Physics 215B
Winter 2000

Problem 1

Suppose that the Coulomb potential in the hydrogen atom were modified to a screened one (Yukawa potential)

$$V(r) = -\frac{\alpha\hbar c}{r} e^{-r/b} .$$

The parameter b is much larger than the Bohr radius a_0. Using perturbation theory, find to lowest order in a_0/b the splitting between the S and P states of the $n = 2$ level of hydrogen.

Ignore the fact that the S and P levels are degenearate and use non-degenerate perturbation techniques. A useful formula:

$$\langle n, l, m | r | n, l, m \rangle = a_0 \frac{3n^2 - l(l+1)}{2}$$

Problem 2

The "relativistic dependence of mass on velocity" adds a correction to the Hamiltonian of a hydrogen atom

$$H_1 = -\frac{p^4}{8m^3c^2} .$$

(This may be obtained by expanding $\sqrt{m^2c^4 + p^2c^2} - mc^2$ in powers of $1/c^2$.) To first order in H_1 find the corrections to the energies of the $1S$, $2S$ and $2P$ levels of hydrogen. Compare this to the spin-orbit splitting. The calculations are easier if you write H_1 as

$$H_1 = \frac{-1}{2mc^2}[H_0 - V(r)]^2$$

and expand out the squared term.

Problem 3

Consider a "perturbed" one dimensional harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2 + \epsilon x .$$

This problem can be solved exactly. Nevertheless, find, to second order in ϵ, the corrected energy levels and compare to the exact solution.

Problem 4

A perturbation $gm\omega^2 xy$ is added to the Hamiltonian for a two dimensional harmonic oscillator

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega_0^2(x^2+y^2)\,.$$

Find, to first order in g, the correction to the energy of the (degenerate) first excited states and compare to the exact solution.

Problem 5

Sakurai, problem 5.10, p. 347

Problem 6

Two coupled oscillators are described by the Hamiltonian

$$H = -A\hbar^2\left(\frac{\partial^2}{\partial\phi_1^2} + \frac{\partial^2}{\partial\phi_2^2}\right) - B\cos(\phi_1 - \phi_2)$$

with ϕ_i equivalent to $\phi_i + 2\pi$.

a. What are the eigenfunctions and eigenvalues for the case $B = 0$?

b. For $B << A\hbar^2$, find to lowest non-vanishing order, the correction to the energy of the ground state.

Problem 7

Suppose you didn't know the wave function for the ground state of the hydrogen atom, but guessed that it is of the form

$$\psi(r) = Ne^{-br}\,,$$

with N and b constants. Use the variational method to find these constants.

Problem 8

Sakurai, problem 5.21, p. 350.

Problem 9 [30 points]

Later in the quarter we shall need he ground state energy of an electron moving in the potential of two fixed charges $+e$ located a distance R apart at coordinates $\vec{R}/2$ and $-\vec{R}/2$.

$$H = \frac{-\hbar^2}{2m}\nabla^2 - \frac{\alpha\hbar c}{|\vec{r} - \vec{R}/2|} - \frac{\alpha\hbar c}{|\vec{r} + \vec{R}/2|}\,.$$

A "good" guess for a variational wave function is to take it to be a superposition of hydrogenic wave functions around the two charges,

$$\psi_\pm(\vec{r}) = N_\pm \left[\exp\left(-\frac{|\vec{r}-\vec{R}/2|}{a_0} \right) \pm \exp\left(-\frac{|\vec{r}+\vec{R}/2|}{a_0} \right) \right] ,$$

where $N_\pm$ is a normalization factor. Evaluate $\langle H \rangle$ for both wave function. Which is lower? Several integrals in this problem are straight forward; you will encounter the following ones which are more difficult:

$$I(R) = \int d^3r \exp\left(-\frac{|\vec{r}-\vec{R}/2|}{a_0} \right) \exp\left(-\frac{|\vec{r}+\vec{R}/2|}{a_0} \right) ,$$

$$J(R) = \int d^3r \frac{\exp\left(-\frac{-|\vec{r}-\vec{R}/2|}{a_0} \right) \exp\left(-\frac{|\vec{r}+\vec{R}/2|}{a_0} \right)}{|\vec{r}-\vec{R}/2|} ,$$

$$K(R) = \int d^3r \frac{\exp\left(-2\frac{|\vec{r}+\vec{R}/2|}{a_0} \right)}{|\vec{r}-\vec{R}/2|} .$$

First express your answer in terms of $I(R), \cdots$ without evaluating these explicitly. These may be evaluated by first Fourier transforming the exponentials.

Problem 10

A particle of mass m is acted on by a central three dimensional potential

$$V(r) = \frac{k}{2}(r - r_0)^2 ,$$

where r is the distance from the origin and

$$r_0{}^2 >> \frac{\hbar}{\sqrt{mk}} .$$

Determine the spectrum of the low lying states of this system. Discuss and justify whatever approximations you are using. [Hint: Do not use perturbation theory, but think of an analogy with diatomic molecules.]

Problem 11

In units where the reduced mass of some diatomic molecule is set to $M_{\text{red}} = 1$ and $\hbar = 1$ the energies of some levels are found to be (somewhat idealized):

1.0	3.0	5.0
1.01	3.01	5.01
1.03	3.03	5.03

In these units, find the location of the minimum of the potential and the second derivative of the potential at the minimum.

Problem 12

A certain attractive, three dimensional potential has a ground state with energy $-\epsilon_0$ and wave function $\phi_0(\mathbf{r})$ as well as an excited state with energy $-\epsilon_1$ and wave function $\phi_1(\mathbf{r})$. Two identical bosons (wave function is symmetric under the interchange of the coordinates of the two particles) of mass m are in the excited are in the excited state. A perturbing interaction of the form

$$V = a^3 V_0 \delta(\mathbf{r}_1 - \mathbf{r}_2)$$

is turned on; $\mathbf{r}_{1,2}$ are the coordinates of the two bosons while a and V_0 are constants with the dimensions of length and energy respectively. V allows for the possibility of dropping one of the bosons into the ground state and exciting the other one into the positive energy continuum (autoionization).

What is the condition on ϵ_0 and ϵ_1 for this process to be energetically possible?

Find the expression for the rate of this process. You may treat the continuum states as plane waves and express your answer in term of an integral over products of wave functions.

Problem 13

A heavy nucleus, (Z, A) decays into a lighter one $(Z, A-2)$ by emitting two neutrons (ignore the symmetrization or antisymmetrization of the wave function). The matrix element for this process is

$$\langle \vec{p}_1,\ \vec{p}_2,\ (A, Z-2)|V|(Z, A)\rangle = M\,,$$

where $\vec{p}_i$ are the momenta of the neutrons and M is a constant. The energy released to the neutrons is Δ. What is the rate for this decay?

Problem 14

A neutron of mass m and momentum $\vec{p}_0$ collides with a heavy nucleus. A neutron with momentum $\vec{p}_n$ and a proton of mass m and momentum $\vec{p}_p$ are emitted, leaving behind a nucleus with one proton less. The matrix element for this reaction is

$$\langle \vec{p}_n,\ \vec{p}_p|H_{\text{int}}|\vec{p}_0\rangle = C\vec{p}_n \cdot \vec{p}_p\,,$$

with C a constant. The energy released to the outgoing neutron and proton is $p_0^2/2m + \Delta$.

What is the cross section for the outgoing neutron to be in the energy interval E_n to $E_n + dE_n$?

Problem 15

Sakurai, 5.27, p. 352

Problem 16

Sakurai, 5.28, p. 353

Problem 17

Calculate, to first order in the Born approximation, the differential and total cross section for scattering a particle of mass m by a Yukawa potential,

$$V(r) = g\frac{\exp(-\mu r)}{r}\,.$$

Problem 18

To first order in the Born approximation, find the differential cross section for the scattering of a particle of charge q from a charge distribution $\rho(\mathbf{r})$. Using this result find the differential cross section for the case $\rho(\mathbf{r})$ represents a total charge Q uniformly distributed inside a sphere of radius R.

Problem 19 [20 pts]

In this problem we will obtain some of the properties of the Legendre functions. These are related to the spherical harmonics via

$$P_l(\cos(\theta) = \sqrt{\frac{4\pi}{2l+1}}Y_{l,0}(\theta,\phi)\,.$$

and satisfy the differential equation

$$\frac{d}{dz}(z^2-1)\frac{d}{dz}P_l(z) - l(l+1)P_l(z) = 0\,.$$

[a.] Show that $F_l(z) = \frac{d^l}{dz^l}(z^2-1)^l$ satisfies the above equation. [Hint: Expand $(z^2-1)^l$, perform all the subsequent differentiations and show that the equation is satisfied for each power of z.]

[b.] We will be interested in obtaining $\int_{-1}^{1} dz\ F_l^2(z)$; to do this, first define

$$I_l = \int_{-1}^{1} dz\ (z^2-1)^l$$

and show, by considering $\int_{-1}^{1} dz\ z\frac{d}{dz}(z^2-1)^l$, that $(2l+1)I_l = -2lI_{l-1}$.

[c.] Show that $I_l = 2(-4)^l(l\,!)^2/(2l+1)!$ satisfies the above recursion relation and has the correct value for $l=0$.

[d.] Find

$$\int_{-1}^{1} dz\ F_l^2(z)\,.$$

[Hint: First integrate by parts till all the derivatives are on one of the $(z^2-1)^l$'s.]

[e.] Knowing that

$$\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\ Y_{l,0}^2(\theta,\phi) = 1\,,$$

find the relation between $P_l(z)$ and $F_l(z)$.

[f.] Show that $P_l(1) = 1$. [Hint: Write $(z^2-1)^l$ as $(z-1)^l(z+1)^l$ and perform some of the derivatives in the definition of $F_l(z)$.]

Problem 20

At a some small momentum $p = \hbar k$ only $l = 0$ and $l = 1$ phase shifts are important. Show that

$$\sigma_{\text{Total}} \le \frac{16\pi}{k^2}\,.$$

Problem 21

At some momentum p the scattering of a particle of mass m depends only on S and P wave phase shifts (l=0 and 1). Assume that we know the total cross section, σ_T, and the differential cross section at 90^0, $d\sigma/d\Omega(\theta = 90^0)$. In term of these, what is the scattering amplitude at all angles?

Problem 22

The scattering amplitude for a particle of mass m and energy E scattering off some potential is

$$f(\theta,\phi) = c\cos(\theta) + i\,,$$

where c is a real number.

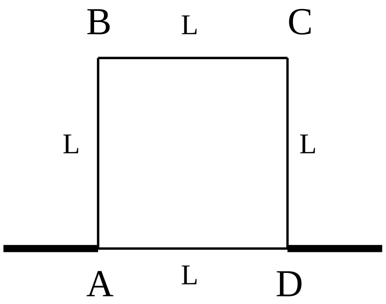
a. In terms of c, what is the differential cross section?

b. What is the total cross section?

c. What is c? [Use the optical theorem.]

Problem 23

A beam of charged particles of mass m, charge q and momentum $p = \hbar k$ is split into two beams at point A and recombined at point D. One beam goes directly from A to D while the other one goes along the edges of the ABCD square. The length of each edge is L.



[a.] If no other potentials are present, what is the ratio of the intensity at D to the intensity at A?

[b.] Now let there be a magnetic field perpendicular to the square. For what values of magnetic flux, Φ, going through the square ABCD will the ratio of the two intensities be one?