Physics 211
Fall 2003
Problems

Problems marked with [M], [F] or [Q] appeared in previous midterms, finals or qualifying exams respectively.

Problem 1

A curve of length l passes through the points $(-1, 0)$ and $(1, 0)$. Find the shape of such a curve that maximizes the area between it and the x-axis.

Problem 2 [20 points]

Find the shape of a soap film that is stretched between two coaxial, parallel circular rings of radius a; the rings are a distance d apart. You may ignore gravity. The soap film will take the shape that minimizes its area. The parameters determining the shape require the solution of a transcendental equation. You do not have to solve this equation; however, what happens when d/r becomes large?

Problem 3

The action for a particle in a uniform gravitational field is

$$S = \int_0^1 dt \left[\frac{1}{2}m\dot{x}^2 - mgx\right] .$$

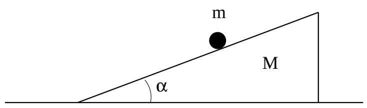
At time $t = 0$ the particle is at $x = 0$ and at time $x = x_f$ it is at $t = 1$. You "suspect" the solution is of the form

$$x(t) = at^2 + bt\,.$$

Find b as a function of a and then find a by minimizing the action with respect to a. Compare to the well known solution of this problem.

Problem 4

A particle of mass m slides without friction on a wedge of angle α and mass M that



can move without friction on a horizontal surface. Find and solve the equations of motion. Show that the the momentum in the x direction is conserved.

Problem 5

Consider the problem of a mass m constraint to move on the surface of a sphere using Cartesian coordinates; include the constraint by means of a Lagrange multiplier. Eliminating the multiplier show that

$$\begin{aligned} \frac{d}{dt}(x\dot{y} - y\dot{x}) &= 0\,; \\ \frac{d}{dt}(y\dot{z} - z\dot{y}) &= 0\,; \\ \frac{d}{dt}(z\dot{x} - x\dot{z}) &= 0\,. \end{aligned}$$

As we shall learn later, this is just angular momentum conservation.

Problem 6 [20 points]

The Lagrangian for a charged particle in a magnetic field $\vec{B}(\vec{x})$ is

$$L = \frac{1}{2}m\vec{v}^2 - \frac{q}{c}\vec{v}\cdot\vec{A}(\vec{x})\,;$$

where q is the charge, c is the velocity of light and the vector potential $\vec{A}(\vec{x})$ is related to the magnetic field by

$$\vec{B}(\vec{x}) = \vec{\nabla}\times\vec{A}(\vec{x})\,.$$

From the above we note that $\vec{A}(\vec{x})$ and $\vec{A}'(\vec{x})$, with $\vec{A}'(\vec{x}) = \vec{A}(\vec{x}) + \vec{\nabla}\Lambda(\vec{x})$ for some function $\Lambda(\vec{x})$, describe the same magnetic field (gauge invariance).

a. Show that $L(A)$ and $L(A')$ differ by a total time derivative.

b. Obtain the equations of motion for this system.

Problem 7 [M] [20 points]

Four equal mass particles move in one dimension; their coordinates are x, y, z and w. The Lagrangian is:

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{w}^2 + \dot{z}^2\right) - V(xz - yw)\,.$$

a. Show that this Lagrangian is invariant under the following transformation:

$$\begin{aligned} x &\to \frac{x}{\sqrt{1+u^2}} + \frac{yu}{\sqrt{1+u^2}}\,, \\ y &\to \frac{y}{\sqrt{1+u^2}} - \frac{xu}{\sqrt{1+u^2}}\,, \\ w &\to \frac{w}{\sqrt{1+u^2}} + \frac{zu}{\sqrt{1+u^2}}\,, \\ z &\to \frac{z}{\sqrt{1+u^2}} - \frac{wu}{\sqrt{1+u^2}}\,; \end{aligned}$$

u is a real number.

b. What is the conserved quantity corresponding to this invariance?

Problem 8 [Q]

The following Lagrangian describes a system of two variables x and y

$$\mathcal{L} = \frac{1}{2}e^{2y}\left(\frac{\partial x}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial y}{\partial t}\right)^2 - V(e^y x)\,.$$

(i) Show that this Lagrangian is invariant under

$$x \to ax, \quad y \to y - \ln(a)\,.$$

(ii) By considering the above transformation for $a = 1 + \epsilon$ with ϵ small, find the conserved quantity related to the above transformation.

(iii) Using the equations of motion, show explicitly that the time derivative of quantity you obtained in part (ii) is zero.

Problem 9 [20 points] [M]

A satellite is in a <u>circular</u> orbit of radius R about a planet of mass M.

a. What is the velocity of the satellite?

At some point a rocket is fired in the direction of motion of the satellite reducing its velocity by a factor η ($v(\text{after}) = \eta v(\text{before})$).

b. Obtain the equation of the new orbit. [Hint: Use conservation of energy and momentum.]

Problem 10

Find the orbit of a particle in a potential

$$V(r) = -\frac{\alpha}{r} + \frac{h}{r^2}\,;$$

h is a positive constant. In particular show that r can be put into a form

$$r = \frac{p}{1 + e\cos a\phi}\,.$$

Obtain p, e and a explicitly.

Problem 11 [Q]

The cross section for the scattering of some particle off some spherically symmetric potential is

$$\frac{d\sigma}{d\Omega} = \alpha + \beta\cos(\theta)\,;$$

α and β are positive constants with $\alpha \geq \beta$.

(i) Find an expression for the impact parameter b as a function of the scattering angle θ. You may assume that at $b = 0$ the particle is scattered into the backward direction.

(ii) At what radius does the potential vanish?

Problem 12 [F]

A beam of particles of mass m and velocity v is incident on a potential $V(r) = -\alpha/r$ for $r > a$ and something else for $r < a$. As the impact parameter b decreases from infinity, at what value of b does the beam begin to sense the deviation from the α/r potential? What scattering angle does this correspond to? [You may use whatever helpful expressions you find in your book or in your notes. You do not have to rederive these.]

Problem 13

Usual discussions of scattering found in most text books limit themselves to central potentials, where the cross section depends only on the polar angle θ and is independent of the azimuthal angle ϕ.

$$d\sigma(\theta) = b(\theta)\left|\frac{\partial b(\theta)}{\partial\theta}\right| d\theta d\phi\,,$$

where $b(\theta)$ is the impact parameter at which an incoming particle will scatter through an angle θ.

For non central potentials the situation is more complicated. The cross section will depend on both the polar and azimuthal angles. Let b and ϕ_0 be the impact parameter and azimuthal angle of an incident particle that scatters into the direction θ, ϕ. The relation between these variables is given by the functions $b(\theta, \phi)$ and $\phi_0(\theta, \phi)$. Write the expression, analogous to the one above, for $d\sigma(\theta, \phi)$ in using these two functions.

Problem 14 [20 points]

A simple model for the potential due to an atom (nucleus plus electrons) is

$$V(r) = \begin{cases} -\alpha\left(\frac{1}{r} - \frac{1}{R}\right), & \text{for } r \le R; \\ 0, & \text{for } r > R. \end{cases}$$

Obtain the relation between the impact parameter and the scattering angle. What is the total cross section?

Problem 15

Find the principal moments of inertia about the center of mass of a flat 45^0 right triangle with side of length a and mass M. What are the principal axes?

Problem 16 [Q]

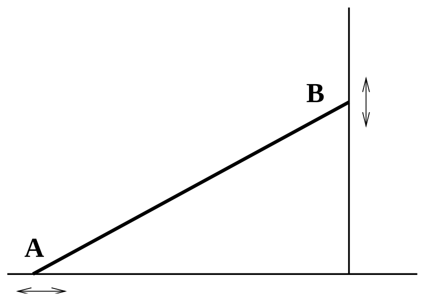
A thin flat uniform ring with inner radius a and outer radius b has mass m. Use a coordinate system with the ring in the x-y plane and z perpendicular to it and through its center.

(i) Find the moment of inertia tensor.

(ii) Determine the angular momentum if the ring is instantaneously rotating with angular velocity ω about the direction in the y-z plane that make a 45^0 angle to the y axis.

Problem 17 [F]

A uniform rod of length l and mass m has end A slide along a horizontal floor and end B up and down a vertical bar. The system is acted on by gravity.



a. Using appropriate coordinates write down the Lagrangian for this system.

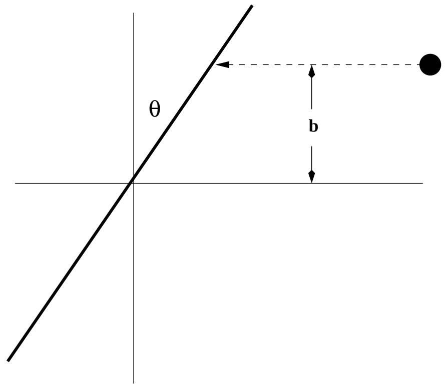
b. The system is released from rest with the end B at a height h above the floor. What is the speed of this end when it hits the floor?

Problem 18

In class we obtained the components of the rotational velocity along axes fixed in the body. Find these components along axes fixed in space. Use Fig. 47 of Landau and Lifshitz and discuss each component in detail.

Problem 19 [20 points]

A rod of length l and mass M is initially inclined at an angle θ to the vertical. A small projectile of mass m and velocity v hits it horizontally at an impact parameter b and sticks to it. Find the subsequent motion of rod-projectile system. In a 'sticking' process energy is not



conserved (it goes into the chemical reactions responsible for the sticking) but momentum and angular momentum are.

Problem 20 [20 points]

A bar of negligible weight and length l has equal mass points m at the two ends. The bar is made to rotate uniformly about an axis passing through the center of the bar and making an angle θ with the bar.

a Using Euler's equations find the components of the torque driving the bar (i.e. components along the principal axes of the bar).

b Using $\vec{K} = d\vec{M}/dt$, find the components of the torque along axes fixed in space. Show that the answer is consistent with part [a]. [Hint: $I_{i,j}$ is time dependent.]

Problem 21 [20 points]

Two masses, m, are located at the vertices of an equilateral triangle; the third vertex is fixed. The vertex, as well as the two masses are all connected by springs with equilibrium distances l and spring constants k. The two masses are free to move in a plane. Find the eigenmodes and eigenfrequencies of this system.

Problem 22 [20 points] [F]

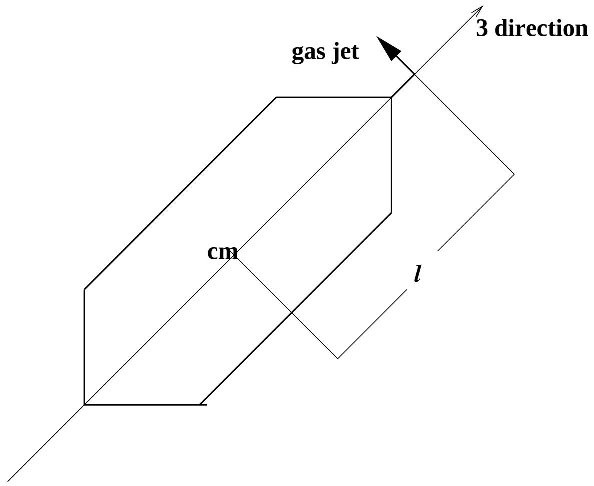
A thin, uniform rod of length l and mass m is free to rotate in all directions around its center; the position of the center is fixed. The total energy of this system is E and the angular momentum along the fixed $\hat{z}$ direction is M_z.

a. What is the range of θ, the angle the rod makes with the $\hat{z}$ axis?

b. For θ close to $\pi/2$ and for the given value M_z, with what frequency does θ oscillate around $\pi/2$?

Problem 23 [F]

A rigid body with moments of inertia $I_1 = I_2 \neq I_3$ is free to rotate about its fixed center of mass. A gas jet emerges from a point on the I_3 axis at a distance l above the center of mass. The direction of the jet is fixed in the body and is perpendicular to the I_3 axis; the jet exerts a recoiling force F on the body. The jet has no effect on the moments of inertia.



(i) Write the equations that determine the time evolution of the rotations Ω_i (rotations around the body fixed axes).

(ii) For $\Omega_3 \neq 0$ there are solutions with all Ω_i constant in time. Find these solutions explicitly.

Problem 24 [20 points] [Q].

A particle of mass m is constrained to move on a sphere of radius R. A potential

$$V(\theta) = \frac{\kappa}{mR^2}\sin(\theta)$$

acts on this particle; κ is a constant and θ, ϕ are the usual polar and azimuthal angles.

(i) What is the Lagrangian for this system?

(ii) For L_z, the z-component of the angular momentum, sufficiently small there is a solution of the equations of motion with $\theta(t) = \theta_0$ independent of time. Find θ_0 as a function of L_z and the other parameters in this problem.

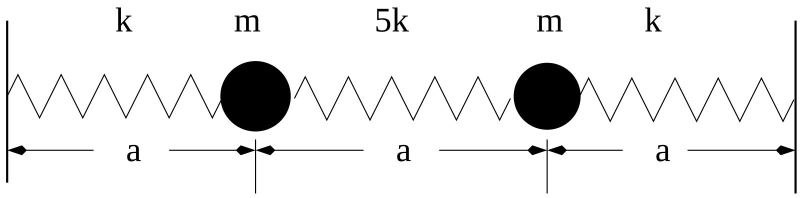
(iii) What is the frequency of small oscillations around θ_0?

Problem 25

A particle is thrown up vertically with initial speed v_0. It reaches a maximum height and falls back to ground. Show that the Coriolis deflection when it again reaches the ground is opposite in direction and four times greater in magnitude than the Coriolis deflection when it is dropped at rest from the same maximum height. As you proceed, keep in mind that $\omega^2 r \ll g$. [Look at Landau and Lifshitz problem 39.1]

Problem 26 [Q]

Two equal masses m are fastened to each other and to two rigid walls by three springs, as shown in the figure. The two outer springs have spring constant k while the middle spring's

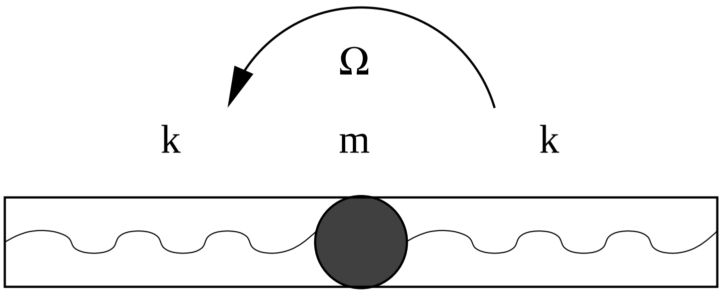


constant is $5k$. All three springs have unstretched lengths a and the system oscillates in one dimension.

Calculate the eigenfrequencies and eigenmodes of this system. Include a sketch showing the behavior of the masses in each mode.

Problem 27 [20 points] [Q]

A particle of mass m is constrained to move on a frictionless track, as shown in the figure. The particle is attached to each end of the track by a spring with spring constant k. The track can rotate around a vertical axis passing through the equilibrium position of the particle.



(i) Assume the track is not rotating. What is the frequency of oscillation of the particle?

(ii) Now let the track rotate with an angular frequency Ω. What is the frequency of oscillation now?

(iii) The track rotating at an angular frequency Ω; as a function of the velocity of the particle along the track, what is the normal force exerted by the track on the particle?

Problem 28 [F]

Two particles with masses m_1 and m_2 move on two concentric circles of radius r_1 and r_2 respectively; $r_2 > r_1$. They are connected by a spring with spring constant k and equilibrium length equal to zero. Find the frequencies of oscillation of this system.

Problem 29

Show that the following $(A_0, \vec{A})$'s transform as four vectors.

1. $(E, \ \ \vec{p}c^2)$,
2. $(1/\sqrt{1-v^2/c^2}, \ \ \vec{v}/\sqrt{1-v^2/c^2})$.

Problem 30

An elementary particle of mass M is moving with some momentum $\vec{P}$ when it decays into two particles of mass m_a and m_b with momenta $\vec{p}_a$ and $\vec{p}_b$. Show that

$$M^2 = m_a^2 + m_b^2 + 2E_aE_b/c^4 - 2\vec{p}_a \cdot \vec{p}_b/c^2 \,.$$

Problem 31

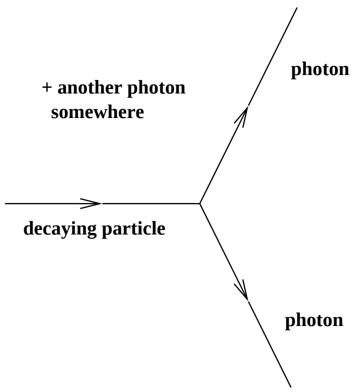
A photon is a particle with zero rest mass. Show that if a photon of energy E scatters of a free electron (mass m_e) the scattered photon will have an energy

$$E' = E\left[1 + \frac{E}{m_e c^2}(1 - \cos\theta)\right] ;$$

where θ is the angle through which the photon scatters.

Problem 32 [F]

A particle with momentum $p\hat{x}$ decays into <u>three</u> photons ($m_{\text{photon}} = 0$). You observe only <u>two</u> of these photons. The magnitude of the momenta of both photons is $2p$ and the directions are at 60^0 to the x-axis, one above it and one below, as illustrated in the figure. Although



we haven't measured everything, we can determine all other kinematic variables.

(i) What is the momentum of the third photon?

(ii) What is the mass of the decaying particle?

Problem 33 [20 points] [F]

A rocket ship accelerates with an acceleration that is constant in its own rest frame. The four-vector for the acceleration is

$$a_\mu = (0, g\hat{x}) .$$

a. What is the acceleration four-vector for for an observer at rest for whom the rocket is moving with velocity $v\hat{x}$?

b. Write out the velocity and acceleration four-vectors in terms of the rapidity η; $\tanh\eta = v/c$.

c. The relativistic relation between velocity and acceleration is

$$\frac{1}{\sqrt{1 - v^2/c^2}} \frac{dv_\mu}{dt} = a_\mu \, ;$$

solve for the rapidity as a function of time.

Problem 34

We discussed the Lagrangian for the gyroscope

$$L = \frac{1}{2} I_1 (\dot\theta^2 + \sin^2\theta \, \dot\phi^2) + \frac{1}{2} I_3 (\cos\theta \, \dot\phi + \dot\psi)^2 - Mgl \cos\theta \, .$$

a. Find the Hamiltonian $H(p_\theta, \, p_\phi, \, p_\psi, \, \theta, \, \phi, \, \psi)$.

b. Obtain the equations of motion from this Hamiltonian.

Problem 35 [20 points] [Q]

The Lagrangian for some system is

$$\mathcal{L} = \frac{1}{2} m \left(\dot x^2 + \dot y^2\right) + a \dot x y^2 \, .$$

(i) Obtain the equations of motion.

(ii) Find the Hamiltonian of this system.

(iii) Obtain the energy as a function $\dot x, \ \dot y, \ x$ and y.

(iv) Using (i) show explicitly that the energy in (iii) is conserved.

Problem 36

The action for a relativistic particle in an external electromagnetic field is

$$S = \int dt \left[-mc^2 \sqrt{1 - \frac{v^2}{c^2}} - q\Phi(\vec r, t) - \frac{q}{c} \vec v \cdot \vec A(\vec r, t) \right] \, ;$$

Φ is the electrostatic potential and $\vec A$ is the vector potential. Obtain the Hamiltonian.