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ELECTRICITY AND MAGNETISM QUALIFYING EXAM

Fall 2011

You may consult only one book: Jackson's 3rd Edition. Do all six (6) problems. The exam is worth 100 points. In this exam we follow the unit conventions of Jackson's 3rd Edition. That is, for non-relativistic problems (Numbers 1 through 5), we use SI units, and for the relativistic problem (Number 6), we use cgs units.

Note that different problems are worth different amounts; you should budget your time accordingly. Write directly on this exam; do not use a blue book. Every other page of the exam is blank to provide additional work space. In order to receive credit you must show all of your work.

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Total: ____________________

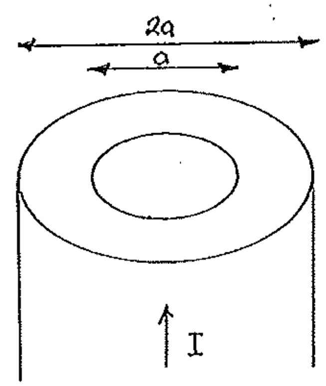
1. [10 points] We use cylindrical coordinates in this problem. A cylindrical shell runs parallel to and is centered on the z-axis, extends to infinity in both directions, and has inner radius a and outer radius $2a$. A current I flows along the $+z$-axis. We assume there is a uniform number density n of charge carriers in the cylinder, each having charge q and velocity v. A magnetic field is present in the cylinder:

$$\mathbf{B}(\rho) = B_0(a/\rho)\,\hat{\phi}$$

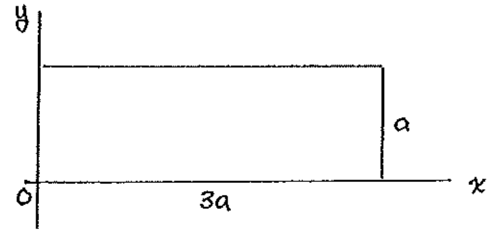
(a) If the charge carriers are positive, in which direction are they deflected by the magnetic field?

(b) Find the potential difference in equilibrium between the outer and inner surfaces of the cylinder in the sense $\Phi_{outer} - \Phi_{inner}$. Assume that the *net* volume charge density in the cylinder is zero everywhere and that the total charge per unit length on the outside surface of the cylinder is equal and opposite to that on the inside surface.

(c) How do your answers to parts (a) and (b) change if the charge carriers are negative?



2. [15 points] A rectangular waveguide runs parallel to the z-axis and has walls defined by $x = 0$, $x = 3a$, $y = 0$, and $y = a$. The walls are perfect conductors. A TM$_{21}$ (transverse magnetic, $m = 2, n = 1$) wave of angular frequency ω moves through the guide in the $+z$-direction. The first index refers to x and the second index refers to y. For each of the following field components, state whether or not the field component vanishes everywhere and at all times. If it does not vanish everywhere, find the planes of constant x and/or constant y for which it vanishes at all times. These planes should be identified by their constant x or y values. Note that the planes in question may or may not coincide with the walls of the guide. [We exclude H_z from consideration because it is identically zero by definition.]



(a) E_x

(b) E_y

(c) E_z

(d) H_x

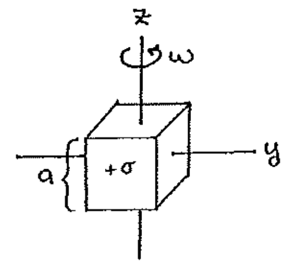
(e) H_y

3. [20 points] A cube of side length a is centered on the origin and rotates at constant velocity ω about the z-axis. Two of its faces always remain perpendicular to the z-axis. One of the two faces that is perpendicular to the x-axis at time $t = 0$ is covered with a uniform charge density $+\sigma$. The opposite face is covered with a uniform change density $-\sigma$. Except for these two faces there are no other charges in the problem. By the symmetry of the problem, it is sufficient to consider field points in the $y = 0$ plane for time-averaged quantities.

(a) Restricting your attention to distant points in the $y = 0$ plane, find the direction or directions in which $<dP/d\Omega>$ has its minimum value. Specify the direction(s) in terms of the usual polar angle θ. What is this minimum value of $<dP/d\Omega>$?

(b) Find the time-averaged power per unit solid angle $< dP/d\Omega >$ received at a far away point in the direction given by the unit vector $\hat{n} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$.

(c) Consider a far away point in the direction given by the unit vector $\hat{n} = (1, 0, 0)$. At this point is the direction of polarization of the observed radiation fixed or does it rotate about the line of sight? If it is fixed, what is the observed direction of polarization? If it rotates, does it rotate clockwise or counter-clockwise about the line of sight?



4. [15 points] A cylindrical shell runs parallel to and is centered on the z-axis, extends to infinity in both directions, and has inner radius a and outer radius $2a$. The shell has a permanent non-uniform magnetization (magnetic dipole moment per unit volume) given by:

$$\mathbf{M}(\rho) = \frac{M_0 \rho}{a}\,\hat{\phi}$$

where ρ is the usual radial cylindrical coordinate (not a volume charge density), $\hat{\phi}$ is a unit vector in the ϕ direction, and M_0 is a constant. There are no free currents anywhere in the problem; there may be implied bound volume and bound surface current densities J_b and K_b, but these symbols should not appear in any of your final answers. Use cylindrical coordinates.

(a) Find the magnitude and direction of the magnetic field $\mathbf{B}$ for $0 < \rho < a$.

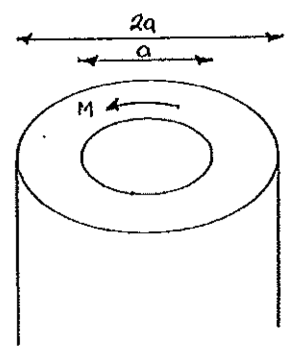
(b) Find the magnitude and direction of the magnetic field $\mathbf{B}$ for $a < \rho < 2a$.

(c) Find the magnitude and direction of the magnetic field $\mathbf{B}$ for $\rho > 2a$.

(d) Find the magnitude and direction of the field $\mathbf{H}$ for $\rho < a$.

(e) Find the magnitude and direction of the field $\mathbf{H}$ for $a < \rho < 2a$.

(f) Find the magnitude and direction of the field $\mathbf{H}$ for $\rho > 2a$.



5. [20 points] We use spherical coordinates (r, θ, ϕ) throughout this problem. A sphere of radius a is centered on the origin. The potential on the surface of the sphere is maintained as follows:

$$\begin{aligned} \Phi &= 0 & \quad 0 \le \theta \le \tfrac{1}{4}\pi \\ \Phi &= V & \quad \tfrac{1}{4}\pi \le \theta \le \tfrac{1}{2}\pi \\ \Phi &= 0 & \quad \tfrac{1}{2}\pi \le \theta \le \pi \end{aligned}$$

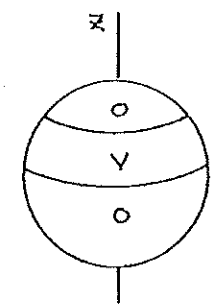
where V is a positive constant.

(a) For $r \gg a$ find the first two non-vanishing terms in an expansion of the potential in terms of a/r for a point with $r > a$ and arbitrary θ.

(b) Find the magnitude and direction of the electric field at the origin.

(c) Find the net charge on the sphere.

(d) Find the dipole moment of the sphere.

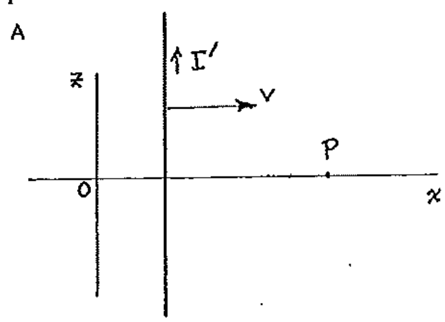


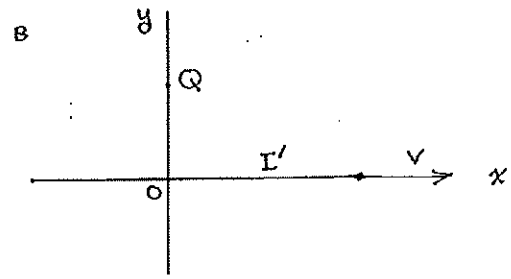
6. [20 points] Consider an infinitely long straight wire. In its rest frame, the wire carries a current I' and no net charge density.

(a) Suppose the wire moves with velocity $v = \frac{5}{13}c$ through the laboratory such that it crosses the origin at laboratory time $t = 0$. The wire remains parallel to the z-axis at all times and the current flows in the $+z$-direction. Consider an observation point $P = (x, 0, 0)$ in the lab frame at a laboratory time t such that the wire is to the left of P. Find *all* components of the electric and magnetic fields measured in the laboratory in terms of I', x, t, and fundamental constants.

(b) Suppose that the wire is now coincident with the x-axis and the current (still I' in the rest frame of the wire) now flows in the $+x$-direction. The velocity of the wire is still $v = \frac{5}{13}c$ in the $+x$-direction. Find *all* components of the electric and magnetic fields measured in the lab frame at the point $Q = (0, y, 0)$ in terms of I', y, and fundamental constants. All components should now be independent of time. Assume $y > 0$.

(c) In Part (b), the fields are purely magnetic in the frame of the wire, and mixed (magnetic plus electric) in the lab frame. Find a rocket frame for Part (b) in which the fields are purely electric or prove that no such frame exists.





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QUALIFIER EXAM
Classical Mechanics
Fall 2011

This is an open book exam. You may refer to *Mechanics* by Landau and Lifshitz. Write directly on this examination and not in a Blue Book. Ever other page is left blank to provide sufficient work space. To receive credit you must show all of your work. Make an effort to present clear and organized solutions. Little or no credit will be given for obscure mathematics.

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Classical Mechanics PhD Qualifying Exam, Fall 2011

1. (15 points) The mass shown in the Figure is resting on a frictionless horizontal table. Each spring has a force constant k and unstretched length l_0, and the length of each string when the mass is at its equilibrium at the origin is a (not necessarily the same as l_0).

   (a) Write down Lagrangian for this system and the expression for potential energy for small fluctuations from the origin as a function of x and y coordinates.

   (b) Show that if $a < l_0$ the equilibrium position is unstable. Explain why and find stable equilibrium position(s).

   (c) Find oscillation frequencies for small fluctuations around stable equilibrium position when $a > l_0$.

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2. (15 points) Consider a particle moving in a potential

$$U(r) = \frac{\beta}{r^2} - \frac{\gamma}{r^4},$$

where $\beta > 0$ and $\gamma > 0$.

(a) Find the minimal value of E for which the particle can fall to the center of the potential.

(b) Calculate the cross-section for the particle with energy E to fall to the center of the potential.

(c) Find the trajectory of the particle approaching the potential from infinity as a function of its initial energy E and impact parameter ρ. (Be careful to distinguish cases where particle may or may not fall to the center.)

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3. (8 points) A dumbbell is made of two identical balls attached to each other. While the dumbbell is at rest it is hit by a third identical ball whose velocity $\overrightarrow{V}$ is perpendicular to the axis of symmetry of the dumbbell and is directed towards the center of one of the balls. The collision is perfectly elastic. Find velocities of the dumbell and the third ball after the collision.

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4. (12 points) Two equal masses, $m_1 = m_2 = m$, are joined by a massless string of length L that passes through a hole in a frictionless horizontal table. The first mass slides on the table while the second hangs below the table and moves up and down in a vertical line.

   (a) Assuming the string remains taut, write down the Lagrangian and Euler-Lagrange equations for this system.

   (b) Write down expressions for all the conserved quantities.

   (c) Determine the trajectory of the first particle. Leave the final integral unevaluated.

   (d) If the first particle moves in a circular path, what is the radius r_0 of that circle?

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QUALIFIER EXAM
Quantum Mechanics
Fall 2011

This is an open book exam. You may refer to Lectures on Quantum Mechanics by Baym & Quantum Mechanicsby Sakurai. There are blank pages available to provide sufficient work space. To receive credit you must show all of your work. Make an effort to present clear and organized solutions. Little or no credit will be given for obscure mathematics.

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Quantum Mechanics Qualifying Exam—Fall 2011

1. *[20 points]* An electron with mass m is constrained to move along a ring of radius R as shown in Fig. 1. A magnetic flux Φ passes through a cylinder (represented by the shaded region in the figure) located at the center of this ring. Outside of the cylinder, the vector potential corresponding to this flux can be written as

$$\mathbf{A}(r,\theta) = \frac{\Phi}{2\pi r}\hat{\theta}. \tag{1}$$

When $\Phi = 0$ the Hamiltonian for the electron (written in the position basis) is

$$H_0 = -\frac{\hbar^2}{2mR^2}\frac{d^2}{d\theta^2}. \tag{2}$$

Construct the Hamiltonian appropriate for $\Phi \neq 0$, and diagonalize it to obtain the energies and corresponding wavefunctions in the presence of the flux. The set of energies you find should be periodic in Φ. What is the period?

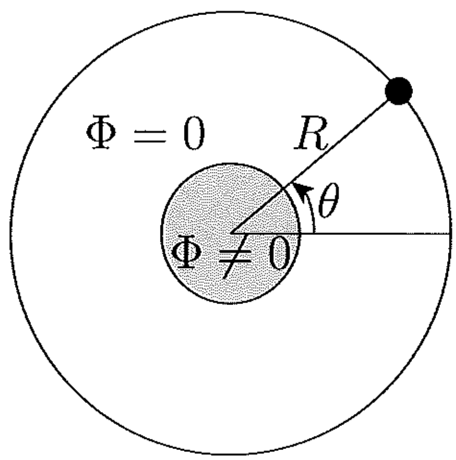

FIG. 1: An electron moves along a ring of radius R, while a magnetic flux Φ passes through the shaded region in the center.

2. An electron in a one-dimensional harmonic oscillator potential is subjected to a uniform electric field $\mathcal{E}$. The Hamiltonian for the problem reads

$$H_0 = \frac{P^2}{2m} + \frac{m\omega^2}{2}X^2 - e\mathcal{E}X. \tag{3}$$

(a) *[15 points]* Suppose that when $\mathcal{E} = 0$ the eigenvalues of H_0 are E_n and the corresponding wavefunctions in the position basis are $\psi_n(x)$. Find the eigenvalues and wavefunctions when $\mathcal{E} \neq 0$ in terms of E_n and $\psi_n(x)$.

(b) *[15 points]* Let's now add an anharmonic potential so that the Hamiltonian becomes

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2}X^2 - e\mathcal{E}X + \delta H \tag{4}$$

$$\delta H = \lambda X^3. \tag{5}$$

Treat δH using perturbation theory, and find the correction to the energies to first order in λ. Feel free to use standard results for the harmonic oscillator without rederiving them. Also, you can express your answer in terms of an expectation value of X^2 for an ordinary quantum harmonic oscillator with $\mathcal{E} = \lambda = 0$.

3. A spin-1/2 particle subjected to a magnetic field directed along the x-axis is described by the Hamiltonian

$$H = -bS^x, \tag{6}$$

where $\mathbf{S} = S^x\hat{\mathbf{x}} + S^y\hat{\mathbf{y}} + S^z\hat{\mathbf{z}}$ is a spin-1/2 operator.

(a) *[15 points]* Working in a basis of S^z eigenstates, find the energies and eigenvectors of H.

(b) *[15 points]* Suppose that at $t = 0$ the spin is prepared into the state $|\psi(t=0)\rangle = |\uparrow\rangle$, where $|\uparrow\rangle$ is the 'spin up' S^z eigenstate. Apply the time-evolution operator to this state to find $|\psi(t)\rangle$. Then compute the probability of finding the particle at time t in the state $|\downarrow\rangle$, where $|\downarrow\rangle$ is the 'spin down' S^z eigenstate.

4. **Hint: for both parts, avoid trying to solve the problem by brute force.** (a) *[10 points]* The behavior of an electron moving in three dimensions is governed by the following Hamiltonian:

$$H = \frac{\mathbf{P}^2}{2m} + V(\mathbf{X}) + \lambda[\mathbf{E}(\mathbf{X}) \times \mathbf{P}] \cdot \mathbf{S}, \tag{7}$$

where the potential is $V(\mathbf{X}) = \alpha[1 + \beta\mathbf{X} \cdot \mathbf{X}]^2$, $\mathbf{E}(\mathbf{X})$ is the corresponding electric field felt by the electron, $\mathbf{S}$ is a spin-1/2 operator, and the λ term represents spin-orbit coupling. Two physicists are debating the ground state degeneracy for this problem. One claims that the ground state must be degenerate. The other counters,"I agree with you when $\lambda = 0$, because then spin up and spin down states are clearly degenerate. But a non-zero λ mixes these spin states and produces a unique ground state." Who is right, and more importantly why?

(b) *[10 points]* Two spin-1/2 particles (described by operators $\mathbf{S}_{1,2}$) and a spin-1 particle (with operator $\mathbf{S}_3$) are all coupled together via the Hamiltonian

$$H = aS_1^xS_2^x + bS_2^yS_3^y + cS_3^zS_1^z. \tag{8}$$

At time $t = 0$, this three-spin system is prepared into the ground state of H. Let $\mathbf{J} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3$ be the total spin operator. Find the probability that at time t the spins will be found in a state with total spin quantum number $j = 1/2$ and explain your answer. Don't forget the hint above!

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QUALIFIER EXAM
Statistical Mechanics
Fall 2011

This is an open book exam. You may refer to *Statistical Mechanics* by R.K Pathria. There are blank pages available to provide sufficient work space. To receive credit you must show all of your work. Make an effort to present clear and organized solutions. Little or no credit will be given for obscure mathematics.

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Statistical Mechanics

1. Monatomic classical ideal gas is trapped in a cylindrical container that has radius R and height L. The system spins about the axis of the cylinder and reaches thermal equilibrium at temperature T.
   a. What is the Hamiltonian of an atom (mass m, angular speed ω, distance from axis, r)?
   b. What is the partition function of the system (particle number N)?
   c. What is the average distribution density at radius r?

2. For a system with N identical molecules that have three internal motion energy levels: 0, ε, and 10ε, find the contributions of internal motions to the
   a. average internal energy, $\langle E \rangle$,
   b. specific heat, C_v.

3. A chain has N ($N >> 1$) identical segments with length ℓ and width 0 (zero). Suppose each segment can adopt either along or perpendicular position, the length of the chain is $N\ell x$ with a tension F in the chain. Find
   a. the entropy of the chain as a function of x,
   b. the connection between x and F at temperature T.
   c. Under what condition we can get the Hook's law.

4. Electron gas with N particles is confined in a box of volume V. At T=0, find the
   a. specific heat
   b. Pauli magnetic susceptibility
   c. pressure on the wall
   d. average kinetic energy

5. Find the chemical potentials of ideal gas with N indistinguishable atoms moving
   a. in a box with volume V (as gas, $\varepsilon = \dfrac{p^2}{2m}$),
   b. on a surface with area A (as adsorbates, $\varepsilon = \dfrac{p^2}{2m} + \varepsilon_0$).
   c. What is the equilibrium adsorption coverage ($n = N_{ad}/A$) at temperature T and pressure P (use $PV = N_{gas}kT$).