}_f \gamma^\mu \nu_f \tag{6.1}$$

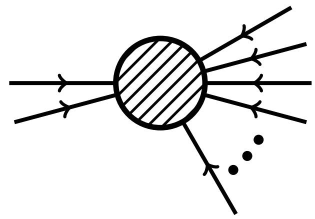
$$j_B^\mu = \sum_f \frac{1}{3} \bar{u}_f \gamma^\mu u_f + \frac{1}{3} \bar{d}_f \gamma^\mu d_f. \tag{6.2}$$

These have a divergence that take an unsurprisingly familiar form. For G generations,

$$\partial_\mu j_L^\mu = \partial_\mu j_B^\mu = -\frac{Gg^2}{16\pi^2} \text{tr} F_{\mu\nu} \widetilde{F}^{\mu\nu}. \tag{6.3}$$

Now we can make *exactly* the same arguments that we've done above. In particular, we have sectors of different weak winding number and instantons tunneling between them. These instanton backgrounds yield 't Hooft operators that will violate baryon and/or lepton number. In particular, for $G = 3$ we can have interactions of the form

$$Q_1 \, Q_2 \to \bar{Q}_1 + 3\bar{Q}_2 + 3\bar{Q}_3 + L_1 + L_2 + L_3 \tag{6.4}$$



Here the Q and L refer to quark and lepton $\mathrm{SU}(2)_L$ doublets. The 't Hooft vertex is an interaction between 12 fermions (including color factors for the quarks). Examples include

$$u + d \to \bar{d} + \bar{s} + 2\bar{c} + 3\bar{t} + e^+ + \mu^+ + \tau^+ \tag{6.5}$$

$$u + d \to \bar{u} + 2\bar{s} + \bar{c} + \bar{t} + 2\bar{b} + \nu_e + \nu_\mu + \tau^+ . \tag{6.6}$$

We remark that $B - L$ must be conserved, and indeed we see that $\Delta B = \Delta L = -3$. These events are weak, with amplitudes that have the semiclassical factor $\exp(-16\pi^2/g_2^2) \sim 10^{-169}$.

We note that there are two main differences between the case of electroweak symmetry versus QCD. First, we know that electroweak symmetry is broken and most of the $\mathrm{SU}(2)_L$ gauge bosons obtain masses. This makes them short range forces whose profiles should go as something like e^{-Mr}. As far as our qualitative assessment is concerned, this is fine: the instanton configurations are modified, but they still interpolate between vacua. Secondly, we remark that the couplings of leptons L and quarks Q to the weak force involve only the left-handed doublets. Thus couplings include projectors P_L which *mix* vector and axial couplings. Thus both vector and axial currents can be anomalous. In particular, baryon and lepton number are vector currents.

Let us end this discussion with an even more parenthetical remark. Because the electroweak sector is much more weakly coupled than QCD, instanton effects are expected to be rather small. However, there exist cousins of instantons that mediate similar effects called **sphalerons**. These are shown heuristically in Fig. 5. Instead of tunneling between the topological vacua like instantons, sphalerons are events that are actually energetic enough to jump over the potential barrier. Such events are expected to have occurred in the early universe when the temperature was hot enough for fields to access neighboring topological vacua.

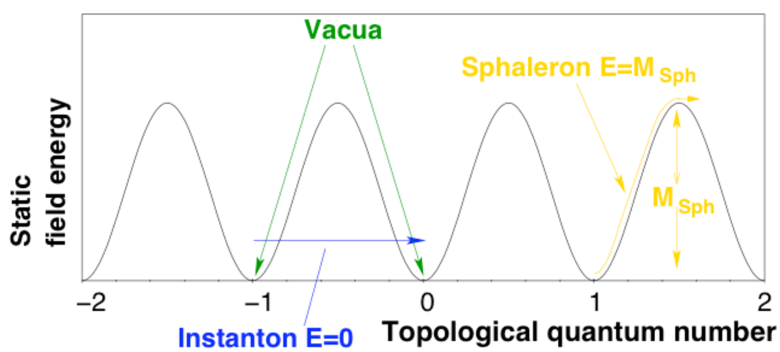


Figure 5: Heuristic description of sphalerons versus instantons. Image from [20].

7 *Dessert*: SQCD

Now we switch gears completely and review the key aspects of supersymmetric $SU(N)$ gauge theory with F flavors, in particular its nonperturbative for various values of F and N. We will find that instanton effects play a surprising role in the determination of the Affleck-Dine-Seiberg (ADS) superpotential for *all* values of F and N. We'll dispense with the usual pleasantries of motivating supersymmetry; no time to wax poetic about its virtues. Let's jump straight into the fracas!

7.1 Moduli space

SYM theories one typically finds **flat directions** or **moduli** in the field space. These are directions in the scalar fields with vanishing potential. When supersymmetry is broken these tree-level flat directions are often lifted through quantum corrections, i.e. by the Coleman-Weinberg potential. In that case these directions are called **pseudomoduli**. We can now study how these flat directions arise in super QCD. At the bare minimum this theory will have a D-term potential since it is a gauge theory. It needn't necessarily have any superpotential, so we will ignore the superpotential contribution for now[16].

7.1.1 Case $F < N$

We assume that we have an $SU(N)$ theory with $F < N$ flavors of 'quarks' ϕ_{im} in the fundamental and 'antiquarks' $\overline{\phi}^{im}$ in the anti-fundamental, where $i = 1, \cdots, F$ and $m = 1, \cdots, N$. The D-term for this theory are

$$\begin{aligned}
D^a &= \sum_i \phi_i^\dagger T^a \phi_i + \overline{\phi}_i^\dagger \overline{T}^a \overline{\phi}_i \\
&= \left[\sum_i \left(\phi^\dagger\right)^{in} \phi_{im} - \sum_i \overline{\phi}^{in} \left(\overline{\phi}^\dagger\right)_{im} \right] \left(T^a\right)_n{}^m .
\end{aligned}$$

where we understand that the ϕs really mean $\langle \phi \rangle$. We can define the $N \times N$ matrices $D^n{}_m$ and $\overline{D}^n{}_m$,

$$\begin{aligned}
D^n{}_m &= \left(\phi^\dagger\right)^{in} \phi_{im} \\
\overline{D}^n{}_m &= \overline{\phi}^{in} \left(\overline{\phi}^\dagger\right)_{im} .
\end{aligned}$$

The condition that our D-term scalar potential vanishes (the 'D-flatness condition') then imposes $D^a = 0$. Since the generators T^a are traceless, a solutions is

$$D^n{}_m - \overline{D}^n{}_m = \alpha \mathbb{1}$$

for some overall constant α. We may now use an $SU(N)$ gauge transformation to diagonalize the D and $\overline{D}$ matrices. In the case $F < N$. Then from their definition we see that the D and $\overline{D}$

[16] In general the superpotential is highly constrained by the global symmetries of the theory.

matrices can have at most F nonzero eigenvalues. Thus they must take the form

$$D = \text{diag}(v_1^2, v_2^2, \cdots, v_F^2, \underbrace{0, \cdots, 0}_{(N-F)}).$$

Imposing $D - \overline{D} = \alpha \mathbb{1}$ then imposes that $\overline{D}$ must also be a diagonal matrix. By the structure of the zero and non-zero entries, we establish that the D-flatness condition can only be satisfied for $\alpha = 0$. From this we may write the solutions for our quark fields,

$$\langle \phi \rangle = \langle \overline{\phi}^\dagger \rangle = \begin{pmatrix} v_1 & & \\ & \ddots & \\ & & v_F \\ \hline 0 & \cdots & 0 \end{pmatrix}. \tag{7.1}$$

This spontaneously breaks $SU(N) \to SU(N-F)$. We observe the **super Higgs mechanism** at work: we started with $(2F) \times N$ chiral superfields and found a vev where we have a number of broken generators

$$(N^2 - 1) - \big((N-F)^2 - 1\big) = 2NF - F^2,$$

each of which 'eats' a chiral superfield. The number of D-flat directions is then the number of chiral superfields minus the number of broken generators,

$$(2NF) - (2NF - F^2) = F^2.$$

In the usual Higgs mechanism a massless vector eats a massless Goldstone boson. The exact same effect occurs here, but due to supersymmetry the entire superfields must be included. Conceptually the actual 'coupling' of the two superfields occurs between the massless vector component and the Goldstone scalar, so one can think of the super Higgs mechanism as the joining of two superfields due to the mixing of one of each of their components due to the regular Higgs mechanism. After this feast, the remaining F^2 massless degrees of freedom are parameterized by an $F \times F$ meson field,

$$M_i^{\ j} = \overline{\phi}^{jn} \phi_{ni}. \tag{7.2}$$

There is actually a more general theorem by Luty and Taylor [21] regarding this.

Theorem 7.1 (Luty-Taylor)**.** *The classical moduli space of degenerate vacua can always be parameterized by independent, holomorphic, gauge-invariant polynomials.*

Proof. A heuristic proof is provided in Intriligator and Seiberg's lecture notes on Seiberg duality [22]. Setting the [D-term] potential to zero and modding out by the gauge group is equivalent to modding out by the complexified gauge group. The space of chiral superfields modulo the complexified gauge group can be parameterized by the gauge invaraint polynomials modulo any classical relations. Then, Intriligator and Seiberg claim, this theorem follows from geometrical invariant theory [23]. For a proper proof the reader is directed to the original paper by Luty and Taylor [21]. □

7.1.2 Case $F \geq N$

Before moving on let's quickly cover the case $F \geq N$. As before the D-flatness condition is still $D - \overline{D} = \rho \mathbb{1}$, where ρ is some constant. We can again use the $SU(N)$ gauge degree of freedom to diagonalize the $D = (\phi^\dagger)^i \phi_i$ and $\overline{D}$ matrices, though now they are of full rank and we may use the D-flatness condition to write $\overline{D}$ in terms of the eigenvalues of D and the constant ρ,

$$D = \begin{pmatrix} |v_1|^2 & & \\ & \ddots & \\ & & |v_N|^2 \end{pmatrix} \qquad \overline{D} = \begin{pmatrix} |v_1|^2 - \rho & & \\ & \ddots & \\ & & |v_N|^2 - \rho \end{pmatrix}. \tag{7.3}$$

This implies that we may write the $\langle \phi \rangle$ and $\langle \overline{\phi} \rangle$ matrices as

$$\langle \phi \rangle = \left(\begin{array}{ccc|c} v_1 & & & \\ & \ddots & & 0 \\ & & v_n & \end{array} \right) \qquad \langle \overline{\phi} \rangle = \left(\begin{array}{ccc} v_1 & & \\ & \ddots & \\ & & v_N \\ \hline & 0 & \end{array} \right). \tag{7.4}$$

Now we see that $SU(N)$ is completely broken at a generic point on the moduli space. This means that we have (N^2-1) broken generators and thus $[2NF - (N^2-1)]$ light D-flat directions in field space. Again we parameterize these degrees of freedom by 'gauge-invariant polynomials',

$$M_i^{\ j} = \overline{\phi}^{jn} \phi_{ni} \tag{7.5}$$

$$B_{i_1 \cdots i_N} = \phi_{n_1 i_1} \cdots \phi_{n_N i_N} \epsilon^{n_1 \cdots n_N} \tag{7.6}$$

$$\overline{B}_{i_1 \cdots i_N} = \overline{\phi}^{n_1 i_1} \cdots \overline{\phi}^{n_N i_N} \epsilon_{n_1 \cdots n_N}. \tag{7.7}$$

But wait! We find that we have too many degrees of freedom. That's okay. We've forgotten to impose the classical constraints to which these fields are subject,

$$B_{i_1 \cdots i_N} \overline{B}^{j_1 \cdots j_N} = M_{[i_1}^{\ \ j_1} \cdots M^{j_N}_{\ \ i_N]} \sim \det M \tag{7.8}$$

7.2 The holomorphic gauge coupling

Recall that the action for a vector superfield is conventionally written as

$$\mathscr{L} = \frac{1}{4} \int d^2\theta \, \mathbb{W}^{a\alpha} \mathbb{W}^a_\alpha + \text{h.c.} \tag{7.9}$$

In this case, the gauge coupling g shows up in the kinetic term for the chiral superfields

$$\mathscr{L}_{\text{kin}} = \int d^4\theta \, \phi^\dagger e^{gV^a T^a} \phi. \tag{7.10}$$

We can redefine $\mathbb{W}$ by absorbing the coupling into the vector superfield,

$$\widetilde{V}^a = g V^a, \tag{7.11}$$

where we are no longer canonically normalized, but we are in some sense using a natural normalization[17]. Then the vector Lagrangian takes the form

$$\mathscr{L} = \frac{1}{4g^2}\int d^2\theta\, \mathbb{W}^{a\alpha}\mathbb{W}^a_\alpha + \text{h.c.} \tag{7.12}$$

We know that there are also non-perturbative effects that contribute to this Lagrangian, i.e. the Θ_{YM} term. We can include this effect by defining a **holomorphic gauge coupling**[18],

$$\tau \equiv \frac{4\pi i}{g^2} + \frac{\Theta_{\text{YM}}}{2\pi} \tag{7.13}$$

Our vector superfield Lagrangian finally takes the form

$$\mathscr{L} = \frac{1}{16\pi i}\int d^2\theta\, \tau\mathbb{W}^{a\alpha}\mathbb{W}^a_\alpha + \text{h.c.} \tag{7.14}$$

Since τ only appears under the $d^2\theta$ of the superpotential, it is manifestly a holomorphic parameter. Recall the RG equations for the perturbative coupling,

$$\mu\frac{dg}{d\mu} = -\frac{b}{16\pi^2} \tag{7.15}$$

$$\frac{1}{g^2(\mu)} = -\frac{b}{8\pi^2}. \tag{7.16}$$

Applying this to τ, we may write

$$\tau_{\text{1-loop}} = \frac{1}{2\pi i}\left[b\log\left(\frac{|\Lambda|}{\mu} + i\Theta_{\text{YM}}\right)\right] \tag{7.17}$$

$$= \frac{b}{2\pi i}\log\left(\frac{\Lambda}{\mu}\right), \tag{7.18}$$

where have defined the **holomorphic dynamical scale**

$$\Lambda = |\Lambda|e^{i\Theta_{\text{YM}/b}}. \tag{7.19}$$

Theorem 7.2. *The holomorphic coupling is only perturbatively renormalized at one loop. It does, however, receive non-perturbative corrections from instanton effects.*

Proof. We've written the one-loop renormalization of g in Eq. (7.18). We now have to show that this only gets corrections from instantons. The key will be to consider the Θ_{YM} dependence. We know that Θ_{YM} is a term which multiplies an $F\widetilde{F}$ in the Lagrangian,

$$F\widetilde{F} = 4\epsilon^{\mu\nu\rho\sigma}\partial_\mu \text{Tr}\left(A_\nu\partial_\rho A_\rho + \frac{2}{3}A_\nu A_\rho A_\sigma\right). \tag{7.20}$$

[17]This can be understood, for example, by considering the renormalization of the gauge coupling in ordinary (non-supersymmetric) field theory. The only diagrams that contribute to this renormalization come from loop contributions to the gauge field propagator. This tells us that g is 'really' something associated to the vector field, not necessarily the coupling of the vector to fermions.

[18]There seem to be many 'standard' normalizations for τ which differ by factors of, e.g., 2π. I audibly groan every time I read a paper with a different normalization.

This is a total derivative and has no effect in perturbation theory (as expected from a non-perturbative instanton effect). However, this term contributes to a topological winding number, n,

$$\frac{\Theta_{\mathrm{YM}}}{32\pi^2}\int d^4x\, F\widetilde{F} = n\Theta_{\mathrm{YM}}. \tag{7.21}$$

In the path integral $\int dA \exp(iS) \sim \int dA \exp(in\Theta_{\mathrm{YM}})$. Thus we see that the Θ_{YM} must be periodic in 2π, i.e. $\Theta_{\mathrm{YM}} \to \Theta_{\mathrm{YM}}+2\pi$ must be a symmetry of the theory. Under this transformation the dynamical scale goes as

$$\Lambda \to e^{2\pi i/b}\Lambda. \tag{7.22}$$

This, in turn, affects the effective superpotential $W_{\mathrm{eff}} = \tau/(16\pi i)\mathbb{W}^2$ through the dependence of the holomorphic coupling on Λ,

$$\tau = \frac{b}{2\pi i}\log\left(\frac{\Lambda}{\mu}\right) + f(\Lambda,\mu), \tag{7.23}$$

where the first term is the one-loop result that we derived and the second term represents an arbitrary function that would include higher-loop corrections. Under the transformation of Λ in (7.22), the one-loop term is already shifted by one unit. Thus the first term already saturates the correct behavior, so the second term must be invariant under the transformation. We can then write out the second term as

$$f(\Lambda,\mu) = \sum_{n=1}^{\infty} a_n \left(\frac{\Lambda}{\mu}\right)^{bn}, \tag{7.24}$$

where the form is set by demanding weak coupling as $\Lambda \to 0$ (we want the perturbative result in this limit). Terms of this form, however, just represent instanton effects. Recall the instanton action,

$$S_{\mathrm{inst}} = \frac{8\pi^2}{g^2} \qquad \Rightarrow \qquad e^{S_{\mathrm{inst}}} \sim e^{2\pi i\tau} = \left(\frac{\Lambda}{\mu}\right)^{b}. \tag{7.25}$$

Thus instanton effects in SUSY gauge theories will always appear with a prefactor of $(\Lambda/\mu)^b$. Thus we have the result that τ is only [perturbatively] renormalized at one-loop order. □

One can also determine the instanton corrections. For example, Seiberg and Witten famously found exact expressions for the a_n coefficients in $\mathcal{N}=2$ SYM. For review see, e.g., [24].

7.3 The NSVZ β-function

If you are doing everything well, you are not doing enough.
– Howard Georgi, personal motto [5]

There is a lovely discussion of the role of instantons in the NSVZ β function. In fact, the story of the NSVZ β function is delightful in itself. Unfortunately, this is beyond the scope of the current work and the author is hopelessly out of time to add anything extra.

7.4 $F < N$: the ADS superpotential

We now review the famous result by Affleck, Dine, and Seiberg in the 1980s that instantons generate the so-called ADS superpotential [25]. Along the way we'll learn how to use the moduli space to go to regions in parameter space where we can make definitive statements that carry over to the nonperturbative regime. In the following section we'll make use of the tools that we've developed to go over the $F \geq N$ case. This should prepare the reader for next step—well beyond the scope of this paper—a discussion of Seiberg duality.

7.4.1 Holomorphic scale as a spurion

The trick that we will employ is to promote the instanton power of the holomorphic scale Λ^b to a spurion for anomalous symmetries. In particular, anomalies from instantons appear via the **'t Hooft operator**,

$$\mathcal{O}_{\text{'t Hooft}} = \Lambda^b \prod_i {\psi_i}^{2T_i}, \tag{7.26}$$

where $T_i = T(\Box) = 1/2$ for the fundamental representation. For a one-instanton background and under a chiral rotation, i.e. a rotation that acts independently on each chiral fermion ψ_i,

$$\psi_i \to e^{i\alpha q_i}\psi_i \tag{7.27}$$

$$\Theta_{\text{YM}} \to \Theta_{\text{YM}} - \alpha \sum_r n_r \cdot 2T(r) \tag{7.28}$$

$$\Lambda^b \to \Lambda^b e^{-i\sum_r n_r(2T(r))}. \tag{7.29}$$

If we recall that $\Lambda = |\Lambda| \exp(i\Theta_{\text{YM}}/b)$, we note that we can assign a fake (i.e. spurious) charge to Λ so that the 't Hooft operator preserves the chiral symmetry,

$$q_\Lambda = -\sum_r 2n_r T(r). \tag{7.30}$$

7.4.2 The ADS Superpotential

Our goal is to write down the effective superpotential. We know that this is given by gauge-invariant polynomials. In fact, the symmetries of the theory allow us to further constrain the superpotential. Let's explicitly write out the representations of the relevant fields under all of these symmetries using a funny table of boxes,

	$SU(N)$	$SU(F)$	$SU(F)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
Q	$\Box$	$\Box$	1	1	0	0
$\overline{Q}$	$\overline{\Box}$	1	$\Box$	0	1	0
Λ^b	1	1	1	F	F	$-2F+2N$

The Λ^b charges under $U(1)$ and $U(1)_2$ are given by the prescription above. Note that all of the $U(1)$ symmetries are anomalous, though two combinations are anomaly-free. (We don't have to worry about this for now.) Since the holmorphic scale is the only quantity carrying R-charge, we know that the superpotential must go as

$$W \sim \Lambda^{\frac{3N-F}{N-F}}. \tag{7.31}$$

Invariance under the $U(1)$ symmetries forces additional factors,

$$W \sim \left(\frac{\Lambda^{3N-F}}{Q^F \overline{Q}^F}\right)^{\frac{1}{N-F}}. \tag{7.32}$$

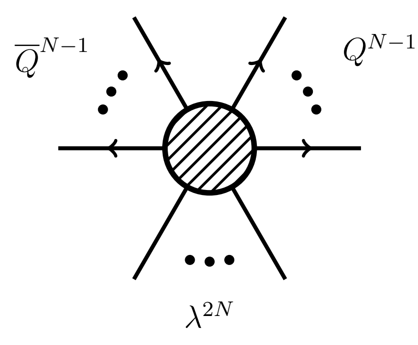
Imposing flavor invariance and writing the superpotential in terms of gauge invariant polynomials (which parameterize the moduli space), we get the **ADS superpotential**,

$$W_{\text{ADS}} = C_{NF} \left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}}, \tag{7.33}$$

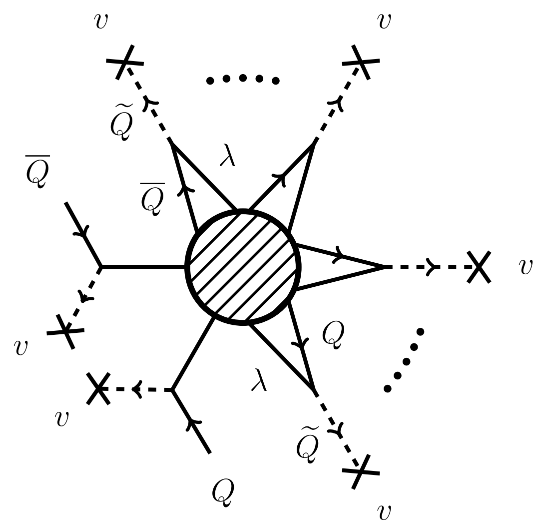
where we've written M to be the gauge-invariant meson field and C_{NF} is a coefficient that we have to determine. We'll now do this for the particular case $F = N-1$ and then we'll show that there are neat tricks we can do to derive more general combinations (F, N).

7.4.3 ADS: $F = N - 1$

For $F = N-1$, $W \sim \Lambda^{3N-F} = \Lambda^b$, so that the ADS superpotential smells like an instanton effect. In this case the $SU(N)$ gauge symmetry is completely Higgsed to $SU(N-(N-1)) = SU(1) =$ nothing. Does this buy us anything? It sounds bad, this puts us in an asymptotically free ($\beta > 0$), strongly interacting region. However, we can go to a region in moduli space where $\langle M \rangle$ is very large. In particular, we can go to a theory where the gauge group breaks *before* the theory becomes strongly coupled so that our instanton calculations are reliable in this weakly interacting regime. Before we jump ahead of ourselves, though, let's convince ourselves that these really are instanton effects. The 't Hooft operator can be drawn as a vertex with an external leg for each zero mode fermion: the quarks, anti-quarks, and the gauginos.



This doesn't quite look like our superpotential. However, we can go along the flat directions to points in the moduli space where the squarks have very large vevs, v. Now recall that we have the coupling between squarks and gauginos, $\lambda Q\widetilde{Q}^*$ and $\lambda\overline{Q}\widetilde{\overline{Q}}^*$. We can use these couplings to connect the λ and $Q, \overline{Q}$ legs of the 't Hooft operator. We have two gaugino legs left over, which we may convert into quarks as shown in the diagram[19]



This rather complicated diagram gives us a contribution to the 'quark' mass (where we're being lax about v versus v^*)

$$v^{2N}Q\overline{Q}\Lambda^{2N+1}. \tag{7.34}$$

To get the right term for the ADS superpotential we need to suppress by the length scale of the instanton. In the presence of the squark vev, this length scale is

$$\rho^2 \sim \frac{b}{16\pi^2|v|^2},$$

and so we can write our instanton-background Lagrangian as

$$\mathscr{L} \sim v^{2N}Q\overline{Q}\Lambda^{2N+1}\left(\rho\right)^{2N} \tag{7.35}$$

$$= v^{-2N}Q\overline{Q}\Lambda^{2N+1}. \tag{7.36}$$

This is just the fermion mass term that we get from the ADS superpotential.

$$W_{\text{ADS}} = \frac{\Lambda^{2N+1}}{\det M} \quad \rightarrow \quad \sim \Lambda^{2N+1}\frac{Q\overline{Q}}{\left(\widetilde{Q}\widetilde{\overline{Q}}\right)^N} \sim \Lambda^{2N+1}\frac{Q\overline{Q}}{v^{2N}}. \tag{7.37}$$

Thus we see that the ADS superpotential for $F = N - 1$ is really just a one-instanton term. Grown-ups can do the exact instanton calculation. I don't know how they do it, and for the

[19]I drew this diagram myself using the TikZ/PGF library in LaTeX. I am very proud of myself.

moment I don't really care. The magical result however, is that the coefficient C_{NF} for $F = N-1$ is... drum-roll...

$$C_{N,N-1} = 1. \tag{7.38}$$

Now we understand what we need for the particular case $F = N-1$. That's useful for very specific models, but we are more ambitious.

7.5 Exploring the moduli space

Now that we have the instanton-generated solution for the particular case $F = N - 1$, we would like to determine the ADS coefficient for arbitrary values of F and N by deforming the theory and seeing where we can find ourselves by following the moduli space.

7.5.1 Give one squark a large vev

Our first trick will be to assign a large vev to one squark flavor,

$$\langle q_F \rangle = \langle \overline{q}_F \rangle = v. \tag{7.39}$$

We thus have two scales in the theory that we'd like to relate via the Wilsonian renormalization group. The original theory has an $SU(N)$ gauge group with F flavors, while the low-energy Higgsed theory has $SU(N) \to SU(N-1)$ and one flavor eaten, i.e. $SU(N-1)$ with $(F-1)$ flavors. Thus this Higgsing has taken us from (N, F) to $(N-1, F-1)$. By matching these two theories, we can find a way to relate the coefficients $C_{N,F}$ and $C_{N-1,F-1}$.

We can now perform usual EFT matching of the low energy (with a subscript L) and UV couplings at the scale v. T

$$\frac{8\pi}{g_L^2(v)} = \frac{8\pi}{g^2(v)} \quad \Rightarrow \quad b_L \log\left(\frac{v}{\Lambda_L}\right) = b \log\left(\frac{\mu}{\Lambda}\right) \quad \Rightarrow \quad \left(\frac{\Lambda_L}{v}\right)^{b_L} = \left(\frac{\Lambda}{v}\right)^{b}. \tag{7.40}$$

The value of the β-function coefficients are well known in SUSY QCD,

$$b = 3N - F \quad \Rightarrow \quad b_L = 3(N-1) - (F-1), \tag{7.41}$$

from which we obtain the so-called **scale-matching** conditions,

$$\Lambda_{N,F}^{3N-f} = v^2 \Lambda_{N-1,F-1}^{3N-F-2}. \tag{7.42}$$

We can represent the $(F-1)^2$ light [scalar] degrees of freedom as an $(F-1) \times (F-1)$ matrix M_L. This can be related to the analogous $F \times F$ matrix in the original (UV) theory via

$$\det M = v^2 \det M_L. \tag{7.43}$$

Going back and plugging (7.40 - 7.43) into the ADS superpotential in (7.33), we get

$$\begin{aligned} C_{N,F}\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{1/N-F} &= C_{N,F}\left(\frac{\cancel{v^2}\Lambda_{N-1,F-1}^{3N-F-2}}{\cancel{v^2}\det M_L}\right)^{1/N-F} \\ &\equiv C_{N-1,F-1}\left(\frac{\Lambda_{N-1,F-1}^{3N-F-2}}{\det M_L}\right)^{1/N-F}, \end{aligned} \tag{7.44}$$

where in the last line we've reminded ourselves of the form of the ADS potential with $N-1$ colors and $F-1$ flavors. They take precisely the same form. Coincidence? No, the Higgsed theory is *exactly* the same as the $(N-1, F-1)$ theory at low energies since in this limit the effects of the Higgsed flavors decouples. (This is the lesson of Wilsonian renormalization.) Thus what we've discovered is that

$$C_{N-1,F-1} = C_{N,F}. \tag{7.45}$$

In particular, this means that C only depends on $(N-F)$, i.e. $C_{N,F} = C_{N-F}$. Thus thanks to our $N = F-1$ solution, we now have a set of solutions for $(N-F) = -1$. It turns out there's still one more trick we can play.

The astute reader will wonder how we came to find such a simple relation in (7.45). What ever happened to the usual complications, namely threshold effects? Usually when we integrate out a field, we get some remnant of the matching in the solutions to the RG equations. The matching we've written without any threshold effects implicitly reflects a choice of the $\overline{DR}$ subtraction scheme. In other words, the threshold effects are absorbed into the particular definition of the cutoff scale.

7.5.2 Exploring the moduli space: mass perturbations

The general principle is clear now: how do we can perturb the UV limit of a super QCD and work out the consequences for the low energy theory. In that limit the UV perturbations are negligible effects so that the IR theory characterized by (N', F') is 'really' the (N', F') super QCD theory. We can match the C coefficients of the two theories to obtain a relation between $C_{N,F}$ and $C_{N',F'}$.

The next perturbation we have at our disposal is to give mass m to a flavor without Higgsing the group,

$$\Delta W = m Q_F \overline{Q}_F. \tag{7.46}$$

This allows us to integrate out that flavor in the low energy theory, $(N, F) \to (N, F-1)$. We can go ahead and play our scale matching game (really just effective field theory),

$$\left(\frac{\Lambda}{m}\right)^b = \left(\frac{\Lambda_L}{m}\right)^{b_L} \quad \Rightarrow \quad \left(\frac{\Lambda_{N,F}}{m}\right)^{3N-F} = \left(\frac{\Lambda_{N,F-1}}{m}\right)^{3N-(F-1)}$$

so that we finally obtain

$$\Lambda_{N,F-1}^{3N-F+1} = m \Lambda_{N,F}^{3N-F}. \tag{7.47}$$

Now we would like to solve the equation of motion in the presence of the mass term,

$$W_{ADS} = C_{N,F}\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}} + m M_{FF}. \tag{7.48}$$

I'll skip the magnificent details which are presented in [26], the result is the rule

$$C_{N,F-1} = (N-F+1)\left(\frac{C_{N,F}}{N-F}\right)^{(N-F)/(N-F+1)}. \tag{7.49}$$

7.6 Gaugino condensation

If you are doing everything well, you are not doing enough.
– Howard Georgi, personal motto [5]

Unfortunately there is no time (and it is too far away from the original prompt) to discuss the case $F < N - 1$, in which case we get the phenomenon of gaugino condensation. This is especially a shame since it naturally interfaces with another A-exam question I've received on the Klebanov-Strassler warped throat which is the gravity dual to a 'cascade' of Seiberg dualities leading to a confinement scale with, among other features, gaugino condensation.

8 *Digestif*: Conclusions

We have presented a pedagogical introduction to instantons in quantum mechanics, Yang-Mills theory, quantum chromodynamics, electroweak theory, and supersymmetry. We've been selective about what we present, but we have hopefully demonstrated the significance of instantons in quantum field theory.

There are many topics which I am sorry to not have been able to cover. These include:

- The [possible] role of instantons in confinement in both pure Yang-Mills and supersymmetric Yang-Mills theories (also SQCD with gluino condensation)
- A more thorough presentation of the relevant differential geometry and topology (in particular a bundle-based analysis of the anomaly)
- The role of the cluster decomposition principle [27]
- A discussion of chiral symmetry breaking in duality cascades; this topic segues naturally into a concurrent A-exam paper on Klebanov-Strassler throats in string theory.
- A discussion of the role of the instanton 'bounce' solution for vacuum decay in field theory. This topic is particularly interesting to me for its role in metastable supersymmetry breaking models
- The night before this exam was submitted, a new paper on surprising topological results in supergravity was posted [28]

Given the time constraints that are part of the nature of this examination, I have limited my discussion to the topics suggested in the prompt—though I have erred off course when necessary.

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Most importantly, I would like to thank Liz Craig and Kasi Dean for their friendship and support during this examination and all the (un)related stresses that came with it. They provided food, laughter, and transportation back and forth from the library—without any of which this exam would never have been completed.

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A Notation and Conventions

4D Minkowski indices are written with lower-case Greek letters from the middle of the alphabet, $\mu, \nu, \cdots$. We use the particle physics ('West Coast,' mostly-minus) metric for Minkowski space, $ds^2 = (+,-,-,-)$. As self-respecting grown-ups, we will generally set $c = \hbar = 1$. Occasionally we will descend into infancy and make $\hbar$ explicit, e.g. when we want to emphasize the semiclassical expansion.

We will roughly follow the conventions of [1] and [13]. All formulae should be expected to be accurate up to factors of 2, i, g, etc. Signs should only be trusted where they are important... and even then they should be taken with a grain of salt.

B Discussion of references

Nearly all of the content in this work came from the references discussed below, though the errors are the fault of the author.

The canonical introduction to instantons are Coleman's lectures [1]. It is less-well known that the NSVZ authors also have an excellent—albeit somewhat dated—review that was written to be complementary to Coleman's lectures [2]. Both of these served as my primary references for general instanton queries. More recently, Rajaraman's textbook provides a nice technical treatment with good discussion [29].

General quantum field theory textbooks with particularly helpful discussions include those by Cheng and Li [8], Ryder [7], and Weinberg (volume II) [6]. The supersymmetry (and general beyond the Standard Model) texts by Terning [26] and Dine [30] also had particularly well-written concise discussions of anomalies and instantons with an eye for modern applications.

A nice gradual introduction to instantons starting with quantum mechanics and building up to QCD can be found in Forkel [31]. Gerard 't Hooft's lectures at the 1999 Saalburg school provide an idiosyncratic (thus useful) overview of monopoles and solitons. These lectures were typeset by Falk Bruckmann, who also has since written his own review of topological objects in QCD [4]. There are two rather encyclopedic reviews that were helpful references: Schäfter and Shuryak [32] and Vandoren and Nieuwehuizen [9]. The latter are the most modern set of review material on this topic. Both provide excellent insights, though the author finds that instanton neophytes are better served using these as references while following a more pedagogical texts.

The author's original motivation to learn more about instantons was a personal interest in differential geometry. (Part of this came from a desire to better understand the geometric nature of superspace and to clarify issues about the curved space Dirac operator.) Particularly well-written works that constitute the author's bed-time, bus-time, and bored-time reading include Göckeler and Schucker [33] and the lectures by Collinucci and Wijns [1]. In addition the usual complement of mathematical physics references have been helpful: Frankel [34], Nakahara [35], and Eguchi-Gilkey-Hanson [36]. More detailed mathematical treatments on particular topics can be found in Bilal [37], Harvey [38], Alvarez-Gaume [39], Zinn-Justin [40], and Stora [41].

Papers that have been particularly helpful for the U(1) problem include Creutz' recent review [42], 't Hoofts papers [18, 19], and Weinberg's review [15].

I have benefitted from courses on related subjects from Michael Peskin [43], Paul Sutcliffe, and Csaba Csáki. I thank them all for lending their wisdom and insight.

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