Direct calculation of 3 body phase space

$$d\Phi_3 = \prod_{i=1}^{3} \frac{d^3 p_i}{2E_i} (2\pi)^4 \delta^{(4)}\left(P - \sum_{i=1}^{3} p_i\right)$$

$$= \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2} \frac{1}{2E_3} (2\pi)\, \delta\left(E - \sum E_i\right)\Big|_{\sum p_i = 0}$$

NOW USE ISOTROPY TO ALIGN p_2 ALONG $\hat{z}$ AXIS & AVERAGE OVER OVERALL ROTATIONS.

$$d^3 p_2 = \frac{1}{(2\pi)^3} p_2^2\, dp_2\, d\cos\theta_2\, d\phi_2$$

$$= \frac{p_2^2}{2\pi^2}\, dp_2 \underbrace{\frac{d\cos\theta_2}{2} \frac{d\phi_2}{2\pi}}_{\text{AVERAGES TO 1}}$$

$$= \frac{d^3 p_1}{2E_1} \frac{p_2^2}{2\pi^2} dp_2 \frac{1}{2E_2} \frac{1}{2E_3} (2\pi)\, \delta(E - \sum E_i)\Big|_{\sum p_i = 0}$$

NOW DO p_1 ANGULAR INTEGRAL W/RT p_2 AXIS

$$= \frac{1}{(2\pi)^3} p_1^2\, dp_1\, d\cos\theta\, d\phi \frac{1}{2E_1}$$
$$\times \frac{1}{2\pi^2} p_2^2\, dp_2 \times \frac{1}{2E_2} \frac{1}{2E_3} (2\pi)\, \delta(E - \sum E_i)$$

NOW SPECIALIZE TO THE CASE OF DEGENERATE MASSES

$$E_i^2 = p_i^2 + m^2$$

$$E_3^2 = (p_1 + p_2)^2 + m^2$$
$$= p_1^2 + p_2^2 + 2 p_1 p_2 \cos\theta + m^2$$

$$\delta(E - \Sigma E_i) = \delta(E - E_1 - E_2 - (P_1^2 + P_2^2 + 2P_1P_2\cos\theta + m^2)^{1/2})$$

recall that $\delta(g(x)) = |g'(x_i)|^{-1}\,\delta(x - x_i)$
where x_i are zeroes of g

$$= \left| -\frac{1}{2}\frac{2P_1P_2}{\sqrt{\cdots}} \right|^{-1} \delta(\cos\theta - \cos\bar{\theta})$$

angle s.t. E conserved for a given P_1 & P_2

$$= \frac{E_3}{P_1P_2}\Big|_{\theta=\bar{\theta}}$$

assuming such a $\bar{\theta}$ exists
→ we'll get to that

$$d\Phi_3 = \frac{1}{(2\pi)^3} P_1^2\, dP_1\, d\phi\, \frac{1}{2E_1} \times \frac{1}{2\pi^2} P_2^2\, dP_2\, \frac{1}{2E_2}$$

$$\cancel{\frac{d\cos\theta}{2E_3}}\,(2\pi)\cdot\frac{\cancel{E_3}}{P_1P_2}\Big|_{\theta=\bar{\theta}}$$

$$= \frac{1}{(2\pi)^4}\,\frac{P_1\,dP_1}{2E_1}\,\frac{P_2\,dP_2}{2E_2}\,\underbrace{d\phi}_{\to 2\pi}$$

$$\boxed{d\Phi_3 = \frac{1}{(2\pi)^3}\,\frac{P_1\,dP_1}{2E_1}\,\frac{P_2\,dP_2}{2E_2}\,\Big|_{\exists\bar{\theta}\text{ s.t. } -E+\Sigma E_i=0}}$$

Pretty simple, <u>except</u> we now have to determine the correct region of integration.

GIVEN $\sqrt{s} = E$, WANT $\exists \bar{\theta}$ s.t.

$$E = E_1 + E_2 + \sqrt{p_1^2 + p_2^2 + 2p_1p_2\cos\bar{\theta} + m^2}$$

$$E_i = \sqrt{p_i^2 + m^2}$$

this limits the support of $dp_1 dp_2$

$$(E - E_1 - E_2)^2 = p_1^2 + p_2^2 + 2p_1p_2\cos\theta + m^2$$

$$\boxed{\left|\frac{(E - E_1 - E_2)^2 - (p_1^2 + p_2^2) - m^2}{2p_1p_2}\right| < \underset{|\cos\theta|}{1}}$$

in the massless limit

$$\frac{(E - (p_1 + p_2))^2 - (p_1^2 + p_2^2)}{2p_1p_2} = \cos\theta$$

Some useful variables (TO MATCH TO DALITZ of KINEMATICS REVIEW IN PDG)

$$p_{ij} = p_i + p_j \qquad m_{ij}^2 = p_{ij}^2$$

$$m_{12}^2 + m_{23}^2 + m_{13}^2 = E^2 + m_1^2 + m_2^2 + m_3^2$$

$$E^2 = s = (\textstyle\sum_i p_i)^2$$

then: $m_{12}^2 = (P - p_3)^2 = E^2 + m_3^2 - 2\underbrace{EE_3}_{P \cdot p_3}$

s.t.: $dm_{12} = \frac{-E}{m_{12}} dE_3$ or: $dE_3 = \frac{-1}{2E} dm_{12}^2$

& $E_3 dE_3 = p_3 dp_3$

ok—see PDG if you want to do Dalitz analysis

$$\sqrt{s} = E_1 + E_2 + (P_1^2 + P_2^2 + 2P_1P_2\cos\theta + M^2)^{1/2}$$

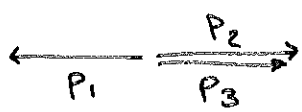
So I WANT $\begin{array}{l} RHS^{max} \geq \sqrt{s} \\ RHS^{min} \leq \sqrt{s} \end{array}$ w/rt $\cos\theta$

THE DOMAIN OF $(P_1, P_2) = (|\vec{P}_1|, |\vec{P}_2|)$ is

① $\sqrt{P_1^2 + M^2} + \sqrt{P_2^2 + M^2} + \sqrt{(P_1+P_2)^2 + M^2} \geq \sqrt{s}$

② ——————— " ——————— $+ \sqrt{(P_1 - P_2)^2 + M^2} \leq \sqrt{s}$

NOW : WHAT CONFIGURATION <u>MAXIMIZES P_1</u> ?



By symmetry, $P_3 = P_2$

$$\Rightarrow \boxed{P_1 = 2P_2 = 2P_3}$$

then E conservation :

$$\sqrt{P_1^2 + M^2} + \underbrace{2\sqrt{\frac{P_1^2}{4} + M^2}}_{\sqrt{P_1^2 + 4M^2}} = \sqrt{s}$$

this is E_1^{max}

$$P_1^2 + 4M^2 = s + P_1^2 + M^2 - 2\sqrt{s}\,E_1^{max}$$

$$\boxed{E_1^{max} = \frac{\sqrt{s}}{2} - \frac{3M^2}{2\sqrt{s}}}$$

NOW THAT WE HAVE AN UPPER LIMIT ON $E_1(P_1)$, WE CONSIDER BOUNDS ON P_2 FOR A VALID P_1.

LET $\mathcal{E}(P_1, P_2, \cos\theta) = E_1 + E_2 + E_3(P_1, P_2, \cos\theta)$

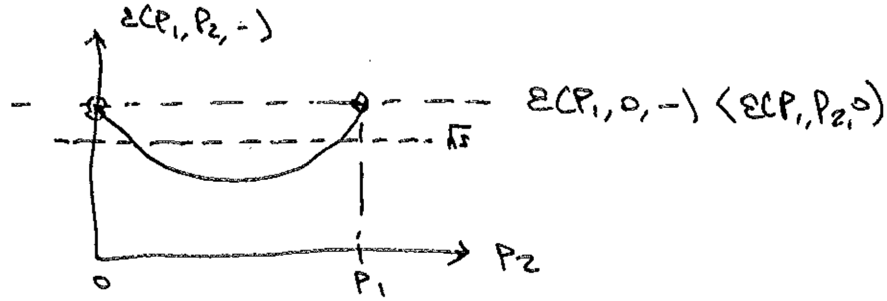
A. IF $\mathcal{E}(P_1, 0, -) > \sqrt{s}$

then condition ① is automatically satisfied

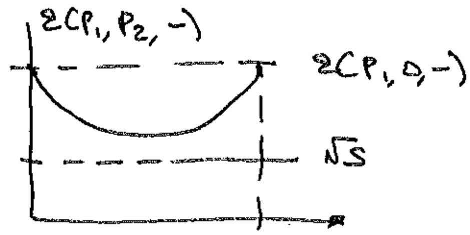
BC $\mathcal{E}(P_1, 0, -) < \mathcal{E}(P_1, P_2, +)$

SO $\mathcal{E}(P_1, 0, -) > \sqrt{S} \Rightarrow \mathcal{E}(P_1, P_2, +) > \sqrt{S}$

HEURISTICALLY:



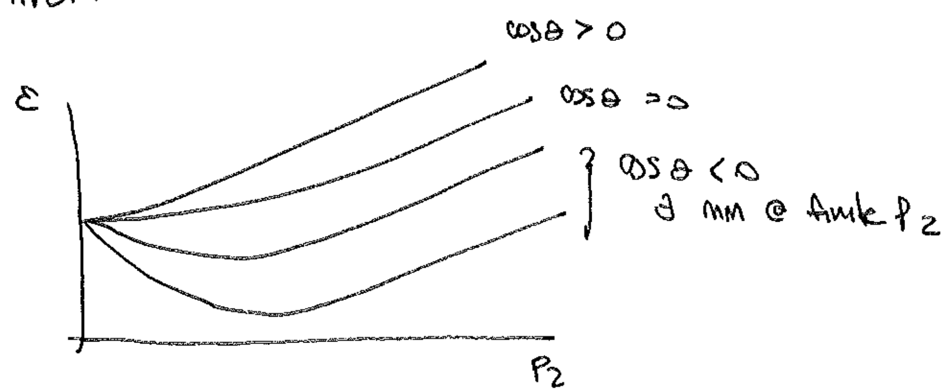
WHY NOT :



well! $\mathcal{E}(P_1, 0, -) > \sqrt{S}$ IS A STATEMENT ABOUT THE LARGENESS OF P_1 ASSUMING $P_1 < P_1^{MAX}$, THEN THIS IS A REGIME WHERE P_1 IS BIG, BUT NOT TOO BIG THAT THERE'S NO POSSIBLE SOLUTION.

THE "WHY NOT" SCENARIO IS WHAT HAPPENS WHEN P_1 IS TOO BIG TO HAVE ANY SOLUTION TO $\mathcal{E}(P_1, P_2, \cos\theta) = \sqrt{S}$. So BY ASSUMPTION THAT $P_1 < P_1^{MAX}$, THIS CANNOT HAPPEN.

IN FACT:
FIXED P_1



THEN: we want the range of P_2 given a value of P_1.

"BIG" VALUE OF P_1 THAT'S NOT "TOO BIG"

(still under the assumption $\mathcal{E}(P_1, 0, \pm) > \sqrt{s}$)

the ~~minimum~~ maximum of P_2 corresponds to $\cos\theta = -$

$$\sqrt{P_1^2+m^2} + \sqrt{P_2^2+m^2} + \sqrt{P_1^2+P_2^2+2P_1P_2\cos\theta+m^2} \stackrel{?}{=} \sqrt{s}$$

so more negative $\cos\theta \to$ smaller LHS
$\to$ compensate w/ larger P_2

s.t. ~~minimum~~ most neg val of $\cos\theta$ $(=-1)$
corresponds to maximum of P_2

so: $\boxed{\mathcal{E}(P_1, P_2^{\max}, -) = \sqrt{s}}$ (*)

~~the maxim~~ what is the minimum P_2?

if $\mathcal{E}(P_1, 0, \pm) \equiv \sqrt{s}$, then $P_2^{\min} = 0$

when $P_2 = 0$, $\cos\theta$ irrelevant

otherwise, for $\mathcal{E}(P_1, \overset{0}{\cancel{P_2}}, \cos\theta) > \sqrt{s}$
NEED SOME FINITE (MINIMUM) VALUE OF P_2
TO REDUCE $\mathcal{E}$.

if $\cos\theta > 0$, then increasing P_2 from 0 makes $\mathcal{E}$ bigger, not smaller
$\Rightarrow \cos\theta < 0$

$\hookrightarrow$ if $\cos\theta = -(\text{small})$, then P_2 HAS TO BE BIGGER (BUT $< P_1$) TO REDUCE $\mathcal{E}$ MORE. SO THIS IS NOT A MINIMUM OF P_2.

$\Rightarrow$ SO MIN OF P_2 COMES FROM $\cos\theta = -1$

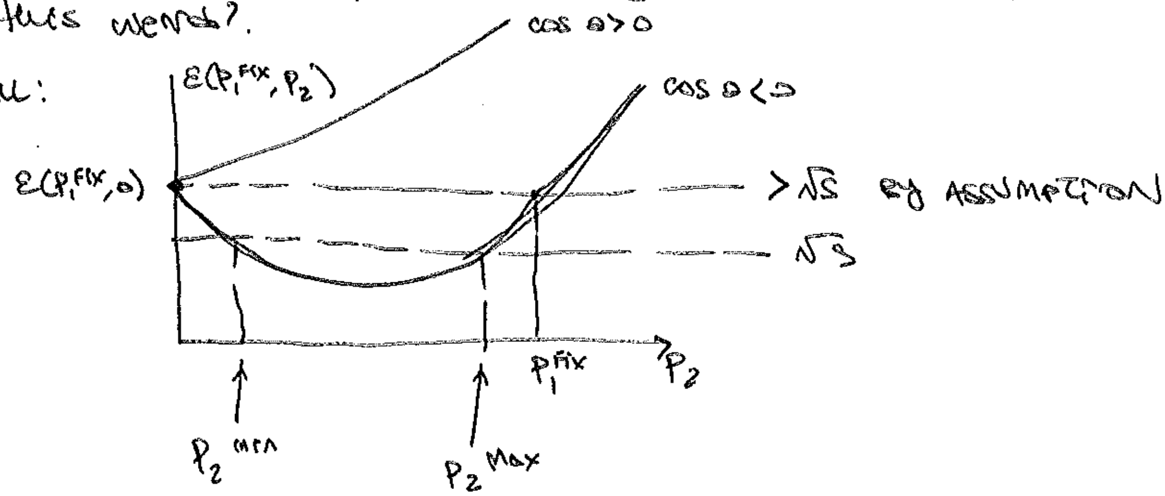
BUT THIS IS JUST THE SAME AS (*)!

$\boxed{\mathcal{E}(P_1, P_2^{\min}, -) = \sqrt{s}}$

$\Rightarrow$ P_2 IS GIVEN BY $\mathcal{E}(P_1, P_2, -) = \sqrt{s}$

So the max & min P_2 is set by the same condition. Is this weird?

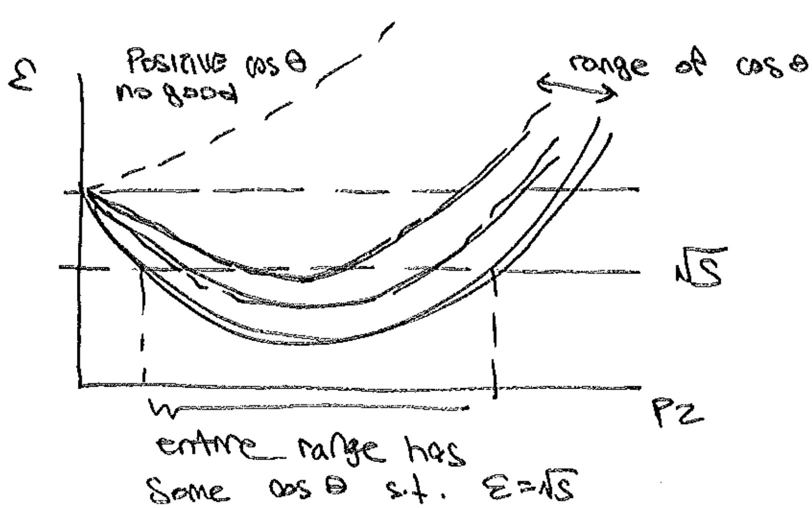
RECALL:



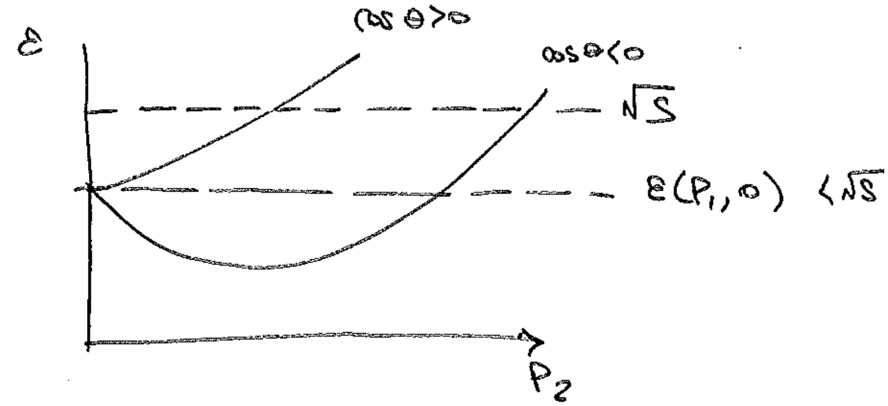
$P_2^{\,MAX}$: P_2 BIG ENOUGH S.T. P_2^2 GROWS BIGGER THAN $+P_1P_2\cos\theta < 0$ TERM. SO INCREASING P_2 <u>INCREASES</u> $\mathcal{E}$.

$P_2^{\,MIN}$: P_2 SMALL ENOUGH S.T. $P_1P_2\cos\theta < 0$ IS BIGGER THAN $P_2^{\,2}$ & INCREASING P_2 GIVES A BIGGER <u>NEGATIVE</u> CONTRIBUTION, <u>DECREASING</u> $\mathcal{E}$.

@ any rate, $\exists$ 2 solutions & a <u>range</u> of P_2 values for which $\exists$ $\cos\theta$ to satisfy $\mathcal{E} = \sqrt{s}$



B **OTHER CASE** : $\underline{\mathcal{E}(P_1, 0, \cos\theta) < \sqrt{S}}$



WHAT IS THE <u>MINIMUM</u> OF P_2?

from the plot above: this corresponds to $\cos\theta = 1$

$P_2 = 0$ GIVES TOO SMALL $\mathcal{E}$, $\cos\theta = +1$ MAXIMIZES THE $\Delta\mathcal{E}$ FOR EACH ΔP_2.

$$\boxed{\mathcal{E}(P_1, P_2^{MIN}, +) = \sqrt{S}}$$

WHAT IS THE <u>MAXIMUM</u> P_2? $\longrightarrow$ $\cos\theta = -1$

in this regime, $\cos\theta = -1$ decreases $\mathcal{E}$ as much as possible as we take the $P_2 \to$ BIG limit.

$$\boxed{\mathcal{E}(P_1, P_2^{MAX}, -) = \sqrt{S}}$$

SO WE HAVE 2 EQNS TO SOLVE : $\mathcal{E}(P_1, P_2, \pm) = \sqrt{s}$

↳ write this as

$$(P_1 - P_2)^2 + m^2 = (\underbrace{\sqrt{s} - E_1}_{} - \sqrt{P_2^2 + m^2})^2$$

$$= (a - \sqrt{P_2^2 + m^2})^2$$

$$P_1^2 + \cancel{P_2^2} - 2P_1P_2 + \cancel{m^2} = a^2 - 2a\sqrt{P_2^2 + m^2} + \cancel{P_2^2} + \cancel{m^2}$$

$$2a\sqrt{P_2^2 + m^2} = -P_1^2 + 2P_1P_2 + a^2$$

$$4a^2(P_2^2 + m^2) = (\underbrace{a^2 - P_1^2}_{} + 2P_1P_2)^2$$

$$4a^2P_2^2 + 4a^2m^2 = (b^2 + 2P_1P_2)^2$$

$$= b^4 + 4b^2P_1P_2 + 4P_1^2P_2^2$$

$$\underbrace{4(a^2 - P_1^2)}_{A}P_2^2 \underbrace{- 4b^2}_{B}P_1P_2 + \underbrace{4a^2m^2 - b^4}_{C} = 0$$

$$P_2 = \frac{1}{2A}\left(-B \pm \sqrt{B^2 - 4AC}\right)$$

$$A = 4(a^2 - P_1^2) = 4\left[(\sqrt{s} - E_1)^2 - P_1^2\right]$$

$$B = -4b^2P_1 = -4\left((\sqrt{s} - E_1)^2 - P_1^2\right)P_1 = -AP_1$$

$$C = 4a^2m^2 - b^4 = 4(\sqrt{s} - E_1)^2m^2 - \left((\sqrt{s} - E_1)^2 - P_1^2\right)^2$$

$$P_2 = \frac{P_1}{2}\left(1 \mp \sqrt{1 - \frac{4AC}{B^2}}\right)$$

$$-\frac{4AC}{B^2} = -\frac{4C}{AP_1^2}$$

cont'd :

$$P_2 = \frac{P_1}{2}\left(1 \mp \sqrt{1 - \frac{4C}{AP_1^2}}\right)$$

$$C = 4a^2m^2 - \left(\tfrac{1}{4}A\right)^2$$

$$\frac{4C}{AP_1^2} = \frac{16a^2m^2}{AP_1^2} - \frac{A}{4P_1^2}$$

$$= \frac{4a^2m^2}{(a^2-P_1^2)P_1^2} - \frac{a^2-P_1^2}{P_1^2}$$

$$= \frac{4a^2m^2}{(a^2-P_1^2)P_1^2} - \frac{a^2(a^2-P_1^2)}{(a^2-P_1^2)P_1^2} + 1$$

$$= \frac{a^2(4m^2-a^2+P_1^2)}{(a^2-P_1^2)P_1^2} + 1$$

$$P_2 = \frac{P_1}{2}\left(1 \mp \sqrt{\frac{-a^2(4m^2-a^2+P_1^2)}{(a^2-P_1^2)P_1^2}}\right)$$

$$\mp \frac{a}{P_1}\sqrt{\frac{-4m^2+(a^2-P_1^2)}{(a^2-P_1^2)}}$$

$$\boxed{P_2 = \frac{P_1}{2}\left(1 \mp \frac{a}{P_1}\sqrt{1 - \frac{4m^2}{a^2-P_1^2}}\right)}$$

matches Arunt. ✓

now do $\mathcal{E}(p_1, p_2, +) = \sqrt{s}$
following the minus sign :

$$\underbrace{4(a^2 - p_1^2)}_{A} p_2^2 + \underbrace{4b^2 p_1}_{B' = -B} p_2 + \underbrace{4a^2 m^2 - b^4}_{C}$$

$= +Ap_1$

then : $P_2 = \frac{-p_1}{2}\left(1 \pm \frac{a}{p_1}\sqrt{1 - \frac{4m^2}{a^2 - p_1^2}}\right)$

matches Arvind. But: must pick sign such that $p_2 > 0$.

$\hookrightarrow$ so $\mp$ sign.

$$\boxed{P_2 = \frac{-p_1}{2}\left(1 - \frac{a}{p_1}\sqrt{1 - \frac{4m^2}{a^2 - p_1^2}}\right)}$$