Physics 214A

Final

Fall 1997

Useful (and useless) Mathematical Results

Bose-Einstein Integrals

$$
\begin{aligned}
g_\nu(z) &= \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}dx}{z^{-1}e^x - 1}\,, \\
g_\nu(z) &= z + \frac{z^2}{2^\nu} + \frac{z^3}{3^\nu} + \cdots\,, \\
g_2(1) &= \frac{\pi^2}{6}\,, \\
g_4(1) &= \frac{\pi^4}{90}\,, \\
g_{3/2}(1) &= 2.612\ldots\,.
\end{aligned}
$$

Fermi-Dirac Integrals

$$
\begin{aligned}
f_\nu(z) &= \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}dx}{z^{-1}e^x + 1}\,, \\
f_\nu(z) &= z - \frac{z^2}{2^\nu} + \frac{z^3}{3^\nu} - \cdots\,, \\
f_{3/2}(1) &= 0.765\ldots\,, \\
f_2(1) &= \frac{\pi^2}{12}\,, \\
\int_0^\infty \frac{\phi(x)dx}{e^{x-\xi}+1} &= \int_0^\xi \phi(x)dx + \frac{\pi^2}{6}\left(\frac{d\phi(x)}{dx}\right)_{x=\xi} + \cdots\,.
\end{aligned}
$$

I.

Using the relations

$$\left.\frac{\partial F}{\partial T}\right)_{V,N} = -S; \quad \left.\frac{\partial F}{\partial V}\right)_{T,N} = -P; \quad \left.\frac{\partial F}{\partial N}\right)_{T,V} = \mu$$

and the fact that $F[T, V, N]$ is extensive in V and N (not in T) obtain the Gibbs-Duhem relations.

II.

A solid contains N magnetic atoms having spin 1/2. At sufficiently high temperatures each spin is randomly oriented. At low temperatures the spins become oriented along the same direction (the details of the interaction responsible for this are unimportant). The heat capacity, as a function of the temperature, is approximated by

$$C(T) = \frac{AT}{(1+T)^2};$$

A is a constant. Find A. [Hint: First determine the entropy for large and for small temperatures.]

III.

In a very simplified model, excitations in ^{4}He are quanta with momentum $\mathbf{p}$ and energy $\epsilon(p) = c_s p$; there is only one such mode for each $\mathbf{p}$ (no transverse-longitudinal distinction). This spectrum persists to some maximum energy, ϵ_M. At high temperatures the specific heat, $C_v \rightarrow \frac{3}{2} N k_B$; N is the number of helium atoms in a volume V.

a. Express ϵ_M as a function of the density and c_s.

b. What is the specific heat at low T?

IV.

We know that at high temperatures or low densities the relation between the fugacity, z, and the density of a Fermi gas approaches the classical one

$$z = \lambda^3 \rho\,;$$

λ is the thermal wavelength. The first quantum mechanical correction is of the form

$$z = \lambda^3 \rho + a(\lambda^3 \rho)^n\,;$$

n is an integer and a is a constant. Find a and n.

V.

An electron gas has a Fermi energy μ_F. At what temperature will its chemical potential $\mu = 0$?