Boxy Spectra Kinematics

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This is just a quick exercise in relativistic kinematics. Suppose dark matter annihilates through an s-channel mediator, φ. We could look for the spectra of photons coming from this decay. Suppose that either φ directly decays into photons or that we know the spectrum $\mathrm{dN}_\gamma/\mathrm{dE}_\gamma$ of secondary photons in the φ rest frame.

Simple case: φ goes directly to photons

In the φ rest frame, each final state photon has four-momenta $p_i = (|\boldsymbol{p}_i|, \boldsymbol{p}_i)$. Now suppose that in the lab frame, φ is boosted. Choose our coordinate system such that the boost is in the x direction. The boost matrix is:

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

The value of β and $\gamma = (1-\beta^2)^{-1/2}$ is fixed by the masses of the φ and its parent. Suppose $\chi\chi \to \varphi\varphi$ with the φ on-shell and the χ essentially at rest. Then the φ has four-momentum $(m_\chi, \boldsymbol{q})$ with $m_\varphi^2 = m_\chi^2 - |\boldsymbol{q}|^2$. This gives

$$\begin{aligned} \gamma &= m_\chi/m_\varphi \\ \beta &= \sqrt{1-\gamma^{-2}} \\ &= \sqrt{1-\frac{m_\varphi^2}{m_\chi^2}}. \end{aligned}$$

Boosting the φ decay products gives $p_i \to p_i'$,

$$p_i' = \begin{pmatrix} \gamma|\boldsymbol{p}_i| - \gamma\beta p_i^x \\ -\gamma\beta|\boldsymbol{p}_i| + \gamma p_i^x \\ p^y \\ p^z \end{pmatrix}.$$

In spherical coordinates with the poles aligned along the x-axis, $p^x = |\boldsymbol{p}_i|\cos\theta$, so that the energy is

$$E_i' = \gamma|\boldsymbol{p}_i|(1-\beta\cos\theta).$$

By isotropy of the annihilations, θ can take any value. There is thus a box-like spectrum. For $\beta = 1$ this is a range of energies from zero to $2\gamma|\boldsymbol{p}_i|$. For $\beta \ll 1$ this is still a sharp peak at $|\boldsymbol{p}_i|$.

Don't over-think this: you might think that we need to weight each value of θ by $\sin\theta$ for polar coordinates since there's more volume for $|\theta| \approx \pi$ than $\theta \approx 0$. This is true relative to a specific axis, but we are implicitly averaging over the boost directions due to the isotropy of $\chi\chi$ collisions, so this difference washes out.

Normalizing integrated functions

Now we get to the business of normalization. Write the photon distribution as

$$\frac{\mathrm{dN}_\gamma}{\mathrm{dE}_\gamma} = f(E_b, E_\gamma) \;\; \equiv \;\; g(E_b) h(E_\gamma).$$

For monochromatic bs, e.g. in $\chi\chi \to bb$, the b spectrum is

$$g(E_b) \;\; = \;\; \delta\Big(E_b - \frac{m_\chi}{2}\Big)\,\mathrm{dE}_b.$$

For the case of $\chi\chi \to \varphi\varphi$, with $\varphi \to bb$ on shell, we have a box-like spectrum where there is equal probability for $E_b \in [\gamma E_0(1-\beta), \gamma E_0(1+\beta)]$, where $E_0 = m_\chi/2$, the rest frame photon energy. Normalizing this gives

$$g(E_b) dE_b \;\; = \;\; \frac{2}{\gamma E_0\, 2\beta}\, dE_b$$

when E_b is in the above specified range, and 0 otherwise. Note the factor of 2 because there are two bs per χ.

Realistic case: $\varphi \to bb$

I was sloppy above. The actual direct decay products of the φ are on-shell bs which in turn shower into photons. The boost is the same, but now instead of boosting a light-like momentum, one has to boost $(E_b, \boldsymbol{p}_b)$ with $E_b = m_\varphi/2$ and $|\boldsymbol{p}_b|^2 = E_b^2 - m_b^2$ so that the b energy in the φ frame is

$$E_b' \;\; = \;\; \gamma E_b - \gamma\beta (E_b^2 - m_b^2)^{1/2} \cos\theta$$

So that the spectrum of E_b' ranges from

$$\frac{\gamma m_\varphi}{2}\left(1 - \beta\sqrt{1 - \left(\frac{2m_b}{m_\varphi}\right)^2}\right) \;\; - \;\; \frac{\gamma m_\varphi}{2}\left(1 + \beta\sqrt{1 - \left(\frac{2m_b}{m_\varphi}\right)^2}\right).$$

The normalization is

$$g(E_b') dE_b' \;\; = \;\; \frac{2}{\gamma E_0\, 2\,\beta\, \sqrt{1 - (2m_b/m_\varphi)^2}}$$

where the factor of 2 again comes from the 'two bs per χ' observation. As a reminder,

$$\begin{aligned} E_0 &= \frac{m_\varphi}{2} \\ \gamma &= \frac{m_\chi}{m_\varphi} \\ \beta &= \sqrt{1 - \gamma^{-2}}. \end{aligned}$$