TITLE: [CONSISTENT, NON-ABELIAN] Anomaly Geometry

PLAN:
I. REVIEW: BRST – what you learned in QFT
II. CONSISTENT ANOMALY: WZ consistency condition
III. DESCENT: solving the WZ condition + more
IV. REMARKS on physics + geometry
& how they come together

GOAL: VERY HEURISTIC LOOK AT THE MATHEMATICS "UNDER THE HOOD" OF THE NONABELIAN ANOMALY.

WE WILL CONNECT TO MANY OF THE GEOMETRIC TOPICS THAT HAVE COME UP THIS ~~[illegible]~~ SEMESTER IN OUR JOURNAL CLUB, BUT THE GOAL IS THAT YOU SHOULD BE ABLE TO APPRECIATE THE DISCUSSION WITHOUT HAVING PAID TOO MUCH ATTENTION.

References

TEXTBOOKS:
NAKAHARA, CH. 13 (11 FOR TECHNICAL BG)
BERTLMANN, *Anomalies in QFT* CH. 8 & 9 (CH. 7 FOR TECHNICAL BG)

SEE ALSO: GÖCKELER & SCHÜCKER
de AZCÁRRAGA & IZQUIERDO (Advanced)

WEINBERG, CH. 15.7-8, 22.6

REVIEWS & LECTURES:

- Alvarez-Gaume & Ginsparg
- BILAL 0802.0634
- ZUMINO, "CHIRAL ANOMALIES + DIFFERENTIAL GEOMETRY"
- STORA, GIFT LECTURES
- MAÑES, STORA, ZUMINO, "ALG. STUDY OF XRAL ANOMALIES"

ALSO USEFUL: A.J. BRUCE'S MSc THESIS (05)
SH SHAO'S QFT FINAL PRESENTATION (10)

PART I: A REVIEW OF THE YM LAGRANGIAN : { PATH INTEGRAL / FADEEV POPOV / BRST

EVERYONE ALREADY KNOWS THIS!

- NAIVE ACTION DOESN'T WORK:

$$\mathcal{L}_{YM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad \left(\text{or } -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}\right)$$

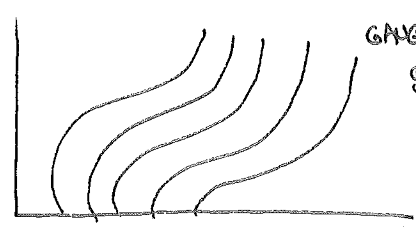
$$= \frac{1}{2}A^\mu \underbrace{(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)}_{}A^\nu$$

KINETIC OPERATOR
INVERSE SHOULD GIVE THE GAUGE FIELD PROPAGATOR

$\boxed{\text{BUT THIS OPERATOR HAS NO INVERSE!}}$

- WE KNOW WHY: PATH INTEGRAL OVER A_μ (unconstrained) CONTAINS ALL GAUGE-REDUNDANT CONFIGURATIONS

  ↳ ie WE ARE INTEGRATING OVER A SPACE THAT IS 'INFINITELY' LARGER THAN THE PHYSICAL CONFIGURATION SPACE.



ALONG A GAUGE ORBIT THE ACTION IS CONSTANT (BY GAUGE INVARIANCE) THUS FOR EACH PHYSICALLY DISTINCT CONFIGURATION IN THE PATH INTEGRAL, THERE IS A MULTIPLICATION BY ∞ ASSOCIATED WITH THE VOLUME OF THE GAUGE GROUP.

THIS IS WHAT IS MAKING OUR NAIVE ACTION MISBEHAVE.

- BUT THIS MEANS $Z = (\text{volume of } G)\, Z_{phys}$ ← PI OVER PHYSICAL CONFIGURATIONS. (GAUGE FIXED)

naive: $Z = \int dA\, e^{iS[A]}$

formally infinite

BUT OVERALL PREFACTOR — EVEN INFINITE ONE — DOES NOT AFFECT THE GENERATING FUNCTION.

SO Z STILL WORKS, WE JUST HAVE TO PULL OUT THE (vol G) FACTOR.

SOLUTION: FADEEV - POPOV PROCEDURE

[We will be very sketchy — you've already done the technical steps in your QFT courses!]

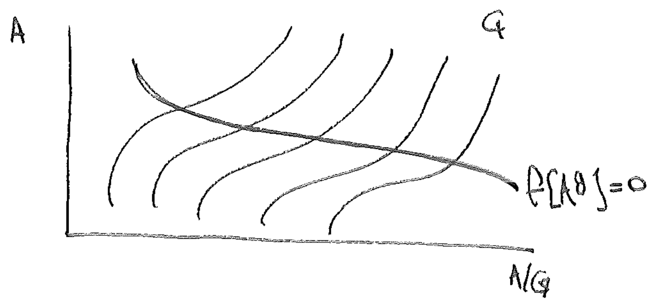
IDEA: INTRODUCE A "LAGRANGE MULTIPLIER" TO ENFORCE A GAUGE CHOICE.

TRICK:
$$1 = \int dg \left(\det \frac{\delta f[A^g]}{\delta g}\right) \delta(f[A^g])$$

INTEGRAL OVER GAUGE ORBITS! THIS IS PRECISELY THE PREFACTOR THAT WE WANT TO PULL OUT

JACOBIAN $\equiv \Delta_{FP}$

$f[A^g]=0$ IS A GAUGE-FIXING CONDITION



CAVEAT: $f[A^g]=0$ MUST HAVE A UNIQUE SOLUTION $g(x)$ $\forall$ $A_\mu(x)$. OTHERWISE, CAN GET PROBLEMS, eg GRIBOV AMBIGUITY.

eg COULOMB GAUGE CANNOT BE GLOBALLY DEFINED.

FYI: $f(A^g)=0$ NOT POSSIBLE $\forall A$, OTHERWISE SPACE IS TOPOLOGICALLY TRIVIAL. $f(A^g)=0$ IS POSSIBLE FOR SMALL A $\longrightarrow$ PERT THY.

THEN WE ARGUE THAT:

$$Z = \int dA\, dg\, \Delta_{FP}[A^g]\, \delta(f[A^g])\, e^{iS[A]}$$

$$= \int dg \cdot \int dA\, \Delta_{FP}[A]\, \delta(f[A])\, e^{iS[A]}$$

bk dA, $S[A]$, dg ARE GAUGE INVT.

$$= \int dg \cdot Z_{phys}$$

$(\mathrm{Vol}\, G)$

GAUGE FIXED.

BUT WE NEED TO MASSAGE IT TO LOOK LIKE A NORMAL PARTITION FUNCTION. HAVE TO SEND $\Delta_{FP}[A]\, \delta(f[A])$ INTO THE EXPONENTIAL.

IN FACT, IT IS EASY TO GENERALIZE TO $f[A]-b(x)=0$ GAUGE FIXING, AND THEN INTEGRATE OVER GAUSSIAN FUNCTIONAL WEIGHT FOR $b(x)$ SO THAT

① WE CAN ENCODE THE δ-FUNCTION INTO THE $\mathcal{L}$

② WE INTRODUCE THE R_ξ GAUGE PARAMETER

$$Z = (\mathrm{Vol}\,G)\int db\, \underbrace{e^{-\frac{i}{2\xi}\int dx\, b^a(x)\, b^a(x)}}_{\substack{\text{GAUSSIAN WEIGHT} \\ \text{(threw out overall normaliz)}}} \int dA\, \Delta_{FP}[A]\, \delta(f[A]-b)\, e^{iS[A]}$$

$$= (\mathrm{Vol}\,G)\int dA\, \underline{\Delta_{FP}[A]}\, e^{iS[A] - \frac{i}{2\xi}\int dx\, f^a[A]\, f^a[A]}$$

$$\Delta_{FP}[A] = \det \frac{\delta f[A^g]}{\delta g}\bigg|_{g=\mathbb{1}} = \det \frac{\delta f[A^\Lambda]}{\delta \Lambda}\bigg|_{\Lambda=0}$$

CONSIDER INFINITESIMAL GAUGE TRANSF : $g = \mathbb{1} + \Lambda$

TO PUT THIS IN THE EXPONENTIAL WE USE THE TRICK THAT <u>FERMIONIC</u> GAUSSIAN INTEGRALS ARE PROPORTIONAL TO DETERMINANTS (cf. BOSONIC GAUSSIANS GO LIKE AN INVERSE POWER OF THE DETERMINANT.)

$$\det \frac{\delta f}{\delta \Lambda} = \int d\bar{c}\, dc\; e^{-i\int dx\, \bar{c}\frac{\delta f}{\delta \Lambda} c}$$

DISAPPEAR IN AXIAL GAUGE, BUT PROPAGATOR IS UGLY.

<u>FP GHOSTS</u> : INDEPENDENT ADJOINT SCALAR FIELDS w/ GRASSMAN STATISTICS.
→ REPRESENTING <u>NEGATIVE</u> DOF.

<u>A GOOD QUESTION</u> : WHY ARE THE FP GHOSTS ANTI-COMMUTING?

WE WILL TRY TO ANSWER THIS GEOMETRICALLY. → WILL IDENTIFY w/ MAURER-CARTAN FORM.

<u>RESULT</u> :

$$Z = (\mathrm{Vol}\,G)\int dA\, d(\text{FERMIONS})\, e^{i\int d^4x\, \mathcal{L}_{YM} + \mathcal{L}_{\psi} + \mathcal{L}_{GF} + \mathcal{L}_{FP} + \text{SOURCES}}$$

- $\mathcal{L}_{FP} = -\bar{c}^a \partial^\mu D_\mu^{ba} c^b$ ⇐ CONTAINS KINETIC PART!

BRS SYMMETRY (or "BRST", but Tyutin never published)

Heuristic motivation: What has happened to gauge invariance? It seems like we've thrown it out completely. Certainly Z_{phys} is gauge fixed.

BUT: We needn't work w/ Z_{phys}. We could still work w/ Z in the form
$$Z = (\mathrm{Vol}\ G)\, Z_{phys} \quad ,$$

Z is equivalent to Z_{phys} as a partition function (overall normalization is irrelevant!), and Z is supposed to be gauge invariant:

$$Z = \int d(\text{gauge orbits}) \cdot Z_{phys}$$
fixed gauge orbit

like: $\int d^4x\ V^\mu(x)$ is not Lorentz inv't.

but: $\int d\Omega \int d^4x\ V^\mu(x)$ is Lorentz inv't.

integrate over orientations of V^μ

(very rough analogy.)

So we could ask: where did the gauge invariance go? Is there any trace of it in Z_{phys}?

"Practical" motivation: We want (need?) some residue of gauge symmetry to prove the renormalizability of YM.

GAUGE TRANSFORMATION : $A^a_\mu \xrightarrow{g(x) = 1 + \lambda^a(x)T^a} A^a_\mu + D^{ab}_\mu \lambda^b(x)$

Claim: We have other fields (semi dynamical) in our Z and we haven't specified their transformation.

We can generalize gauge invariance to include the transformation of the ghost fields

$\to$ BRS (T)

NOTE: FOR THIS TALK, I REALLY DON'T CARE ABOUT THE DETAILS OF THE BRST TRANSFORMATION.

converts into ghost

SIMPLEST FORM:

$$\begin{cases} \delta_c A_\mu = \boxed{D_\mu v} \\ \delta_c c = -\tfrac{1}{2}[v,v] \\ \delta_c \psi = -v\psi \\ \delta_c \bar{\psi} = -\bar{\psi} v \end{cases}$$

this is a gauge transf where the ghost is the trans. param!

adjoint

OBS. δ_c CARRIES GHOST #

REMARKS:
- WHAT ABOUT $\delta_c \bar{v}$?
  YOU CAN DETERMINE THIS MOST EASILY BY INTRODUCING AN AUXILIARY FIELD. SEE, eg, PESKIN § 16.4.
  THIS IS A TECHNICAL DETAIL.
- TRANSF INCLUDE (ie GENERALIZE) 'NORMAL' GAUGE TRANSF.
- "CHOSEN" SO THAT $\boxed{\delta_c^{\,2} = 0}$, NILPOTENT

PHYSICS: THE PHYSICAL CONTENT OF A GAUGE THEORY LIVES IN THE KERNEL OF THE BRST OPERATOR MODULO THE IMAGE OF THE BRST OPERATOR.

↳ I.E. THE PHYSICS IS IN THE COHOMOLOGY.

VERY HEURISTICALLY: FOR EXAMPLE, I CAN ALWAYS ADD A FINITE LOCAL COUNTER TERM FROM THE IMAGE OF δ_c (eg $\delta_c f$). THIS TERM IS INDEPENDENTLY BRST-INVARIANT ($\delta_c^{\,2} = 0$) BUT CAN BE USED TO CANCEL ANY OTHER TERMS IN THE IMAGE OF δ_c.

Physically (see Peskin 16.4): δ_c TRANSFORMS UNPHYSICAL GHOSTS & GLUON POLARIZATIONS INTO ONE ANOTHER, BUT ANNIHILATES PHYSICAL GLUON POLARIZATIONS.

OFF HAND REMARKS

1. WE CAN TAKE THIS FURTHER & USE BRS AS A BASIS for QUANTIZING OUR THEORIES. See Polchinski I, Weinberg II, 15.8
   This is done, for example, in string theory. (It is usually the third way that you learn to quantize the bosonic string.)

2. BRST is important for the manifest unitarity of the S-matrix since it can be used to show that physical polarizations don't leak into unphysical polarizations.
   See Peskin 16.4

"ON HAND" REMARKS

1. GHOSTS ARE STILL "MYSTERIOUS"
   - ⇒ Why do they anticommute?
   - ⇒ Geometric meaning?
   - ⇒ Properties of δ_c?

haven't really done anything, just words

2. THE BRST INVARIANCE WE'VE "SHOWED" HERE IS @ THE CLASSICAL LEVEL.

   A GOOD QUESTION IS: IS THE QUANTUM ACTION BRST INVARIANT?

$$Z = e^{iW}$$

QUANTUM ACTION (BY WHICH I MEAN GENERATING FUNCTIONAL OF CONNECTED DIAGRAMS)

WARD / SLAVNOV-TAYLOR IDENTITY : $\boxed{\delta_c W = 0}$

OR

"ANOMALOUS WARD IDENTITY"

ANOMALY (non-Abelian) : $\boxed{\delta_c W = G \neq 0}$

KEY RESULT (somewhat trivial): WZ CONSISTENCY: $\delta_c^2 W = \delta_c G = 0$

PART II : CONSISTENT ANOMALY

GOAL: INVESTIGATE ANOMALOUS WARD IDENTITIES A BIT MORE
next part: a few words about solving

REVIEW: ANOMALY: SYM OF CLASSICAL ACTION IS NOT A SYM OF THE QUANTUM ACTION

PERTURBATIVELY: ABJ | CHIRAL | SINGLET ANOMALY FROM TRIANGLE DIAGRAMS w/ CHIRAL FERMIONS.

NON-PERTURBATIVELY: FUJIKAWA — COMES FROM THE NON-INVARIANCE OF THE PATH INTEGRAL MEASURE.

~~RELATED TO NON-ABELIAN~~

NON-ABELIAN ANOMALY: RELATED TO GAUGE BACKGROUND of INSTANTONS

in fact, Yang already remarked on this relation in his talk.

see Flip's A EXAM
"LOTS OF PEOPLE ARE READING IT" – Z. KOMARGODSKI

ANOMALIES ARE INTIMATELY TIED TO FERMION ZERO-MODES (massless excitations); the presence of a nontrivial GAUGE CONFIGURATION CAN GENERATE FERMION ZERO MODES. THIS IS ESSENTIALLY WHAT THE 't HOOFT VERTEX DOES. [MORE INTUITIVE: PESKIN'S EXPLANATION of HOW A GAUGE TRANSF CAN SHIFT THE ∞ LADDER ~~of the~~ IN THE FERMION SPECTRUM.]

CAN FURTHER SHOW THAT ANOMALIES MANIFEST THEMSELVES ONLY IN THE PHASE OF THE DIRAC OPERATOR.
SEE NAKAHARA 13.3.

WE ARE PARTICULARLY INTERESTED IN THE NON-ABELIAN ANOMALY & ITS RELATION TO BRS & GEOMETRY.

PHYSICS: AN ANOMALY IN AN HONEST-TO-GOODNESS GAUGE SYMMETRY IS AN INCONSISTENCY IN THE THEORY & CANNOT BE TOLERATED. (this is why SUPERSTRINGS LIVE IN 10 DIM.) THIS IS BECAUSE GAUGE SYM IS NOT A SYM, IT IS A REDUNDANCY IN OUR DESCRIPTION. HOWEVER, FOR THIS ANALYSIS WE MAY "WEAKLY GAUGE" AN ANOMALOUS GLOBAL SYMMETRY.

There is a <u>lot</u> of physics that I am skipping re: anomalies.
FORTUNATELY, YOU ALL ALREADY KNOW ALL OF IT (see Weinberg II or the book by Fujikawa.)

MARIO WILL COVER GRAVITATIONAL ANOMALIES IN 2 WKS. IF YOU HAVE ANY QUESTIONS, PAUL GINSPARG IS ONE OF THE EXPERTS ON ALL OF THIS.

<u>SOME FAMILIAR RESULTS</u> (perturbative analysis)

$$\partial^\mu j^5_\mu = \frac{1}{4\pi^2}\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,\partial_\mu\left(A_\nu\partial_\rho A_\sigma + \tfrac{2}{3}A_\nu A_\rho A_\sigma\right)$$

$$D_\mu j^{\mu a}_{NA} = \frac{1}{24\pi^2}\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,T^a\partial_\mu\left(A_\nu\partial_\rho A_\sigma + \tfrac{1}{2}A_\nu A_\rho A_\sigma\right)$$

the one we care about in this talk

same rough structure... interesting factor of 2/3 vs 1/2

1/2 FROM $\frac{1}{2}\gamma_5$ IN P_L PROJECTION
2/3 FROM RG SCHEME WHICH RESPECTS GAUGE SYMS, BUT NOT GLOBAL (# OF GAUGE VERTICES IN TRIANGLE)

[from perturbative point of view we can see where these numerical factors come from... is there something deeper?]

IT IS MORE NATURAL TO WRITE THIS IN GROWN UP NOTATION:

$$D{*}j^a = \frac{1}{24\pi^2}\,\mathrm{Tr}\,T^a\,d\left(A\,dA + \tfrac{1}{2}A^3\right)$$

GAUGE FIELDS ARE ONE-FORMS

CURRENT IS MOST NATURALLY A 3-FORM $*j$
~~SO $d{*}j$ IS A 4-FORM THAT CAN GIVE CHARGE WHEN INTEGRATED.~~

<u>REMARKS</u>

WE CAN WRITE $d{*}j^5 = \frac{1}{4\pi^2}\mathrm{Tr}\,F^2 \sim F\tilde{F}$ AS EXPECTED.

BUT WE <u>CANNOT</u> WRITE $D{*}j^a$ AS SOMETHING LIKE $F\tilde{F}$.
THIS SOUNDS WRONG! COVARIANCE REQ. $D{*}j^a \sim F\tilde{F}$, RIGHT?

TURNS OUT THERE ARE <u>TWO WAYS</u> TO WRITE NON-ABELIAN ANOMALY
1. <u>CONSISTENT</u>: $\delta_\alpha W = 0$ BUT $D{*}j^a$ NOT COVARIANT
2. <u>COVARIANT</u>: $D{*}j^a \sim F\tilde{F}$, BUT $\delta_\alpha W \neq 0$

THEY DIFFER BY A LOCAL <u>BARDEEN-ZUMINO CURRENT</u> (regularization)
THIS TALK: ONLY #1

WHAT DOES THE DIVERGENCE of THE CURRENT $D * j^a \sim T_r T^a d(AdA + \frac{1}{2}A^3)$
HAVE TO DO WITH THE WZ CONSISTENCY EQN?

$$Z = e^{iW[A]} = \int d(\text{fields})\, e^{i\int d^4x\, \bar{\Psi}_L i \not{\partial} \Psi_L + A^a_\mu j^{\mu a}}$$

$$\delta W[A] = \int d^4x \underset{\substack{\shortparallel \\ Dc}}{\delta A} \frac{\delta}{\delta A} W[A] = -\int d^4x\, c^b \underbrace{\boxed{D^{ab}_\mu \frac{\delta}{\delta A^a_\mu}}}_{\equiv X} W[A] \equiv c \cdot X W$$

$\equiv X$ INFINITESIMAL GAUGE OPERATOR

ghost plays role of gauge transf. parameter

(recall $D \to g^{-1} D g \Rightarrow A \to A + D\lambda$ FOR $g \sim (1+\lambda)$)

AT LEAST W/RT A TRANSF, $c \cdot X \sim \delta_c$

FROM EXPLICIT EXPRESSION ABOVE, $\boxed{\frac{\delta}{\delta A^a_\mu} W[A] = \langle j^a \rangle}$

$$\Rightarrow \boxed{-D^{ab}_\mu \langle j^{\mu b} \rangle = G^a[A]}$$

RHS OF ANOMALOUS WARD IDENTITY
$\delta_c W = G$

SO INDEED, THE ANOMALOUS WARD IDENTITY IS TELLING US ABOUT A NONVANISHING DIVERGENCE of A CLASSICAL CURRENT. ✓

FOR FUTURE REF: WE CAN ALSO COMPLETE THE BRST TRANSF BY INCLUDING THE GHOST TRANSFORMATION:

$$\delta W[c] = \int d^4x \underset{\substack{\uparrow \\ -\frac{1}{2}[c,c]}}{\delta c} \frac{\delta}{\delta c} W[c] = -\int d^4x\, c^a \tfrac{1}{2} f^{abc} c^b \frac{\delta}{\delta c^c} W[c]$$

$$\equiv c \cdot X_c W$$

THEN $\delta_c = \begin{cases} c \cdot X & \text{ON } W[A] \\ c \cdot X_c & \text{ON } W[c] \end{cases}$

NOTE: SUBSCRIPT c LABELS GHOST IF

FACT: $[X^a(x), X^b(y)] = f^{abc} X^c(x)\, \delta(x-y)$

Pf. ~~KISS MY ASS~~. [IT'S NOT PARTICULARLY ILLUMINATING] LOOK IT UP OR DERIVE IT YOURSELF.

- THE GAUGE (BRS) OPERATORS FURNISH AN INFINITE DIMENSIONAL REPRESENTATION OF THE GROUP.

this is as we expect tautologically

ANOTHER WAY TO CAST THE WZ CONSISTENCY CONDITION IS THUS:

$$\boxed{X^a(x)\, G^b[A](y) - X^b(y) G[A](x) = f^{abc} G^c[A](x)\, \delta(x-y)}$$

HERE WE'VE ONLY USED THE "GAUGE TRANSFORMATION" PART OF THE BRST TRANSFORMATION. WE CAN GO A BIT FURTHER USING THE ENTIRE BRST SYMMETRY + ITS COHOMOLOGY

DEFINE $G[c,A] = \int d^4x\, c^a G^a[A]$

RHS OF ANOMALOUS WARD IDENTITY $\delta_c W[A] = G[A]$

THE WESS-ZUMINO CONSISTENCY CONDITION IS

$$\boxed{\delta_c G[c, A] = 0}$$

one can check that this is indeed equivalent to the boxed equation above

REMINDER: · δ_c CARRIES GHOST # = 1
· MOD OUT BY 'TRIVIAL' SOLUTION $G[c,A] = \delta f[A]$, (LOCAL FUNCTIONAL)
THESE CAN BE REMOVED BY FINITE COUNTERTERMS

SO THE ANOMALY IS THE COHOMOLOGY H^1 OF S @ GHOST NUMBER UNITY. (local functionals of ghost #1 modulo terms like δ(loc func of ghost # 0).

PART III: SOLVING THE WZ CONSISTENCY CONDITIONS

THERE IS A POWERFUL MATHEMATICAL TOOL FOR SOLVING THE WZ CONDITION. IT ALLOWS YOU TO COMPLETELY DETERMINE ~~[illegible]~~ $G(c,A)$ UP TO OVERALL NORMALIZATION & TURNS OUT TO RELATE SEVERAL PHYSICALLY RELEVANT QUANTITIES. IN FACT, THIS IS WHY THE WZ CONDITIONS ARE SO POWERFUL

→ ONE CAN EVEN USE THE WZ CONSISTENCY CONDITION AS A FORMAL DEFINITION OF THE ANOMALY!

I DON'T WANT TO GET BOGGED DOWN IN MATHEMATICAL MACHINERY, SO WE WILL ONLY GO OVER THE TECHNIQUE VERY HEURISTICALLY. FORTUNATELY, YANG'S PREVIOUS BSM JC TALKS ON CHARACTERISTIC CLASSES INTRODUCES THE TECHNICAL FRAMEWORK ADMIRABLY.

THE TECHNIQUE IS CALLED THE "STORA-ZUMINO DESCENT EQUATIONS"

FIRST LET'S GET A FLAVOR OF WHAT'S GOING ON BEHIND ANY FANCY MATH TO SEE THAT ~~IS~~ IT IS PLAUSIBLE THAT THE WZ CONDITIONS COULD BE USEFUL.

"EXAMPLE": WHAT IS THE $G[c,A]$ FOR THE NON-ABELIAN ANOMALY?

remark: seems silly that this is an 'example' — isn't this the whole point? Turns out there's much more.

A NICE ANSATZ:

ghost

$$G[c,A] = \frac{-1}{24\pi^2}\int \mathrm{Tr}\, c\Big((dA)^2 + c_1\, dA\cdot A^2 + c_2 A(dA)A + c_3 A^2 dA + c_4 A^4\Big)$$

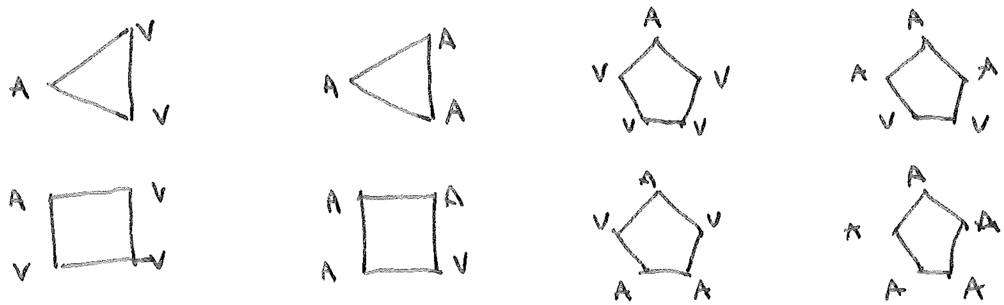
normalization fixed by the triangle diagram

NOW YOU MIGHT JUSTIFIABLY PROTEST: WHY DID WE NEED TO DO A PERTURBATIVE FEYNMAN DIAGRAM CALCULATION TO DETERMINE THE OVERALL COEFFICIENT? DOESN'T THE TRIANGLE DIAGRAM TELL US EVERYTHING WE NEED TO KNOW ANYWAY?

1. IN PRINCIPLE, WE DON'T CARE ABOUT THE OVERALL COEFFICIENT. WE COULD HAVE IGNORED IT & JUST WORRIED ABOUT RELATIVE COEFFICIENTS OF THE TERMS.

2. NAIVELY, THERE ARE OTHER DIAGRAMS

   eg. CONSIDER $SU(3)_V \times SU(3)_A$ OF QCD



FINITE, BUT REQ FOR VECTOR WARD IDENTITY DUE TO DIVERGENT SEAGULLS (Bardeen '69)

IT IS A FACT THAT THE ANOMALY IS 1-LOOP EXACT, BUT IN PRINCIPLE THERE ARE OTHER 1-LOOP DIAGRAMS BESIDE THE TRIANGLE.

[do the boxes actually matter?]

3. FURTHER, WE COULD ALSO ADD LOCAL COUNTER TERMS TO $W[c, A]$ THAT COULD SHIFT THE ANOMALY.

NOW LET'S SEE HOW THIS MIGHT WORK.

$\delta_c G[c, A] = 0$

CONSIDER THE c_4 TERM:

$$\delta_c \mathrm{Tr}[c A^4] \qquad A \to A + Dc$$

some work

$$= \underbrace{\mathrm{Tr}[-c A^4 c]} + \mathcal{O}(A^3, c\,dc)$$

$\nexists$ NO OTHER TERMS IN $\delta_c G$ THAT CAN CANCEL THIS $\Rightarrow c_4 = 0$.

WE CAN KEEP CHUGGING ALONG W/ THE $c_{1,2,3}$ TERMS. WE DON'T NEED THEM TO CANCEL, WE JUST NEED THEM TO BE TOTAL DERIVATIVES UNDER THE d^4x.

WE END UP W/ $c_1 = -c_2 = c_3 = 1/2$

WHICH RECOVERS THE USUAL FORM OF THE NON ABELIAN ANOMALY.

Stora Zumino DESCENT EQS

"CHAIN OF"

ALGEBRAIC MANIFESTATION: STORA, BRST OPERATOR → we'll discuss this
GEOMETRIC MANIFESTATION: ZUMINO → next part (briefly)

TOPOLOGICAL MANIFESTATION: INDEX THM

↳ SEE DEAN'S TALK THIS SEMESTER

PHYSICAL MANIFESTATION: INTER-RELATION OF ALL OF THESE CHERN-THINGIES. → SEE YANG'S TALKS

really pulls together a lot of the ideas that we've been developing this semester.

USUALLY CALLED s

BY NOW YOU SHOULD BE CONVINCED THAT δ_c IS A KIND OF (EXTERIOR) DERIVATIVE.

$\delta_c^2 = 0$ & PHYSICS LIVES IN ITS COHOMOLOGY

SIMILAR TO d

IN FACT: ONE CAN CHECK THAT $\{\delta_c, d\} = 0$

POINCARÉ LEMMA:

GHOST #

$$\delta_c G[c,A] = \delta_c \int d^4x\, c^a G^a[A] = 0 \Rightarrow \boxed{\delta_c (c^a G[A]) = -dQ^2_3(c,A)}$$

FORM #

(AT LEAST LOCALLY, WE WON'T DISCUSS THE GENERALIZATION TO NONTRIVIAL GAUGE BUNDLES ~~OTHER~~ OTHER THAN TO SAY THAT SUCH A GENERALIZATION IS POSSIBLE)

↳ See Bertlmann ch. 9.2

see Yang's talk

Q IS ONE OF THE CHERN-SIMONS FORMS

FOR A SYM INV. POLYNOMIAL $P(F^n)$, $P(F^n) = dQ_{2n-1}(A,F)$

eg $\mathrm{Tr}\, F^n$

Q IS GENERICALLY CALLED A <u>TRANSGRESSION</u>. FROM YANG'S TALK (see also his notes, Nakahara, Bertlmann) WE KNOW THAT THERE IS AN EXPLICIT FORMULA FOR THE TRANSGRESSION:

$$P(F^n) = dQ_{2n-1} \Rightarrow \boxed{Q_{2n-1}[A,F] = n\int_0^1 dt\, P[A,\, tF + (t^2-t)A^2]}$$

DETAILS ARE NOT IMPORTANT. THE POINT IS:

<u>FACT</u>: GIVEN $P(F^n)$, $\exists$ EXPLICIT FORMULA FOR Q_{2n-1}

<u>REMARK</u>: THE FUNNY WAY THAT F APPEARS ON THE RHS COMES FROM <u>TOPOLOGICAL INVARIANCE</u> OF AN INVARIANT POLYNOMIAL. GIVEN TWO CONNECTIONS THAT ARE HOMOTOPICALLY RELATED

↳ ie COULD WRITE $A_t = tA + (1-t)\bar{A}$

$$\int P[F^n] = \int P[\bar{F}^n] \Rightarrow P[F^n] - P[\bar{F}^n] = dQ[\bar{A}, A]$$

TO DERIVE THE FORMULA FOR Q, ONE USES THE HOMOTOPIC CONNECTION A_t. (typically we're interested in the case where $P[\bar{F}^n] = 0$.)

NOW SUPPOSE WE HAVE THE INV'T POLYNOMIAL $P[F^n] = \mathrm{Tr}\, F^n$.

POINCARÉ LEMMA: $\mathrm{Tr}\, F^n = dQ^0_{2n-1}$

CONSIDER $\delta_c Q^0_{2n-1}$: $d(\delta_c Q^0_{2n-1}) = -\delta_c d Q^0_{2n-1} = -\delta_c \mathrm{Tr} F^n = 0$

POINCARÉ LEMMA: $\delta_c Q^0_{2n-1} = -d\underbrace{Q^1_{2n-2}}_{\text{GHOST \# 1 SINCE } \delta_c \text{ HAS GN.}}$ ← $\exists$ some Q^1_{2n-2}

CONSIDER $\delta_c Q^1_{2n-2}$: $d(\delta_c Q^1_{2n-2}) = -\delta_c d Q^1_{2n-2} = \delta_c^2 Q^0_{2n-1} = 0$

POINCARÉ LEMMA ... & SO FORTH! UNTIL $\delta Q^{2n-1}_0 = 0$

<u>DESCENT EQNS</u>:

$$\boxed{\begin{aligned} \mathrm{Tr}\, F^n - dQ^0_{2n-1} &= 0 \\ \delta_c Q^0_{2n-1} + dQ^1_{2n-2} &= 0 \\ \delta_c Q^1_{2n-2} + dQ^2_{2n-3} &= 0 \\ &\vdots \\ \delta_c Q^{2n-1}_0 &= 0 \end{aligned}}$$

eg. FOR $n=2$, $P(F^2)$

$$Q^0_3 = \mathrm{Tr}\,[AdA + \tfrac{2}{3}A^3]$$ ← CHIRAL ANOMALY! NOT A COINCIDENCE

$$Q^1_2 = \mathrm{Tr}\, c\,dA$$

$$Q^2_1 = -\mathrm{Tr}\, c^2 A$$

$$Q^3_0 = -\tfrac{1}{3}\mathrm{Tr}\, c^3$$

[$n=3$ gives NA ANOMALY @ 1st ORDER IN c]

~~FACT~~ OBSERVATION: $\underline{\underline{\mathrm{Tr}\, F^n}} = dQ^0_{2n-1}$

↑ ABELIAN (CHIRAL) ANOMALY IN $2n$ DIMENSIONS!

FACT: Q^1_{2n-2} IS THE NON-ABELIAN ANOMALY IN $(2n-2)$ DIM!

- SANITY CHECK: WHAT HAPPENED TO WZ CONSISTENCY? WE ORIGINALLY STARTED w/

$$\delta_c\,(c^a G^a[A]) = -dQ^2_3\,[c, A]$$

THEN WE WROTE SOME 'ARBITRARY' $P(F^n) = \mathrm{Tr}\, F^n$ AND WROTE OUT A CHAIN OF TRANSGRESSIONS.

WHAT DO THE Q's OF $\mathrm{Tr}\, F^n$ HAVE TO DO WITH THE Q^2_3 APPEARING IN THE WZ FORMULA?

→ JUST INTEGRATE OVER SPACETIME

DESCENT EQ: $\delta_c Q^1_{2n-2} + dQ^2_{2n-3} = 0$

$$\Rightarrow \underbrace{\int d^4x\;\delta_c Q^1_{2n-2} = 0}_{\text{SATISFIES WZ CONSISTENCY! ✓}}$$

↳ OF COURSE, CAN ADD LOCAL FUNCTIONALS AS COUNTER TERMS [physics is in the cohomology]

VERY INTERESTING RELATION BTWN ABELIAN & NONABELIAN ANOMALIES IN DIFFERENT DIMENSIONS!

PART IV: REMARKS (GEOMETRY & OTHER SILLY THINGS)

DESCENT EQUATIONS

- CAN GENERALIZE TO NONTRIVIAL GAUGE BUNDLES.
- COHOMOLOGY FROM CONSISTENCY CONDITION: $H(\delta_c)$

  $\delta_c G[c, A] = 0 \quad w/ \quad \delta_c^2 = 0$; mod by $G \sim \delta_c F[A]$
- COHOMOLOGY FROM DESCENT EQN: $H(\delta|d)$

  $\delta_c Q^1_{2n-2} + d Q^2_{2n-3} = 0$

  mod by $Q^1_{2n-2} \sim s\tilde{Q}^0_{2n-1} + d\tilde{Q}^1_{2n-2}$
- GENERAL SOLUTIONS TO THE DESCENT EQS WERE FOUND BY STORA USING THE "RUSSIAN FORMULA", THIS IS THE WAY IT IS TYPICALLY DONE IN TEXTBOOKS
- CAN ALSO SOLVE THE DESCENT EQS STARTING FROM THE BOTTOM SEE: PIGUET & SORELLA ~~EDS~~ LNP M28 (1995)

PHYSICS OF THE CHERN-SIMONS FORMS

WE MET THESE IN YANG'S TALK. NOW WE SEE THAT THEY ARE ALL LINKED VIA THE DESCENT EQS.

Q^0_{2n-1} : QUANTUM ACTION OF TOPOLOGICAL (CS) THEORIES $\int$ ABELIAN ANOM IN 2n-DIM (SEE YONG-HUI'S TALK, SPRING 2010)

Q^1_{2n-2} : NONABELIAN ANOMALY IN (2n-2) DIM. (UP TO NORMALIZ)

Q^2_{2n-3} : SCHWINGER TERM OF EQUAL TIME COMMUTATORS

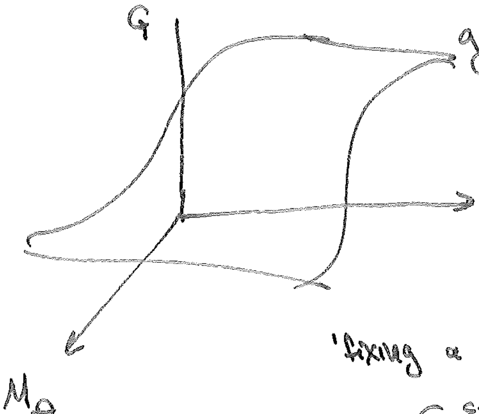
Q^3_{2n-4} : IN QM, RELATED TO FAILURE OF JACOBI IDENTITY @ MAG MONOPOLE

↳ higher terms have no known interpretation, but are commonly believed to be physically significant

ZUMINO'S GEOMETRIC APPROACH

ZUMINO ARRIVED @ THE CONSISTENCY CONDITIONS & DESCENT EQS FROM A MORE GEOMETRIC (& ELEGANT) FORMALISM. WE WON'T GO INTO IT EXCEPT TO HIGHLIGHT GEOMETRICAL INTERPRETATIONS FOR ASPECTS OF THE 'ALGEBRAIC' APPROACH THAT WERE SOMEWHAT AD-HOC.

MAIN IDEA: EXTEND SPACETIME MANIFOLD TO INCLUDE THE PARAMETER SPACE OF THE GAUGE GROUP.



$g(x,\theta): M \times M_\theta \to G$

or $S^4 \times S^\theta \to G$

M MINKOWSKI, $\{x\}$

'fixing a gauge'

GAUGE PARAM SPACE, $\{\theta\}$

EXTERIOR DERIVATIVE IN M_θ DIR

$$\delta = \frac{\partial}{\partial\theta^a}\, d\theta^a \qquad \delta^2 = 0$$

NOW JUST "DO DIFFERENTIAL GEOMETRY" ON THIS.

INFINITESIMAL GAUGE TRANSFORMATION GIVEN BY THE M_θ MAURER-CARTAN FORM

Recall in GEOM OF LIE GROUPS THE MC FORM CONVERTS AFFINE TAN VECTORS TO ELEMENTS OF THE LIE ALGEBRA.

$$c = g^{-1}\delta g = c_a(x,\theta)\, d\theta^a \leftarrow \text{1 FORM!}$$

FADEEV-POPOV GHOST, WHICH PLAYED ROLE OF GENERALIZED GAUGE TRANS PARAM IN BRST.

↪ 1-FORM: "EXPLAINS" ANTICOMMUTING (FERMIONIC) IDENTITY!

EXTERIOR DERIVATIVES WORK AS WE'D EXPECT,

eg VOL FORM = $dx^1 \wedge \cdots \wedge dx^4 \wedge d\theta^1 \wedge \cdots \wedge d\theta^p$

EXTERIOR DIFFERENTIAL OPERATOR ON WHOLE SPACE:

$\Delta = d + \delta$ w/ $\Delta^2 = 0$

↳ 'explains' $d\delta_c + \delta_c d = 0$

IE THE FORM ALGEBRA PROVIDES ANTISYMMETRIZATION

- <u>BRS ALGEBRA</u>: VERY AD HOC WHEN WE INTRODUCED IT. WE REQ. THAT TRANS OF A IS A GAUGE TRANS, BUT THE OTHER TRANSFORMATIONS WERE ENGINEERED TO MAKE δ_c NILPOTENT $(\delta_c^2 = 0)$.

  IN ZUMINO'S PICTURE, THE BRS ALGEBRA ~~...~~ APPEARS <u>AUTOMATICALLY</u> AS THE <u>MAURER-CARTAN STRUCTURE EQS</u>.

  ↓ FROM THE GEOMETRY OF THE SPACE! (eg FROM VIELBEIN THEORIES OF GRAVITY)

eg/

COMES FROM $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega[X,Y]$

FOR A LIE GROUP w/ 1-FORMS θ,

$$d\theta^a(X_b, X_c) = -f^a{}_{bc}$$

BUT: $dx^\nu dx^\mu (X,Y) = X^\nu X^\mu - Y^\nu X^\mu$

$$\Rightarrow \theta^{\bar{b}} \theta^{\bar{c}} (X_b, X_c) = \delta^{\bar{b}}{}_b \delta^{\bar{c}}{}_c - \delta^{\bar{b}}{}_c \delta^{\bar{c}}{}_b$$

$$\Rightarrow \boxed{d\theta^a = -\tfrac{1}{2} f^a{}_{bc} \theta^b \theta^c}$$

↓

$$\Rightarrow \boxed{\delta v = -v^2 = -\tfrac{1}{2}[v,v]}$$

THUS: STRUCTURE EQS ⟶ BRS ALG

↑

DETERMINED BY GEOMETRY