Physics 215A
Fall 2000

In the next several problems the symbol $\{|a_i\rangle\langle b_j|\}$ represents the process of accepting the system in state b_j and turning it into the state a_i. Schwinger and Gottfried use the symbol $M(a_i, b_j)$.

Problem 1

Show that

$$\{|a_i\rangle\langle b_j|\} = \sum_{k,l} \langle c_k|a_i\rangle\langle b_j|d_l\rangle\{|c_k\rangle\langle d_l|\}\,.$$

and

$$\sum_j \langle a_i|b_j\rangle\langle b_j|c_k\rangle = \langle a_i|c_k\rangle\,.$$

Problem 2

Define $p(b_j, c_k, a_i)$ as

$$p(b_j, c_k, a_i) = p(b_j, c_k)p(c_k, a_i)\,.$$

[a.] Give a physical meaning to this expression.

[b.] Show that

$$\sum_k p(b_j, c_k, a_i) \neq p(b_j, a_i)\,.$$

[c.] Discuss why in classical mechanics you would expect the equality to hold.

Problem 3

Show that the identification $\{|a_i\rangle\langle b_j|\} = |a_i\rangle\langle b_j|$ satisfies the properties we wish $\{|a_i\rangle\langle b_j|\}$ to have, namely

$$\begin{aligned}
\sum_i \{|a_i\rangle\langle a_i|\} &= 1 \\
\{|b_k\rangle\langle c_j|\}\,\{|c_i\rangle\langle a_l|\} &= \delta_{i,j}\{|b_k\rangle\langle a_l|\} \\
\{|a_i\rangle\langle b_j|\}\{|c_k\rangle\langle d_l|\} &= \langle b_j|c_k\rangle\{|a_i\rangle\langle d_l|\}\,.
\end{aligned}$$

Problem 4

Show that if $|w\rangle = \mathcal{O}|v\rangle$ then $\langle w| = \langle v|\mathcal{O}^\dagger$.

Problem 5

Show that

$$\begin{aligned}
\mathcal{O}^\dagger &= \mathcal{O}^{*T}\,, \\
(\mathcal{O}_1\mathcal{O}_2)^T &= \mathcal{O}_2^T\mathcal{O}_1^T\,, \\
(\mathcal{O}_1\mathcal{O}_2)^\dagger &= \mathcal{O}_2^\dagger\mathcal{O}_1^\dagger\,.
\end{aligned}$$

Problem 6

Show that for $\mathcal{U} = \sum_k |b_k\rangle\langle a_k|$, $\mathcal{U}^\dagger\mathcal{U} = 1$ ($\mathcal{U}^\dagger = \mathcal{U}^{-1}$). also, show that in general $\mathcal{U} = \sum_{i,j} |a_j\rangle U_{ji}\langle a_i|$ is unitary provided $U^\dagger U = 1$, taken as matrix multiplication. ($U^\dagger_{ij} = U^*_{ji}$).

Problem 7

Show that:
(1) The product of two unitary operators is unitary.
(2) If the trace of an operator $\mathcal{O}$ is defined as

$$\mathrm{Tr}\mathcal{O} = \sum_i \langle a_i|\mathcal{O}|a_i\rangle\,,$$

then $\mathrm{Tr}\mathcal{O} = \mathrm{Tr}(\mathcal{U}^\dagger\mathcal{O}\mathcal{U})$, for any unitary $\mathcal{U}$.
(3) If $\mathcal{A}$ is Hermitian so is $\mathcal{U}^\dagger\mathcal{A}\mathcal{U}$ for any unitary $\mathcal{U}$

Problem 8

Remembering that the probability of obtaining a_i for a measurement of $\mathcal{A}$ in the state $|v\rangle$ is $|\langle a_i|v\rangle|^2$, show that the average value for a measurement of $\mathcal{A}$ in this state is $\langle v|\mathcal{A}|v\rangle$

Problem 9

Find the eigenvalues and normalized eigenfunctions of the following operator

$$\mathcal{A} = 2|a\rangle\langle a| - |b\rangle\langle b| + 2|a\rangle\langle b| + 2|b\rangle\langle a|\,;$$

where the vectors $|a\rangle, |b\rangle$ satisfy

$$\begin{aligned}
\langle a|a\rangle &= \langle b|b\rangle = 1\,, \\
\langle a|b\rangle &= \langle b|a\rangle = 0\,.
\end{aligned}$$

Problem 10

A certain operator has a 3×3 matrix representation as follows:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Find the normalized eigenvectors and corresponding eigenvalues. Is there any degeneracy

Problem 11

Show that

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\phi} = \delta(\phi) \, ,$$

for ϕ restricted to the interval $-\pi \leq \phi \leq \pi$. To do this show that $\delta(\phi)$, as defined above, satisfies

$$\begin{aligned} \int_{-\pi}^{\pi} d\phi \delta(\phi) &= 1 \\ \delta(\phi) &= 0, \text{ for } \phi \neq 0 \, ; \end{aligned}$$

for the latter, do the sum over n explicitly.

Problem 12

Relate $\psi_{mom,S}(p)$ to $\psi_S(x)$ and obtain an expression for $\langle p|X|S\rangle$ in terms of $\langle p|S\rangle$.

Problem 13 [20 points]

(a) Find the propagator for a free particle.
(b) Suppose that

$$\psi(x,0) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{4}} \exp\left[-\frac{(x-x_0)^2}{4\sigma^2} + \frac{ip_0 x}{\hbar}\right] ,$$

find $\psi(x,t)$.

Problem 14

For the potential $V(x) = 0$ for $0 \leq x \leq L$ and $V(x) = \infty$ otherwise, calculate $\langle S,t|[X - (L/2)]|S,t\rangle$ for the case where

$$|S, t=0\rangle = \frac{a|n=1\rangle + b|n=2\rangle}{\sqrt{|a|^2 + |b|^2}} \, ;$$

a and b are complex numbers.

Problem 15

[a.] Show that if E is nondegenerate the eigenfunction of the Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_E(x) + V(x)\psi_E(x) = E\psi_E(x)\,,$$

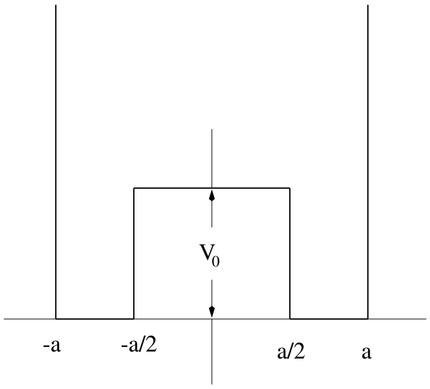
can be chosen to be real.
[b.] Show that the same holds even if E is degenerate.

Problem 16

If in the above problem $V(x) = V(-x)$, show that the eigenfunctions can be chosen to be symmetric or antisymmetric as $x \to -x$.

Problem 17 [20 points]

For the potential illustrated below, namely $V(x) = \infty$ for $|x| > a$, $V(x) = 0$ for $a > |x| > a/2$



and $V(x) = V_0$ for $|x| < a/2$ set up the procedure to find the energy eigenvalues and the corresponding wave functions for both the odd and even parity situations and for energies greater or smaller than V_0. Namely:

a. Write down the form of the wave functions in all the regions.
b. What equations arise as a consequence of the boundary conditions?

Do **not** try to solve these equations.

Problem 18

(a) Calculate the following matrix elements for the harmonic oscillator:

$$\langle n|a|m\rangle\,,$$
$$\langle n|a^\dagger|m\rangle\,,$$
$$\langle n|X|m\rangle\,,$$

$$\langle n|P|m\rangle\,,$$
$$\langle n|X^2|m\rangle\,,$$
$$\langle n|P^2|m\rangle\,.$$

(b) Show that the expectation value of the kinetic energy in some state is equal to the expectation value of the potential energy in that state.

Problem 19.

An electron near the surface of liquid helium feels a potential

$$V(z) = \begin{cases} \infty & \text{for } z \leq 0 \\ -\frac{b}{z} & \text{for } z > 0\,; \end{cases}$$

z is the height above the surface and b is some parameter. Ignore the motion in the x and y directions.

a. Write out the Schrödinger equation for this problem.

b. What is the form of the solution of this equation for large, positive z?

c. What is the behavior of the solution for small, positive z?

d. Combining the results of [b.] and [c.] guess (or derive) the wave function for the ground state and find the energy of that state.

Problem 20

A particle of mass m is bound in a potential $V(x) = gx^4$. Scale the Schrödinger equation and determine the dependence of the energies on m, g and $\hbar$. Find numerically the energies and wave functions of the first two even parity states and the first odd parity state.

Problem 21

At $t = 0$ a harmonic oscillator is in the state $(|0\rangle + |1\rangle)/\sqrt{2}$. What is the probability of finding it in the state $(|0\rangle - |1\rangle)/\sqrt{2}$ at time $t > 0$? The frequency of the oscillator is ω.

Problem 22

For the potential

$$V(x) = \begin{cases} 0 & \text{for } x < 0 \\ V_0 & \text{for } 0 \leq x \leq a \\ 0 & \text{for } x > a\,, \end{cases}$$

with V_0 positive, find the reflection and transmission coefficients for $E < V_0$.

Problem 23

Sakurai, 3.1; p.242

Problem 24

In this problem you derive some useful relations for the Pauli spin matrices.

[a.] Show that $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\,\mathbf{1}$; $\mathbf{1}$ is the unit 2×2 matrix.
[b.] If $\vec{A}$ and $\vec{B}$ are ordinary, *i.e.* commuting, vectors, show that $\vec{\sigma}\cdot\vec{A}\vec{\sigma}\cdot\vec{B} = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot\vec{A}\times\vec{B}$. [Use the result in part [a.]]
[c.] Show that

$$\exp(i\vec{\sigma}\cdot\vec{A}) = \cos(A) + i\frac{\vec{\sigma}\cdot\vec{A}}{A}\sin(A)\ ;$$

A is the magnitude of $\vec{A}$. [Hint. Expand the exponential, use result [a.] and resum the series.]

Problem 25

The deuteron, among other systems, has an intrinsic spin $j = 1$; such systems may be described by a three component object. Obtain an explicit representation of the J_i's as 3×3 matrices.

Problem 26

Evaluate $\langle 1,1|L_x|1,0\rangle$ both algebraically and by doing the integral

$$\int sin\theta\, d\theta\, d\phi\, Y^*_{1,1}(\theta,\phi) L_x Y_{1,0}(\theta,\phi)\,.$$

Problem 27

Sakurai, 3.17; p. 244

Problem 28

Find the energy levels, for each angular momentum state, of the three dimensional harmonic oscillator. Each state $|n,l,m\rangle$ is a linear superposition of the rectangular coordinate solutions $|n_x,n_y,n_z\rangle$. Find explicitly $|n=2,l=2,m=1\rangle$ as a combination of the $|n_x,n_y,n_z\rangle$'s.

Problem 29

$\vec{J}_A$ and $\vec{J}_B$ are two sets of angular momentum operators acting on two different systems. Show that $\vec{J} = \vec{J}_A + \vec{J}_B$ are also angular momentum operators.

Problem 30

Find the eigenvalues and the ground state eigenfunction of the two dimensional hydrogen atom $[V(r) = -\alpha\hbar c/r]$.

Problem 31

The Hamiltonian for a charged particle in a magnetic field is

$$\mathcal{H} = \frac{1}{2m}\left[\vec{p} - \frac{e}{c}\vec{A}(\vec{r})\right]^2 ,$$

where $\vec{A}(\vec{r})$ is the vector potential related to the magnetic field by $\vec{H}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$. What are

$$\frac{d\vec{r}}{dt} \quad \text{and} \quad \frac{d^2\vec{r}}{dt^2}?$$

Compare to the classical equations of motion.

Problem 32

For the above problem find an expression for the probability current $\vec{j}(\vec{r})$ such that

$$\frac{d\rho(\vec{r})}{dt} + \vec{\nabla} \cdot \vec{j}(\vec{r}) = 0 .$$

Problem 33

$\vec{J}_A$ and $\vec{J}_B$ are angular momentum operators referring to two different systems A and B. Obtain the commutators $[J_{A,z}, \vec{J}_A \cdot \vec{J}_B]$, $[J_{B,z}, \vec{J}_B \cdot \vec{J}_B]$ and $[J_{A,z} + J_{B,z}, \vec{J}_A \cdot \vec{J}_B]$.

Problem 34

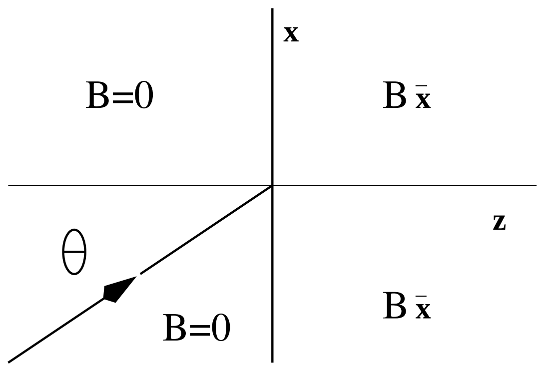
A spin-$\frac{1}{2}$ particle is placed in a magnetic field pointing along the z direction. The Hamiltonian is

$$\mathcal{H} = \frac{ge}{2mc} H S_z ,$$

with $\vec{S} = \hbar\vec{\sigma}/2$. Find and solve the equations of motion for $\vec{S}$.

Problem 35.

The region of space for $z > 0$ contains a magnetic field $\mathbf{B} = B\hat{\mathbf{x}}$. There is no field for $z < 0$. A beam of neutral spin one half particles with mass m and magnetic moment $\mu = \mu\hbar\sigma/2$ is incident from the left of the interface. (The interaction of the magnetic moment with the magnetic field is given by $-\mu \cdot \mathbf{B}$.) The beam has momentum $p = \hbar k$ and makes an angle θ with the $\mathbf{z}$ axis. Take $p^2/2m$ to be greater than $\hbar\mu B/2$.



[a.] Find the wave function for the reflected and transmitted waves for the cases where the spin is parallel and anti-parallel to the magnetic field.

[b.] Write the wave functions for the case the incident spin is parallel to the $\mathbf{z}$ axis.

Problem 36

A spin 1/2 particle, of mass M and Landé factor g is placed in a constant magnetic field pointing in the z direction, $\mathbf{B} = B_0\hat{z}$. In addition, this system is subjected to a rotating magnetic $\mathbf{b}(t) = b\hat{x}\cos(\omega t) + b\hat{y}\sin(\omega t)$. The particle is initially with spin up along the z direction. Find, as a function of time, the probability of finding it with spin down.

Problem 37.

a. Two spin-$\frac{1}{2}$ particles are at rest and initially in a state with total spin-angular momentum zero. In term of the states $|m_1, m_2\rangle$, with $m_i = \pm\frac{1}{2}$ ($\hbar m_i$ is the z-component of the spin of particle i), write out the initial state of this two particle system.

b. A Hamiltonian

$$H = \omega_0(S_{1z} - S_{2z})$$

acts on this system [S_{iz} is the z-component of spin of particle i]. As a function of time, what is the probability of finding the system is a state with total spin-angular momentum equal to $\hbar$?

Problem 38

[a.] Evaluate $[J_z, XY]$, where J_z is the operator for angular momentum in the z direction and X, Y are the x and y components of the position operator.

[b.] Using [a.], above, show that for any $j,\ j',\ m$ and m'

$$(m' - m)\langle j',\ m'|XY|j,\ m\rangle = i\langle j',\ m'|(Y^2 - X^2)|j,\ m\rangle\ .$$

Problem 39

Some process is governed by the matrix elements $\langle j = 1/2, m'|\vec{R}|j = 3/2, m\rangle$. Suppose we know $\langle j = 1/2, m = 1/2|Z|j = 3/2, m = 1/2\rangle = A$. Find all the other matrix elements of Z and of $X \pm iY$.