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Electricity and Magnetism Qualifying Exam
Spring 2012

This is an open book exam. You may refer to Jackson but to no other materials. You may utilize any intermediate result in Jackson that is appropriate. No table of integrals is necessary for this exam.

Do all six (6) problems. The exam is worth 100 points. In this exam we follow the conventions of Jackson's 3rd Edition. That is, for non-relativistic problems (Numbers 1 - 5), we use SI units, and for the relativistic problem (Number 6), we use cgs units. To receive credit you must show all of your work. You may use a calculator. You do not need to determine decimal values for numerical constants, but you should reduce fractions to lowest terms.

DO NOT OPEN THIS EXAM UNTIL YOU ARE TOLD TO DO SO

FOR ADMINISTRATIVE USE ONLY:

1 : ______________________________

2 : ______________________________

3 : ______________________________

4 : ______________________________

5 : ______________________________

6 : ______________________________

Total: ______________________________

1. **[18 points]** A hollow sphere of radius a is centered on the origin. The potential of the surface of the sphere is maintained as follows:

$$
\begin{aligned}
\Phi(a,\theta,\phi) &= V \qquad \text{for} \qquad 0 \le \theta < \tfrac{\pi}{4} \\
\Phi(a,\theta,\phi) &= 0 \qquad \text{for} \qquad \tfrac{\pi}{4} \le \theta < \tfrac{3\pi}{4} \\
\Phi(a,\theta,\phi) &= V \qquad \text{for} \qquad \tfrac{3\pi}{4} \le \theta \le \pi
\end{aligned}
$$

There are no charges inside of the sphere or outside of the sphere. HINT: Some integrals in this problem may be easier to do if you make the substitution $x = \cos\theta$.

(a) For a point inside the sphere with $r \ll a$ find the potential expressed as a power series in r and $\cos\theta$, retaining and explicitly evaluating terms up to and including r^3.

(b) Find the total charge on the sphere.

(c) Find the dipole moment of this charge distribution.

2. **[16 points]** In the quasi-static approximation in electrostatics, we neglect the displacement current term in the Maxwell equations. Use the quasi-static approximation to solve this problem. A hollow infinite cylinder of radius a runs parallel to and is centered on the z-axis and carries a time-dependent surface current density:

$$\mathbf{K} = \frac{K_0\, t}{\tau}\hat{\phi}$$

where $\hat{\phi}$ is a unit vector in the azimuthal direction, t is the time, and K_0 and τ are constants.

(a) Find the magnitude and direction of the magnetic field inside of the cylinder and outside of the cylinder as a function of time.

(b) Find the magnitude and direction of the electric field inside of the cylinder and outside of the cylinder as a function of time.

3. **[16 points]** Give numerical answers for this problem. Take the speed of light to be exactly 3.00×10^8 m/s or 3.00×10^{10} cm/s.

(a) A waveguide with perfectly conducting walls has a rectangular cross-section of dimensions 5.00 cm (width) by 2.00 cm (height). Assume the waveguide has vacuum inside and that a transverse electric mode TE_{mn} is propating down the guide with angular frequency $\omega = 21\pi \times 10^9$ Hz. Find all possible non-trivial transverse electric modes of this frequency that will propagate. Identify the modes by the indices (m, n). We follow the usual convention that the first index refers to the larger dimension.

(b) Repeat Part (a) for the case that the waveguide is filled with a dielectric of permittivity $\epsilon = 2\epsilon_0$.

4. [16 points] A uniform magnetic field of magnitude B_0 points in the $+x$ direction ($\phi = 0$) and fills all of space. Into this field is placed a solid cylinder of radius a and magnetic permeability $\mu = 2\mu_0$, centered on the z-axis and extending to infinity in both directions. There are no free currents. Find the magnetic field components B_ρ and B_ϕ inside of the cylinder.

5. **[16 points]** A charge $-2q$ is located at the origin, a charge $+q$ is located on the x-axis at $x = a$, and another charge $+q$ is located on the x-axis at $x = -a$.

(a) Find the leading non-vanishing term in the electrostatic potential Φ at a point with $r \gg a$ in the $\phi = 0$ plane as a function of r and θ.

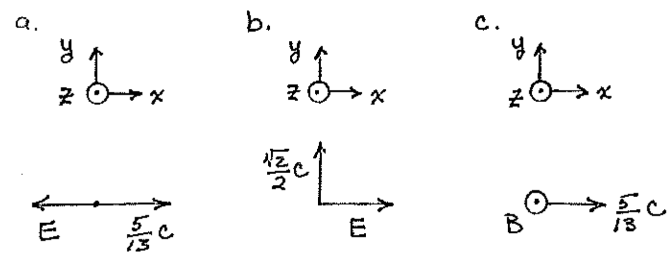
(b) Repeat Part (a), but for the $\phi = \frac{\pi}{2}$ plane.

6. **[18 points]** NOTE: This problem must be done relativistically. It consists of three separate and independent parts, all of which involve a particle of charge $q > 0$ and mass m. Express your answers in terms of a, m, c, E or B, and numerical factors only. Read all parts of the problem carefully as they are similar but involve critical differences. In each case you are asked to find a time $t > 0$. If there is no such time, prove this fact. If there are multiple such times, find the smallest one.

   (a) The particle finds itself in a uniform electric field that points in the $-x$-direction and has magnitude E. The particle has initial velocity $v = \frac{5}{13}c$ in the $+x$-direction, *i.e.* opposite the direction of the electric field. Find the time t at which the particle comes momentarily to rest.

   (b) The particle finds itself in a uniform electric field that points in the $+x$-direction and has magnitude E. The particle has initial velocity $v = \frac{\sqrt{2}}{2}c$ in the y-direction. Find the time t at which the Lorentz factor of the particle reaches the value $\gamma = 2$.

   (c) The particle finds itself in a uniform magnetic field that points in the $+z$-direction and has magnitude B. The particle has initial velocity $v = \frac{5}{13}c$ in the $+x$-direction. Find the time t at which the x-component of the velocity of the particle is $v_x = 0$.



Classical Mechanics PhD Qualifying Exam, Spring 2012

Question:	1	2	3	Total
Points:	15	20	15	50
Score:				

1. (15 points) Consider small oscillations of an anharmonic oscillator with the Lagrangian

$$L = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2 - \alpha x^3\,,$$

where α is small in appropriate units.

(a) Write down the Hamiltonean for the system and Hamilton equations (remember that you can drop total derivatives from the Lagrangian).

(b) Perform a canonical transformation with a generating function

$$\Phi(P, x) = Px + aPx^2 + bP^3$$

and find values of a and b for which the oscillations in new variables are harmonic to leading order in α.

(c) Find small fluctuations of the system and write the solution in terms of the original variable, $x(t)$.

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2. (20 points) Consider a particle moving in the potential

$$U(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2},$$

where both α and β are positive.

(a) Find the trajectory of the particle in the form $r = r(\phi)$ or $\phi = \phi(r)$. Under which conditions is motion finite?

(b) For the case of finite motion, find the angular displacement $\delta\phi$ between two consecutive transitions of the particle through the perihelion of the trajectory (i.e. point with $r = r_{min}$).

(c) Find the period of radial oscillations T_r (i.e. time between two consecutive transitions through the perihilion).

(d) Find the rotational period T_ϕ (i.e. time interval during which the angular coordinate changes by 2π).

(e) Under which condition is the trajectory closed?

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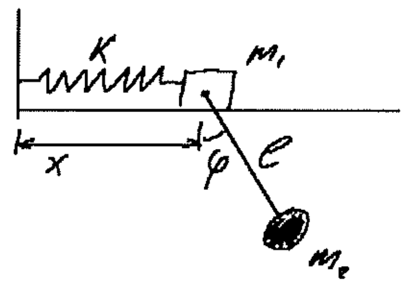
3. (15 points) A pendulum of mass m_2 and length l is attached to a block of mass m_2 which, in turn, is attached to the wall by a spring with spring constant k (see the Figure). Assume that $k = 2(m_1 + m_2)g/l$.



(a) Write down Lagrangian and Euler-Lagrange equations for small fluctuations of the system

(b) Find normal coordinates and eigenfrequencies.

(c) Now assume that the block experiences a friction force $f = -\alpha\dot{x}$ and find the solution for damped oscillations.

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Quantum Mechanics Qualifying Exam—Spring 2012

1. [25 points] A particle moving in one dimension experiences a potential $V(x)$, where $V(x \leq 0) = 0$ and $V(x > 0) = \frac{m\omega^2 x^2}{2}$. Find the energies and eigenstates of the Hamiltonian $H = \frac{P^2}{2m} + V(X)$. For the ground state, compute the expectation value of X. (A possibly useful integral: $\int_0^\infty du\, u^3 e^{-u^2} = 1/2$.)

2. [25 points] A spin-1/2 particle is described by a time-dependent Hamiltonian

$$H(t) = B\cos(\omega t)\mathbf{S}\cdot\hat{\mathbf{z}}. \tag{1}$$

Here B and ω are constants, while $\mathbf{S} = S_x\hat{\mathbf{x}} + S_y\hat{\mathbf{y}} + S_z\hat{\mathbf{z}}$ is a spin-1/2 operator. At time $t = 0$ the spin's expectation value points along the direction $\hat{\mathbf{y}}$. That is, if $|\psi(t = 0)\rangle$ is the initial normalized wavefunction, then

$$\langle\psi(t = 0)|\mathbf{S}|\psi(t = 0)\rangle = \frac{\hbar}{2}\hat{\mathbf{y}}. \tag{2}$$

A measurement of S_x is performed at some later time τ. At which values of τ (if any) is it possible to predict the outcome of this measurement with 100% probability? How does your answer change if one instead performed a measurement of S_z?

3. [25 points] Consider a particle confined to a 3D harmonic oscillator potential. The Hamiltonian reads

$$H = \frac{P_x^2 + P_y^2 + P_z^2}{2m} + \frac{m\omega^2}{2}(X^2 + Y^2 + Z^2). \tag{3}$$

In this problem we will focus on the eigenstates of H that have the lowest and second-lowest energies. Diagonalize $\mathbf{L}^2$ and L_z within this subspace, and deduce the corresponding eigenvalues. (Note that $\mathbf{L}$ is the usual orbital angular momentum operator.)

4. [25 points] A particle with mass m and positive charge q moves in one dimension, but is confined to the region $x > 0$. The particle is subjected to an electric field $\mathbf{E} = -\mathcal{E}\hat{\mathbf{x}}$, with $\mathcal{E} > 0$, as shown in Fig. 1. Argue that for $x > 0$ the wavefunction

$$\psi(x) = \sqrt{\frac{\alpha^3}{2}}\, x e^{-\alpha x/2} \tag{4}$$

provides a reasonable guess for the ground state of the Hamiltonian describing the particle. Find the optimal value of α and extract the upper bound on the true ground-state energy that follows from our guess. [A possibly useful formula: $\int_0^\infty du u^n e^{-u} = (n!)$.]

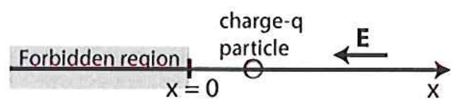

FIG. 1: Geometry for problem 4.

Statistical Mechanics Qualifying

Spring, 2012

Open Book

Problem 1. [10 points]

The canonical partition function of a one dimensional classical hard sphere gas is given by

$$\mathcal{Z}(T, N, L) = \frac{1}{N!}\left[\frac{L - Nb}{\lambda}\right]^N ;$$

L is the length of the one dimensional container, N is the number of molecules whose linear extent is b and $\lambda = 2\pi\hbar/\sqrt{2\pi mkT}$ is the thermal wavelength.

(i) What is the entropy of this gas?

(ii) Obtain the force exerted by this gas on the confining "walls".

(iii) Obtain the chemical potential.

Problem 2 [15 points]

The quantum levels of some system are labeled by the integers, $n = 1, \ldots$ with energies $E_n = an^b$. The degeneracy, g_n of level n is $g_n = cn^d$; a, b, c and d are constants.

(i) What is the relation between energy and temperature for this system?

(ii) What is the specific heat, C as a function of the temperature?

Problem 3 [25 points]

For a system of conserved bosons (i.e. the number of particles, N, is fixed and not determined by setting the chemical potential, $\mu = 0$.) the energy-momentum relation is $E(\vec{p}) = c|p|$. The dimensionality of this system is three.

1. Calculate the relation between the temperature and density for the onset of a Bose-Einstein condensation. You may leave your answer depending on a dimensionless integral, which you do not have to evaluate explicitly.

2. If the energy-momentum relation were $E(\vec{p}) = c|p|^3$ would there be a Bose-Einstein condensation? Give explicit arguments for your answer.