# Flight of the Warped Penguins 

 AND OTHER WEAKLY COUPLED TALES OF STRONG DYNAMICSA Dissertation<br>Presented to the Faculty of the Graduate School of Cornell University<br>in Partial Fulfillment of the Requirements for the Degree of<br>Doctor of Philosophy<br>BY<br>Philip Tanedo<br>August 2013

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# Flight of the Warped Penguins <br> AND OTHER WEAKLY COUPLED TALES OF STRONG DYNAMICS 

Philip Tanedo, Ph.D<br>Cornell University 2013

Strongly coupled phenomena are typically inaccessible to the perturbative methods that are the bread and butter of physicists. In many cases, however, there exist alternate dual descriptions of the strongly coupled system that are perturbative. In this thesis, we explore applications of strong dynamics to models of physics beyond the Standard Model that are tractable due to the existence of weakly coupled descriptions.

The Randall-Sundrum framework for a warped extra dimension, for example, is related by the holographic principle to strongly coupled four dimensional gauge theories. We present a detailed calculation of loop-level flavor-changing 'penguin' diagrams in this scenario, both for the lepton and quark sectors. The phenomenological bounds from these processes are complementary to those from tree-level diagrams. Further, we present a definitive analysis of the one-loop finiteness of these diagrams.

Another handle for strong coupling is chiral perturbation theory. Here one makes use of the symmetry breaking pattern at low energies to determine the low-energy spectrum of Goldstone bosons. We examine a supersymmetric generalization of these types of theories and explore the features that make the 'Goldstone fermion' partner of the Goldstone boson a viable dark matter candidate. We then present a general framework by which one may calculate the low-energy enhancement of the dark matter interaction cross section when the dark matter self-interactions generate a singular potential.

Finally, we explore the phenomenological signatures of gluinos in $R$-parity violating supersymmetry. Signatures based on same-sign dileptons can constrain the scale at which Majorana gluinos are produced. We explore the extent to which jet substructure techniques may be used to improve current and future bounds.

## Biographical Sketch

Philip 'Flip' Tanedo was born in Los Angeles, California. He completed his primary and secondary education in the Los Angeles Unified School District, where he was fortunate to have been influenced by the faculty and students of its Gifted and Talented Education program. In seventh grade he read The Physics of Star Trek and decided that he would become a physicist. This is a shame because later in life he became an avid reader of Batman comics and, in an alternate universe, might have become a superhero. In high school was the slowest member of the track and cross country team but also went on to complete the Los Angeles Marathon in 1998, 2000, and 2001. He supplemented his education with summer and evening courses at Santa Monica College and the University of California at Los Angeles; he took a total of 21 Advanced Placement exams, including 11 in his junior year.

Flip earned a Bachelor of Science with honors and distinction from Stanford University, where a flatmate once described him as "the type of person who would sit on the quad with a physics textbook that was poorly hiding a comic book-except the comic book was, in turn, hiding a more advanced physics textbook." During these years, he participated in research with Accelerator Research Department A at the Stanford Linear Accelerator Center, the Fisher group of the Geballe Laboratory for Advanced Materials, and the Stanford Institute for Theoretical Physics. He was a resident tutor at Granada dormitory, a tour guide with Stanford visitor information services, a member of the Stanford triathlon team (where he learned how to swim), a member of Stanford's first steel pan band, and an active member of the Society of Physics Students. It was at Stanford that Flip discovered a passion for particle physics and decided to be a high energy theorist during the Large Hadron Collider era.

As a Marshall Scholar, Flip completed a Master of Advanced Studies in Mathematics from the Department of Applied Mathematics and Theoretical Physics of the Centre for Mathematical Sciences in Cambridge University and a Master of Science by research as a member of the Institute for Particle Physics Phenomenology at Durham University. Over two years in the United Kingdom, Flip developed a taste for British television and a mild distaste for British food. He started a blog, An American Physics Student in England, which would later lead to an interest in science communication and outreach. During these years Flip developed as strong sense of physics as a community. This was especially nurtured by his Part III colleagues at Cambridge, many of whom have become lifelong friends.

Flip completed his Ph.D. in physics with the particle theory group at Cornell University. During this time he blogged about the science of the Large Hadron Collider at Quantum Diaries, learned how to dress in cold weather, and became a much better swimmer and table tennis player. Flip will continue his research in particle physics at the University of California, Irvine.

In MEMORY OF THE TOO MANY STUDENTS WHO WE LOSE TO SUICIDE EACH YEAR [1].

For each that we have lost, there are dozens who have stood at the precipice. And for each who has stared down the ledge, there are many more who feel lost or broken.

Dedicated to all who have struggled with depression-you are not alone.

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The experience of a Ph.D student and adviser is similar to penguin families, as described in this passage from Things That Are by Amy Leach,

If penguins can stand through the four-day blizzards without getting dashed down and exhausted by snow, if they can live through their first year-as two out of five do not—and grow streamy slick feathers and stop looking like fraying bags; and if their parents, finally, when the chicks are strong, stop arriving with squid, so that the chicks become hungry enough to wobble off, through miles of golden blowing snow, to find the unseen sea, then they will discover, once they leap in, that they have talents besides standing-swimming with sudden winging, wheeling grace in water.
The work presented here owes its existence to many people who have shaped my life over the past five years. Foremost is my adviser, Csaba Csáki, who has been a teacher, role model, mentor, and friend. Along with Csaba, I am grateful to the faculty of the lepp theory group: Yuval Grossman, whose warmth and joviality have made me feel at home from my first day; Liam McAllister, whose intellectual curiosity is contagious; and Maxim Perelstein, whose understated humor always had me in stitches, even more so when the other faculty don't seem to pick up on it. Each have indelibly shaped me as a physicist. My education also owes much to the lepp experiment faculty, in particular Julia Thom, Jim Alexander, Anders Ryd, and Peter Wittich, who have always humored me when they caught me stealing Lepp journal club cookies. I would also like to thank Veit Elser for his enthusiasm and wit as an instructor. From our current and recent postdocs-Andi Weiler, Enrico Pajer, Brando Bellazzini, Monika Blanke, Javi Serra, Paul McGuirk, and Marco Farina-I've learned how to live an appropriately unbalanced physics life. I thank Darren Puigh, Aidan Condle-Rande, and Ben Kreis for their careful explanations of LhC results and their patience when I dig for rumors. Though our research now has zero mutual overlap, I am especially grateful to Suchitra Sebastian, who has been a mentor and friend since I first considered myself a physicist. Her passion and mirth continues to inspire me to be a better scientist and a better person.

A doctoral degree is not something one pursues in vacuum. In addition to my mentors and teachers, I owe much to the colleagues with whom I have shared parts of the journey. At Cornell, I have been especially grateful to have overlapped with Mathieu Cliche, David Curtin, Johannes Heinonen, Sohang Gandhi, David Marsh, Mario Martone, Mike Saelim, Yuhsin Tsai, and Gang Xu. I've been fortunate for the friendship of graduate students at other institutions, especially Steffen Gielen, Evan Keane, David Simmons-Duffin, Matt Dolan, Dave Wilson, Aoife Bharucha, Felix Yu, Tien-Tien Yu, Eder Izaguirre, and Gordan Krnjaic. I would have drowned in paperwork if not for the help of our admins, Ekaterina Malysheva and Elizabeth Gustafson. This thesis could not have been written without physical therapy from Megan Wagenet; I also thank her for all the swimming that I may now return to. And I thank the theory group of the University of California, Irvine who saw something in the work presented that convinced them to give me the opportunity to continue it as a postdoctoral scholar.

I was fortunate to have shared much of my life outside of physics with very special people: Kris Kooi, Nick Ledesma, Liz Craig, Kasi Dean, Liz Ellis, and Bader Obeidat. The most difficult part of a Ph.D isn't research, that's the struggle that we signed up for. Rather, it's doing research while simultaneously composing the rest of one's life. In this respect, this thesis and all it represents owes its existence to the compassion and of those who supported me when I needed it the most: Sharon Mier, whose patience and counseling kept me on my feet; Beth Fiori, whose tireless efforts for Cornell's undergraduates gave me perspective; Craig Beaver, who paternally looked out for us and taught me courage before adversity; Warren Ilchman, Stan Heginbotham, and my brothers and sisters of the Soros Fellowship for New Americans, whose hearts and minds have lent me passion and strength; Melanie McFarland, my lil' sis, who gives me a reason to be a better person; and most of all to my mother, from whom I inherited the best of all that I am and to whom I owe all that I achieve.

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Note added (July 30, 2013): Now that my defense is behind me and all my paperwork has gone through, I can finally put this document behind me. This paperback version of my thesis was prepared to share to those who have been there to help me get to where I am. In addition to those thanked above, I would like to thank Kris Koor-again-for his patience and support
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The this thesis covers research conducted between 2008 and 2013 at the Laboratory for Elementary Particle Physics at Cornell University with additional chapters of introductory material. For accessibility, each chapter begins with an overview meant to introduce and summarize the contents for those outside of the particle theory community. Chapters $2,3,7,8$ present background material that can be found in review articles and textbooks, though are presented in a way that the author felt either highlighted necessary prerequisite material or elucidated topics not commonly addressed in pedagogical literature. Chapter 4 presents explicit calculations for the mixed position-momentum space formalism of the Randall-Sundrum frame work. While the material is not new, it is not readily found in the literature on this topic. These chapters are based on the author's unpublished personal notes as well as an amalgamation of review material.

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J. Rosiek, P. Chankowski, A. Dedes, S. Jager, and P. Tanedo, "SUSY_FLAVOR: A Computational Tool for FCNC and CP-Violating Processes in the MSSM," Comput.Phys.Commun. 181 (2010) 2180-2205, arXiv:1003.4260 [hep-ph]
A. Crivellin, J. Rosiek, P. Chankowski, A. Dedes, S. Jaeger, et al., "SUSY_FLAVOR v2: A Computational tool for FCNC and CP-violating processes in the MSSM," Comput.Phys.Commun. 184 (2013) 1004-1032, arXiv:1203.5023 [hep-ph]
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You're in the Csáki group? You know Johannes Heinonen? Let me give you some advice as a graduating student. If Johannes ever comes up to you and says, "Hey, there's something really interesting that I want to show you in the restroom," do not follow him.

Dan Goldbaum, 19 August 2008

## I

## For non-specialists

In the interest of accessibility, each chapter of this thesis begins with a non-technical introduction meant to be understandable by a general 'science enthusiast' audience. This chapter goes one step further and provides introductory material to a general audience with no science background with the goal of being able to appreciate the non-technical summaries of subsequent chapters.

### 1.1 Act 1: Science

Science is a branch of human knowledge associated with the rational, objective, and empirical study of the natural world. The primary mode of generating such knowledge is the scientific method, by which hypotheses are checked against experiments. Science differs from the humanities in its subject and from the arts in its method.

Scientific fact is based on observation. Causal explanations for these observations are theories that must be rigorously checked against experiment. It is worth highlighting that a "theory," in the scientific sense, both explains observed phenomena and predicts further observable phenomena. In this way scientific theories are falsifiable and differ from the common use of the word "theory" which implies opinion of speculation. A theory may end up being incorrect when subjected to further experiments, but this is a feature rather than a shortcoming of the scientific method.

### 1.2 Act 2: Physics

Physics is the branch of science concerned the fundamental laws of nature. Branches of physics study atoms (and all things subatomic), materials in different phases (condensed matter), dynamics of different systems (e.g. geophysics, general relativity), outer space (astrophysics and cosmology), and applications to other sciences (biophysics, physical chemistry).

Unlike the other sciences, physicists can roughly be divided into theorists and experimentalists. Theorists are primarily concerned with mathematical models of nature that can be used to explain experimental data. Experimentalists are primarily concerned with testing theories and acquiring new data that may point to science beyond current theories. This divide occurs because of the high degree of specialization required to study nature at the level of physics. Theorists must be fluent in advanced mathematical methods while experimentalists must be clever to build apparatuses and interpret data.

### 1.3 Act 3: Particle ('High Energy') Physics

Particle physics is the branch of physics concerned the smallest building blocks of nature. In the past century, the "particles" that physicists considered "smallest" have gone from atoms, to nuclei, to protons, to quarks (not to mention electrons and their cousins). We have also learned how to think of the fundamental forces of nature in terms of force-mediating particles such as the photon.

Why do we study these particles? One reason is that we hope that by studying the basic building blocks of the universe we can understand composite objects better (reductionism). There is also a philosophical/aesthetic appeal associated in understanding what the ultimate basic building blocks of the universe should look like.

The current canon of particle physics is called the Standard Model and was mostly completed in the 1970s. It is based on quantum physics and explains the strong and weak nuclear forces as well as electromagnetism. It has passed every direct experimental test with flying colors and is regarded as a stunning success.

### 1.4 Intermission: Effective Theories

We know, however, that the Standard Model is incomplete. This is not to say that it is wrong, but that it is an effective theory for the distance scales that we have probed. In the same sense, Maxwell's equations are an effective theory for electromagnetism above the atomic scale, where quantum effects become relevant (and another theory is effective: quantum electrodynamics).

The reason why effective theories are reasonable is that nature tends to only care about physics at the scale you are probing. For example, when a chef bakes a cake, there are several chemical reactions that occur as the batter bakes. At the heart of these chemical reactions are statistical and quantum effects which are ultimately explained by the Standard Model, which, in turn, may ultimately be explained my a more fundamental theory such as string theory. The chef, however, does not need to know particle physics, quantum mechanics, or even chemistry to bake the cake; the chef has an "effective theory" of how to bake cakes that is based on measuring cupfuls of ingredients.

In the same way the Standard Model is an effective theory for physics at the length scales we have probed. (Particle physicists measure scales in electron volts, which are inversely proportional to length; we have probed scales up to around the hundreds of giga-electron volt range.) There must be more to the story at smaller scales, but they don't have an appreciable effect on the scale that we've currently been able to study. One of the major missing pieces in the Standard Model is a quantum theory of gravity.

### 1.5 Act 4: Particle Theory

Theoretical particle physics focuses on ways we can understand nature beyond the Standard Model. There are roughly two kinds of particle theory: phenomenology and formal theory. Phenomenologists attempt to study the next level of effective theory by looking for signals of physics beyond the Standard Model in experiments and constructing new models. Formal theorists attempt to answer the bigger question of finding a fundamental "theory of everything" that is a complete theory that describes nature down to the smallest length scales. Most formal theory today focuses on string theory.

Since the characteristic scale of gravity is well beyond anything that is experimentally accessible in our lifetimes, formal theory often comes up against the barrier of experimental assessment. Much of the motivation for string theory comes from the hope that it can be a self-consistent theory of quantum gravity.

This thesis focuses, instead, on phenomenology.

### 1.6 Act 5: Particle Phenomenology

Particle phenomenology often used as a blanket term used to describe theoretical particle physics that is not string theory. This generally refers to particle theory that is more closely related to experiment, with theory and experiment each suggesting
new research directions to the other. It is an exciting time to be in this subfield since the Large Hadron Collider (Lhc) will open up a new sectors of nature to scientific inquiry.

Some phenomenologists study finer details of the Standard Model, these include on-going studies of CP violation and neutrino physics. There is also a subgroup of phenomenologists who work on the theory of strong interactions (i.e. quarks and gluons), called quantum chromodynamics (QCD) which is notorious for being nonperturbative. Most QCD research involves applying new mathematics (such as twistor methods) or computer simulations on discretized space (lattice QCD) to extract more accurate predictions from the theory.

While these are both very promising directions, my primary research interest is what happens when our current effective theory breaks down. The answer is almost certainly that it is replaced by another effective theory, perhaps motivated by string theory, that sheds further light on the structure of nature.

### 1.7 Act 6: Beyond the Standard Model

"Beyond the Standard Model" phenomenology deals with ways to extend the Standard Model past its range of validity and, hopefully, include any new physics we discover at the Large Hadron Collider. There are several sources of data for particle physics, including astrophysics and cosmology, but colliders still represent our best controlled experiments.

There are good reasons to believe that there should be physics beyond the Standard Model within the reach of the lhc even though quantum gravity is well beyond that range. For one, from astrophysical observations we know that there is a class of massive particles called dark matter that is responsible for the clustering of galaxies. Within reasonable assumptions, such a particle should be produced at the LHC. Another reason is the mass of the Higgs boson, which seems to suggest a "UV completion" at the Tev scale.

The two most prominent ideas in 'BSM' phenomenology are supersymmetry and extra dimensions. Supersymmetry (susy) adds extra quantum dimensions to spacetime that lead to each particle having a "supersymmetric partner." This is analogous to each particle having an antiparticle. Extra dimensional scenarios extend our spacetime with classical dimensions, allowing our known particles to resonate in these extra directions to produce new "Kaluza-Klein" particles.

For the past ten or twenty years, bSM phenomenology has been centered around model building, i.e. developing new theories or reworking old theories that can solve the problems of the Standard Model. With the LhC turning on, however, the bSM community has shifted towards developing bottom-up data-driven approaches to new physics. The big question when the LhC turns on will be whether we can identify signals that are beyond the Standard Model. This is not a trivial thing since piecing together experimental signatures at a particle collider is very much a detective mystery in its own right; luckily this task is shared by experimental particle physicists.

As of this writing, the bSM phenomenology community is faced with the prospect that there is yet no evidence for new physics from the LHC. Possible astrophysical signals of dark matter have yet to be accompanied by 'smoking gun' signals from sensitive underground experiments. Searches for the quantum effects of new particles at low energies have also yet to find deviations from the Standard Model. Despite this, there are reasons to be optimistic that this generation of particle physics experiments may show the way beyond the Standard Model.

I tell the undergrads that it is not by mistake that [our department] is in [the college of] Arts and Sciences. All of physics is about the art of making the right approximation. Making the right approximation can teach you a lot.

Yuval Grossman



## Hegemony of the Standard Model

Our present working theory of particle physics, the Standard Model (sm), has survived three decades against every collider constraint, culminating in the 2012 discovery of the Higgs boson at the Large Hadron Collider (Lhc). In this chapter we review the structure of the SM and highlight theoretical and experimental evidence for directions beyond it.

### 2.1 THE THEORY OF QUARKS AND LEPTONS

The Standard Model describes the dynamics of quarks and leptons with respect to the strong and electroweak forces [12, 13]. It is based on a quantum field theory (QFT) with a non-Abelian gauge symmetry which is broken to a subgroup via the Higgs mechanism [14-16]. The theoretical and experimental path to the SM is a story of scientific triumph over several decades that is best told by its makers [17-19]. Since this thesis focuses on our steps beyond the Standard Model, we will be necessarily terse and focus on the finished product rather than its development.

The Standard Model matter content is given in Table 2.1. The SM is the most general renormalizable quantum field theory with this matter content assuming that the potential for $H$ breaks $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}} \rightarrow \mathrm{U}(1)_{\mathrm{EM}}$. The resulting Lagrangian is a sum of terms,

$$
\begin{equation*}
\mathcal{L}=\mathcal{L}_{\text {gauge }}+\mathcal{L}_{\text {kin. }}+\mathcal{L}_{\text {Yuk. }}+\mathcal{L}_{\text {Higgs }} . \tag{2.1}
\end{equation*}
$$

We now review the physics of each of these terms.

### 2.1.1 Gauge structure

Gauge symmetry is a redundancy in the mathematical description of a theory. It appears because we prefer to work with mathematical objects, such as the four-vector potential $A_{\mu}$, which are manifestly covariant with respect to the Lorentz symmetry of spacetime. The redundancy can be seen in the massless vector particle. Of the four degrees of freedom contained in $A_{\mu}$,

- two are physical states (left- and right-polarizations)

| Field | Spin | $\operatorname{SU}(3)_{c}$ | $\operatorname{SU}(2)_{\mathrm{L}}$ | $\mathrm{U}(1)_{\mathrm{Y}}$ |
| :---: | :---: | :---: | :---: | :---: |
| $Q$ | $1 / 2$ | $\square$ | $\square$ | $1 / 6$ |
| $\bar{u}$ | $1 / 2$ | $\square$ | $\mathbb{1}$ | $-2 / 3$ |
| $\bar{d}$ | $1 / 2$ | $\square$ | $\mathbb{1}$ | $1 / 3$ |
| $L$ | $1 / 2$ | $\mathbb{1}$ | $\square$ | $-1 / 2$ |
| $\bar{e}$ | $1 / 2$ | $\mathbb{1}$ | $\mathbb{1}$ | -1 |
| $H$ | 0 | $\mathbb{1}$ | $\square$ | $1 / 2$ |

Table 2.1: Matter content of the Standard Model. Each spin-1/2 field is a left-handed Weyl fermions that transforms under a distinct $\mathrm{U}(3)$ flavor symmetry. The $\mathrm{SU}(2)_{\mathrm{L}}$ doublets contain left-handed fields $Q=(u, d)$ and $L=(v, e)$; when necessary we disambiguate these from the $S U(2)_{\mathrm{L}}$ singlets by writing subscripts $L$ and $R$ for the doublet and singlet respectively.

- one, the longitudinal polarization, is removed by the massless condition
- one is an unphysical gauge redundancy associated with the shift $A_{\mu} \rightarrow A_{\mu}+\partial_{\mu} \lambda(x)$.

In other words, one must identify (or 'mod out') gauge redundant configurations as describing the same physical state. The natural mathematical language for describing this gauge redundancy is that of a principal fiber bundle and its associated bundles; these are reviewed in [20-23]. A convenient way to describe these gauge redundancies is in terms of a 'local' symmetry. Conceptually this helps connect to the global symmetry associated with the gauge redundancy under which particles are charged and which, as we will see in Section 2.1.4, is what 'breaks' in the Higgs mechanism.

The gauge potentials, $A_{\mu}(x)=A_{\mu}^{a}(x) t^{a}$, are Lie algebra valued fields with gauge field strength

$$
\begin{equation*}
F_{\mu \nu}=\partial_{[\mu} A_{v]}+g\left[A_{\mu}, A_{\nu}\right], \tag{2.2}
\end{equation*}
$$

where $g$ is the gauge coupling. The commutator evaluates to $\left[A_{\mu}^{a}, A_{v}^{b}\right]=i f^{a b c} A^{c}$, where $f^{a b c}$ are the structure constants of the Lie algebra. Out of the field strength we may write the pure gauge contributions to the Standard Model Lagrangian (2.1),

$$
\begin{equation*}
\mathcal{L}_{\text {gauge }}=\sum_{i=1}^{3} F_{(i) \mu \nu} F_{(i)}^{\mu \nu}+\Theta_{\mathrm{QCD}} F_{(3) \mu \nu} \tilde{F}_{(3)}^{\mu \nu} . \tag{2.3}
\end{equation*}
$$

Here $F_{(i)}$ labels the field strength for the $\mathrm{SU}(3)_{\mathrm{c}}, \mathrm{SU}(2)_{\mathrm{L}}$, and $\mathrm{U}(1)_{\mathrm{Y}}$ gauge factors for $i=3,2,1$ respectively. The dual field strength, $\tilde{F}_{\mu \nu}=\varepsilon_{\mu \nu \rho \sigma} F^{\rho \sigma}$, appears in the CP-violating phase $\Theta_{\text {CCD }}$. This phase only appears for QCD since the chiral nature of the Standard Model allows one to rotate away the $\Theta$-angle associated with $\operatorname{SU}(2)_{\mathrm{L}}$. It is conventional to write the $\mathrm{U}(1)_{\mathrm{Y}}$ coupling as $g^{\prime}$ and the $\mathrm{SU}(2)_{\mathrm{L}}$ coupling as $g$.

### 2.1.2 MATTER KINETIC TERMS

The matter fields in Table 2.1 each have kinetic terms so that the kinetic Lagrangian terms in (2.1) are

$$
\begin{equation*}
\mathcal{L}_{\text {kin. }}=\sum_{i=\text { fermions }} i \psi_{i}^{\dagger} \vec{D} \psi_{i}+|D H|^{2} . \tag{2.4}
\end{equation*}
$$

Here $D$ is the gauge covariant derivative,

$$
\begin{equation*}
D_{\mu}=\partial_{\mu}-i \sum_{i} g_{i} A_{(i) \mu}^{a} \mu_{(i) r}^{a}, \tag{2.5}
\end{equation*}
$$

where $i$ runs over gauge groups and $t_{r}$ is the $r$ representation of the Lie algebra. Geometrically this is the object that generates horizontal lifts on the fiber bundle; the gauge field appears as a connection for the bundle geometry. The $\vec{D}$ notation indicates a contraction $\bar{\sigma}^{\mu} D_{\mu}$ and is the Weyl spinor analog of the usual $\not D=\gamma^{\mu} D_{\mu}$ for Dirac spinors; see [24] for a review.

Together $\mathcal{L}_{\text {gauge }}$ and $\mathcal{L}_{\text {kin. }}$. describe a non-interacting theory with an $\mathrm{SU}(3)_{\mathrm{c}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$ gauge redundancy and the matter content of Table 2.1. Note that this theory has a classical $U(3)^{5}$ global symmetry corresponding to independent rotations of each fermion species in flavor space.

### 2.1.3 YUKAWA TERMS

The flavor sector of the Standard Model is determined by the Yukawa terms in the Lagrangian $\mathcal{L}_{\text {Yuk }}$,

$$
\begin{equation*}
\mathcal{L}_{\text {Yuk }}=y_{i j}^{e} \bar{L}^{i} H e^{j}+y_{i j}^{d} \bar{Q}^{i} H d^{j}+y_{i j}^{u} \overline{Q^{j}} \tilde{H} u^{j}+\text { h.c. }, \tag{2.6}
\end{equation*}
$$

where the Yukawa matrices $y_{i j}$ are flavor space matrices and $\tilde{H}=i \sigma^{2} H^{*}$ is the $\operatorname{SU}(2)_{L}$ antifundamental that can be formed from the fundamental since $\square=\bar{\square}$ in $\operatorname{SU}(2)$.

The three Yukawa matrices are spurions for flavor symmetry breaking, $\mathrm{U}(3)^{5} \rightarrow \mathrm{U}(1)^{2}$ where the classically unbroken factor is identified with baryon and lepton number, $\mathrm{U}(1)_{\mathrm{B}} \times U(1)_{\mathrm{L}}$. Quantum mechanically, however, this is broken to $U(1)_{\mathrm{B}-\mathrm{L}}$ due to the anomaly coming from the chiral matter content in Table 2.1.

Opon electroweak symmetry breaking, the Higgs obtains a vacuum expectation value (vev) $\langle H\rangle=(0, v / \sqrt{2})$ and the Yukawa terms generate Dirac masses for the quarks and charged leptons. One may use the broken $\mathrm{U}(3)^{5}$ flavor symmetry to rotate the fermions into this basis, $\psi \rightarrow U_{\psi} \psi$ for $\psi=Q, u, d, L, e$. In the quark sector this gives

$$
\begin{equation*}
\mathcal{L}_{\text {Yuk. }} \supset \bar{u}_{\mathrm{L}}^{i} \hat{m}_{i i}^{u} u_{\mathrm{R}}^{i}+\bar{d}_{L}^{i}\left(U_{\mathrm{Q}}^{\dagger} U_{d}\right)_{i}^{\dagger j} \hat{m}_{i j}^{d} d_{\mathrm{R}}^{j} . \tag{2.7}
\end{equation*}
$$

The mismatch between the quark mass and flavor bases are the source of the non-trivial flavor physics in the $S M$, see Section 2.2 below. Observe that in there is no analogous flavor structure in the lepton sector since the SM does not include right-handed neutrinos. Such an extension is straightforward and addressed in Section 2.3.2.

### 2.1.4 Higgs mechanism

The last piece of the Standard Model Lagrangian (2.1) is the 'Mexican hat' Higgs potential, $\mathcal{L}_{\text {Higgs }}=-V(H)$,

$$
\begin{equation*}
V(H)=-\mu^{2} H^{\dagger} H+\lambda\left(H^{\dagger} H\right)^{2} \tag{2.8}
\end{equation*}
$$

This potential has a minimum at non-zero values of $H$ so that the Higgs obtains a vev which we may choose to be $\langle H\rangle=(\mathrm{o}, v / \sqrt{2})$, where

$$
\begin{equation*}
v^{2}=\frac{\mu^{2}}{\lambda} . \tag{2.9}
\end{equation*}
$$

This spontaneously breaks electroweak symmetry ${ }^{1}$ to the electromagnetic subgroup $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}} \rightarrow \mathrm{U}(1)_{\mathrm{EM}}$. This electroweak symmetry breaking (EwSB) is manifested in the masses generated for the gauge bosons associated with the broken generators by the connection terms in the Higgs kinetic term (2.4). The mass eigenstates for the gauge bosons of the

[^0]electroweak sector are the massless photon $A$, the massive electrically charged $W^{ \pm}$, and the massive neutral $Z$,
\[

$$
\begin{equation*}
M_{Z}^{2}=\frac{1}{4}\left(g^{2}+g^{\prime 2}\right) v^{2} \quad M_{W}^{2}=\frac{1}{4} g^{2} v^{2} \tag{2.10}
\end{equation*}
$$

\]

The rotation to the mass basis is given by the Weinberg angle, $\theta_{W}$, and depends on the gauge couplings,

$$
\begin{equation*}
\sin \theta_{W}=\frac{g^{\prime}}{\sqrt{g^{2}+g^{\prime 2}}} \tag{2.11}
\end{equation*}
$$

Observe that the Higgs sector parameters $\mu^{2}$ and $\lambda$ only appear in the combination $v^{2}$ for all interactions that do not involve the other radial Higgs excitation, $(0, v / \sqrt{2}+h(x))$. This is the reason why physicists have been able to tightly confirm the structure of the Standard Model long before knowing the mass of the Higgs boson; the 'Standard Model sans Higgs' is actually a non-linear $\Sigma$ model (NLEM) which depends only on the parameter $v^{2}$. The discovery of the Higgs confirms at this NLEM is indeed completed by a linear $\Sigma$ model.

### 2.2 Flavor

One of curiosities of the Standard Model is its flavor structure. Theoretically, the hierarchy in the spectrum of fermion masses-though technically natural-seems to beg for a dynamical explanation. Experimentally, flavor observables probe nature at high scales and severely constrain generic models of new physics.

### 2.2.1 The CKm matrix

The mismatch between the flavor and mass bases in (2.7) is represented by the Cabbibo-Kobayashi-Maskawa (скм) matrix,

$$
\begin{equation*}
V_{\mathrm{cKM}}=U_{Q}^{\dagger} U_{d} \tag{2.12}
\end{equation*}
$$

The diagonal element of the unitary transformations $U_{Q, u, d}$ allows one to set the mass matrices to be real, but one physical complex phase is left over in the Скм matrix and contributes to CP-violation.

### 2.2.2 The unitarity triangle

The unitarity of the скм matrix implies relations of the form

$$
\begin{equation*}
\sum_{i=u, c, t} V_{i d} V_{i b}^{*}=0 . \tag{2.13}
\end{equation*}
$$

Each term in on the left-hand side can be plotted as a phasor in the complex plane so that the null sum can be represented as a triangle. While most of these triangles are very flat owing to having one term much smaller than the others, the particular relation between the bottom and down quarks in (2.13) is unique because each term in this sum is $\mathcal{O}\left(\lambda^{3}\right)$ in the Wolfenstein parameter. Each side is roughly the same order and the resulting 'unitarity triangle' is robust against experimental errors. Normalizing the sides of the triangle by $V_{c d} V_{c b}^{*}$,

$$
\begin{equation*}
\frac{V_{u d} V_{u b}^{*}}{V_{c d} V_{c b}^{*}}+1+\frac{V_{t d} V_{t b}^{*}}{V_{c d} V_{c b}^{*}}=0 . \tag{2.14}
\end{equation*}
$$

This is shown in Figure 2.2.1.


Figure 2.2.1: The unitarity triangle.

### 2.2.3 The gim mechanism

The GIM mechanism, named after Glashow, Iliopoulos, and Maiani [25] is the observation that because any pair of columns in a unitary matrix are orthogonal, flavor changing neutral currents (FCNCS) prohibited at tree level. Further, loop-level FCNCs are most sensitive to the heavy quarks running in the loops. Because these virtual heavy quarks are off shell, this contradicts the expectation from the Appelquist-Carazzone decoupling theorem [26], which states that the physical effects of a virtual particle vanish as its mass is taken to infinity. Indeed, the heavy particles in FCNC loops do not decouple and the amplitude is a function of the combination $m_{t}^{2} / M_{W}^{2}$.

The reason for this violation of decoupling is that FCNC processes are essentially mediated by the longitudinal part of the $W$ due to the Goldstone boson Equivalence theorem. The $W$ coupling to the fermions in these diagrams is essentially a Yukawa coupling, which goes like the fermion mass. When you have a particle whose coupling is proportional mass, then it is clear that decoupling fails. Thus the gIM mechanism tells us that one-loop diagrams carry factors of $m_{i}^{2} / M_{W}^{2}$, where $i$ is summed over the internal quarks.

Because new physics appearing as virtual particles in loops can affect low energy observables, the GIM observation tells us that the FCNCs are a particularly fertile way to constrain physics beyond the SM. This is because GIM prohibits the leading order Sm contribution to these processes so that both the new physics signal and sm background appear at the same order in a loop expansion.

### 2.2.4 Penguins

Penguins are a class of FCNC diagrams that will be discussed at length in this thesis. The curious etymology of this process is explained in John Ellis' own words in [27]. These diagrams correspond to effective operators of the form $\chi_{i}^{\dagger} \sigma^{\mu \nu} \chi_{j} F_{\mu \nu}$ for a Weyl fermions $\chi$ and gauge field strength $F_{\mu v}$; an example is shown in Figure 2.2.2. One way to understand these diagrams is to consider what would be required for a fermion to change flavor: the GIM mechanism requires a loop-level process, momentum conservation requires the emission of a light particle, spin conservation requires this particle to be a boson and that the fermion chirality flips, and this in turn implies the presence of a mass insertion. Alternately, the mass insertion can be understood as a necessary term to avoid the GIM cancellation in (2.13). Consider, for example, the penguin associated with $b \rightarrow s \gamma$. Naïvely, the amplitude's dependence on the скм matrix takes the form

$$
\begin{equation*}
\mathcal{M} \propto \sum_{i=u, c, t} V_{i b}^{*} V_{i s}=0 . \tag{2.15}
\end{equation*}
$$



Figure 2.2.2: Example of a penguin diagram where arrows refer to fermion chirality. Note that the chiralityflipping mass insertion is required by the dipole operator.

This product vanishes identically by unitarity. This tells us that any term independent of the internal quark mass vanishes so that

$$
\begin{equation*}
\mathcal{M}=\sum_{i} V_{i b}^{*} V_{i s} f\left(m_{i}\right) \tag{2.16}
\end{equation*}
$$

We can expand $f$ in a power series. If $m_{i} \ll M_{W}$ then $f\left(m_{i} \ll M_{W}\right) \propto m_{i}^{2} / M_{W}^{2}$. This limit is valid for the up and charm quarks, but there is no sense in which $m_{t}^{2} / M_{W}^{2}$ is a small parameter-it turns out, however, that the linear approximation is only off by an $\mathcal{O}(4)$ factor [28].

### 2.3 HINTS FOR NEW PHYSICS

The discovery of the Higgs boson was a crowning achievement for the Standard Model [29,30]. However, the many open questions associated with the Sm have been known since long before the first beam at the LhC. Here we review some of the reasons why we expect physics beyond the Standard Model, highlighting particular aspects that are addressed in this thesis.

### 2.3.1 Naturalness and the Hierarchy problem

For the past three decades, one of the main motivations for physics beyond the Standard Model has been the so-called Hierarchy problem. In typical quantum field theories, one may estimate the size of the model's parameters based on their scaling dimensions. The gauge Hierarchy problem is the observation that the expected Higgs squared mass parameter is much larger than what is expected from the observed masses of the electroweak gauge bosons.

In the the Wilsonian picture ${ }^{2}$, a given quantum field theory is defined with a cutoff, an upper limit on its range of validity [31]. Changing this cutoff-or, alternately, probing the theory at different scales within its range of validity-can be recast into a scale transformation on the theory. General field theories, however, are not scale invariant; the physics manifested by the transformed theory is different from that of the original theory. Even terms which are classically invariant under rescaling can pick up quantum ('anomalous') corrections to their naïve 'engineering' dimension. The flow in theory space as one lowers the cutoff-i.e. "integrates out" high scale physics-is known as renormalization. See [32-36] for details.

The flow to the infrared (IR) generates large logarithmic corrections to the parameters of a theory. In order to maintain a perturbative expansion, these large logarithms can be resummed using the renormalization group (RG) or 'improved perturbation theory'. The Callan-Symanzik, or RG, equation for the running couplings is precisely a consistency condition that recursively determines higher order corrections in the large logs to a given order of the coupling [37]. This resummation is a geometric series of the heuristic form $1+\alpha \ln +(\alpha \ln )^{2}+\cdots=(1-\alpha \ln )^{-1}$. The solution of the RG equation prescribes the scale dependence of the parameters of the theory in the so-called 'running couplings.'

[^1]Suppose that a given theory is defined at a cutoff $\Lambda$ where, for example, it maps onto a more fundamental uv theory that completes it. The action for the theory takes the form

$$
\begin{equation*}
S=\int d^{4} x \sum_{i} h_{i} \mathcal{O}_{i} \tag{2.17}
\end{equation*}
$$

with dimensionful coupling $h_{i}$ and operators $\mathcal{O}_{i}$. Note that the action is dimensionless so that $\left[h_{i}\right]+\left[\mathcal{O}_{i}\right]+4=0$, where $[\cdots]$ is the mass dimension. It is convenient to work with dimensionless couplings, $g_{i}=h_{i} \mu^{-\left[h_{i}\right]}$, where $\mu$ is a characteristic scale at which the interaction is probed. For example, a scalar mass operator $\mathcal{O}_{2}=\phi^{2}$ has a 'coupling' $h_{2}=m^{2}$ and a dimensionless parameter $g_{i}=m^{2} / \mu^{2}$. Note that the mass dimension $[\cdots]$ implicitly includes the anomalous dimension coming from quantum corrections.

Naturalness is the principle that the characteristic mass scales of the theory should be on the order of the cutoff $\Lambda$ at which it is defined:

$$
\begin{equation*}
m_{i} \sim \Lambda, \tag{2.18}
\end{equation*}
$$

up to $\mathcal{O}(1)$ factors [38] (see [39,40] for reviews). This gives us a classification of the types of operators in a QFT:

- Irrelevant operators have positive mass dimension $\left[\mathcal{O}_{i}\right]-4>0$ so that their couplings have negative mass dimension $\left[h_{i}\right]<o$ and their dimensionless couplings go like a positive power of the scale $\mu, g_{i} \sim \mu^{\left[\mathcal{O}_{i}\right]-4}$
- Relevant operators have a negative mass dimension $\left[\mathcal{O}_{i}\right]-4<0$ so that their couplings have positive mass dimension $\left[h_{i}\right]>0$ and their dimensionless couplings go like a negative power of the scale $\mu, g_{i} \sim \mu^{-\left[\mathcal{O}_{i}\right]-4}$.
- Irrelevant operators classically have zero mass dimension. If this is corrected quantum mechanically, then they are known as marginally irrelevant or marginally relevant operators and scale logarithmically with $\mu$. Otherwise, they are exactly marginal.

Observe that the $\mu$ scaling of relevant and irrelevant and operators is relative to the characteristic coupling scale $m_{i}$. Thus, according to the natural values of this scale (2.18), the expected scaling is $(\mu / \Lambda)$ to a negative or positive power for relevant and irrelevant operators respectively. If the uv scale is much larger than the probe scale, $\Lambda \gg \mu$, then we may ignore irrelevant operators and restrict ourselves to theories with marginal and relevant couplings subject to the symmetries in the IR. Conversely, relevant operators become large in this limit. This is simply the statement that relevant terms, such as a mass parameter, are expected to be on the order of the UV scale, (2.18). For this reason 4D gravitational effects are highly suppressed and can be ignored in experimental searches for new physics.

The electroweak sector appears to violate naturalness. The naïve uv scale for the Standard Model is the Planck scale $\Lambda \sim M_{P 1}$ where one expects $\mathcal{O}(1)$ effects from quantum gravity. From (2.9), the Higgs squared mass parameter is

$$
\begin{equation*}
\mu^{2}=\lambda v^{2} \tag{2.19}
\end{equation*}
$$

where $v^{2}=(246 \mathrm{GeV})^{2}$ is fixed by the observed $W$ and $Z$ boson masses and weak couplings. The natural value of the $\mu^{2}$ parameter, $\mu^{2} \sim M_{\mathrm{P}}^{2}$, would thus require that the marginal coupling $\lambda$ becomes correspondingly large. However, perturbativity requires

$$
\begin{equation*}
\frac{\lambda}{16 \pi^{2}} \lesssim 1 \Rightarrow \mu \lesssim 4 \pi v=3 \mathrm{TeV} . \tag{2.20}
\end{equation*}
$$

In other words perturbativity and naturalness predict that the physics associated with the order parameter of electroweak symmetry breaking must be within the range of the LHC, and certainly well below the expected 'natural' value of the Planck scale. The several orders of magnitude difference between $\mu^{2}$ and $M_{P 1}$ is the Hierarchy problem and is strong motivation for physics beyond the Standard Model near the TeV scale [38, 41-47].

Observe that at no reference to the existence of a Higgs boson is required to motivate the Hierarchy problem. Indeed, one may have relaxed the requirement of perturbativity and assumed that the 'Higgs squared mass parameter' $\mu^{2}$ is a generated by strong dynamics that causes an electroweak symmetry breaking condensate analogous to the chiral condensate of QCD-these ideas are the basis of technicolor models [44, 48-51]. In this case one would have expected the observation of new states at
the strong coupling scale. The discovery of the Higgs boson [29,30] with mass

$$
\begin{equation*}
m_{h}^{2}=2 \mu^{2}=2 \lambda v^{2}=(125 \mathrm{GeV})^{2} \tag{2.21}
\end{equation*}
$$

largely refutes these sorts of models and has been hailed by many as the 'triumph of perturbativity' and the 'death of technicolor'--though much of this thesis is dedicated to the ways in which strong coupling continues to find applications in post-Higgs model building.

We thus now know the parameters of the Higgs sector,

$$
\begin{aligned}
& \mu=88 \mathrm{GeV} \\
& \lambda=0.13,
\end{aligned}
$$

and must confront the Hierarchy problem, $\mu \ll M_{\mathrm{Pl}}$. In a sense 'naturalness' is an aesthetic principle, albeit formalized by the framework of Wilsonian effective field theory. One might suppose that $\mu^{2}$ just happens to be a 'fine tuned' parameter relative to its natural value. In the past decade this perspective has garnered some more serious thought [52,53] due to observations about the landscape of string theory vacua that appear to essentially offer a continuum of different low-energy parameters [54] (see [55] for a recent examination). This has led to some speculation that string theory might be telling us that perhaps naturalness fails due to some anthropic-like principle [56] akin to Weinberg's proposal for the cosmological constant problem (a much more severe violation of naturalness than the Hierarchy problem) [57]. Depending on the extent to which one subscribes to these ideas, one may skip the remainder of this subsection or this thesis altogether.

In light of the aforementioned theoretical forays away from naturalness and-at the time of this writing-the lack of experimental evidence indicating new physics at the TeV scale, it is worth reviewing a well known particle physics example where naturalness proved true: the mass of the electron [58]; additional examples can be found in [39]. A question that any high school student may ask is what happens when one approaches the 'point charge' distribution of an electron. Since the electrostatic potential energy goes like $V \sim 1 / r$, this energy appears to diverge and the electron seems to want to blow itself apart. Furthermore, $E=m$ at rest, so that this self-energy should appear as a contribution to the electron mass,

$$
\begin{equation*}
m_{\text {physical }}=m_{\text {bare }}+\Delta E_{\text {self }} . \tag{2.24}
\end{equation*}
$$

Regulating the radius of the electron to $r_{e}=10^{-17} \mathrm{~cm}$ gives $\Delta E_{\text {self }}=10 \mathrm{GeV}$. Since $m_{\text {phys }}=511 \mathrm{keV}$, this fixes the bare mass to be

$$
\begin{equation*}
m_{\text {bare }}=-9.999489 \mathrm{GeV} \tag{2.25}
\end{equation*}
$$

Disregarding the strange sign, we observe that the electron mass appears to be fine tuned against this Coulomb self-energy. The resolution to this hierarchy is that 'new physics' appears at the scale $E=2 m_{\mathrm{phys}}$ since at this scale one is sensitive to the vacuum polarization caused by quantum mechanical electron-positron pair production. A full calculation reveals that this screens the electromagnetic self-energy [59] and cancels the $\Delta E_{\text {self }} \sim 1 / r$ behavior. In fact, the leading correction goes like $\Delta E \sim a m_{\text {phys }} \ln \left(m_{\text {phys }} r_{e}\right)$ so that $m_{\text {phys }}$ is multiplicatively, rather than additively, renormalized so that the $\mathcal{O}\left(10^{-4}\right)$ tuning in (2.25) is completely avoided. The naturalness argument for new physics at the LHC assumes that a similar cancellation of quantum corrections occurs with with new states.

Finally, it is important to highlight how symmetry principles can preserve naturalness even when mass parameters take values much less than their expected values, $m_{i} \ll \Lambda$. When this occurs, we say that the theory is technically natural [38]; even though a parameter is tuned relative to its expected mass scale, it is not fine tuned (unnatural) because symmetries control the size of quantum corrections. As demonstrated in the electron self-interaction example above, symmetries render radiative corrections to be multiplicative rather than additive. This multiplicative renormalization achieves technical naturalness in two ways: power law corrections are softened to logarithmic corrections and these corrections are suppressed by the smallness of the smallness of the tuned parameter. Stated more concretely, a dimensionful parameter $g_{i} \sim m_{i}^{|n|}$ is technically natural if the limit $g_{i} \rightarrow$ o restores a symmetry. The two examples in the Standard Model are the masses of the fermions and the electroweak gauge bosons.

- Even the heaviest sm fermion, the top quark, is well below the Planck scale. Loop corrections to the fermion masses, however, are forced to be proportional to the mass itself due to chiral symmetry. In other words, corrections that are
not proportional to the fermion mass do not generate the correct fermion helicity for a mass correction.
- The masses of the gauge boson are also protected because they are generated through spontaneous symmetry breaking. Any quantum correction to this mass must couple to the gauge bosons and carry the symmetry breaking order parameter so that such corrections must go like $g^{2} v^{2} \sim M_{\text {gauge }}^{2}$.


### 2.3.2 Lepton flavor

The observation of neutrino oscillations is a smoking gun for physics beyond the Standard Model [60]. This implies a non-zero neutrino mass which is not included in the SM Lagrangian (2.1). Analogously to to the CKM matrix in (2.12), this implies the existence of a unitary matrix that encodes the difference between the neutrino flavor and mass bases: the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [61-63].

Note that in the quark sector, it is purely conventional that the скм matrix is defined with respect to the down-quark sector. The cкм matrix appears, for example, in the $W$ boson vertex connecting mass basis up- and down-type quarks so that the misalignment between mass and flavor states could have equivalently been shifted to the up-sector in (2.7). In the lepton sector there is a strong physical prejudice that the pmns matrix appear on the neutrino mass terms. This is because the small neutrino mass splittings cause them to oscillate on macroscopic distances, whereas the charged leptons lose coherence at very short distances [64]. This is analogous to meson oscillations: only the $K, D, B$, and $B_{s}$ can oscillate, other charge neutral mesons either decay or their oscillatory components become decoherent [65].

It would be trivial to extend the SM neutrino sector to include a flavor multiplet of right-handed neutrinos, $\bar{N}$, analogous to the right-handed charged leptons so that the structure parallels that of the quark sector. Observe that in order to form the leptonic analogues of the up- and down-type Yukawa interactions, $\bar{N}$ must be a sm gauge singlet. This, in turn, opens up an alternate possibility for neutrino mass: the $\bar{N}$ may have Majorana masses [66]. Note, however, that a net SM singlet state $\bar{N}$ is not necessary for Majorana masses. Indeed, below the scale of electroweak symmetry breaking, the only relevant gauge group is $\mathrm{U}(1)_{\mathrm{em}}$. One can construct higher dimensional (irrelevant) operators containing the Higgs doublet which, upon inserting the Higgs vev, generate a Majorana mass term for the left-handed neutrino inside the $\operatorname{SU}(2)$ doublet $L$ :

$$
\begin{equation*}
\Delta \mathcal{L}_{d=s}=\frac{c_{s}}{\Lambda}(L \cdot H)^{\dagger}(L \cdot H) \rightarrow \frac{1}{2} \frac{c_{5} v^{2}}{\Lambda} v_{L}^{2}+\text { h.c. } \tag{2.26}
\end{equation*}
$$

where $(L \cdot H)$ refers to a contraction of $S U(2)_{L}$ indices by the $\varepsilon$ tensor and the $\dagger$ applies the appropriate conjugations. This is the unique $d=5$ operator made out of SM fields that respects the SM gauge group. This operator can be generated, for example, by the exchange of the right handed neutrino $\bar{N}$. Observe that the smallness of the effective mass term controlled by the ratio of the electroweak scale to the UV scale, $(v / \Lambda)$, which we assume to be very small by the naturalness arguments above.

A particularly intriguing way of tying the smallness of neutrino masses to new UV physics is the see-saw mechanism [67-71]. Here one assumes the existence of the right-handed neutrino $\bar{N}$ with a large Majorana mass term. The mass matrix between the left-handed neutrino $v_{L}$ and the right handed singlet $v_{R}=N$ is

$$
M_{\text {see-saw }}=\left(\begin{array}{cc}
0 & y^{v} v / \sqrt{2}  \tag{2.27}\\
y^{v} v / \sqrt{2} & M_{M}
\end{array}\right)
$$

where for simplicity we've assumed just one flavor with Yukawa coupling $y^{\nu}$. The off-diagonal terms are Dirac masses which are of electroweak scale up to hierarchies in the Yukawa, while the Majorana mass term $M_{M}$ is assumed to be natural and on the order of a large cutoff scale. Note that the fermion masses-even the Majorana mass-are all technically natural no matter what scale they take; here we make the further assumption that the Majorana mass is 'absolutely' natural. The key observation is that given a matrix with the hierarchies above, one finds that the light mass eigenstates have masses on the order of $\sim\left(y^{v} v\right)^{2} / M_{M}$, which is parametrically small. Thus in this case the lightness of the neutrino eigenstates is related to the scale $\Lambda \sim M_{M}$ at which the neutrino sector is augmented by the right-handed singlet states. It is especially compelling that $\mathcal{O}(\mathrm{ev})$ scale neutrino masses point to $M_{M} \sim 10^{15} \mathrm{GeV}$, a scale favored by grand unification schemes (see below).

### 2.3.3 The origin of the Yukawa matrices

In Section 2.2 we quickly surveyed the flavor structure of the Standard Model. The orders of magnitude difference between the lightest and heaviest fermion masses indicates a curious hierarchy in the Yukawa eigenvalues. While this hierarchy is technically natural-i.e. it is protected against quantum corrections-it is a curiosity in the Standard Model.

### 2.3.4 Dark Matter

A second piece of evidence for physics beyond the Standard Model comes from astrophysics: approximately one quarter of the energy density of the universe-and the large majority of the matter density-is non-baryonic and (to good approximation) non-luminous. There are now many independent checks of this hypothesis that strongly suggest an additional 'dark' sector that augments the Standard Model and which contains a stable dark matter particle. We briefly summarize these below, for further details see reviews in [72-74].

- The dm hypothesis was first proposed to explain the radial velocity dispersion of galaxies in the Coma cluster [75, 76], a phenomenon that was soon discovered in the Virgo cluster [77] and later in the local group [78].
- The rotational velocity curves of spiral galaxies: The outer regions of these galaxies rotate with higher velocities than would be expected if their matter distribution is composed only of luminous matter [79, 80].
- Comparing the matter content required for hydrostatic equilibrium to the luminous matter determined from X-ray emissions of elliptical galaxies further confirms that dark matter is not exclusive to spiral galaxies [81]. Additional refinements of these searches are summarized in [82].
- One can further probe gravitational signals of dark matter through lensing phenomena. This effect can be seen at different magnitudes depending on the gravitational potential of the lensing object. Strong lensing refers to easily visible distortions of an individual light source. Weak lensing, on the other hand, requires a statistical analysis of a large number of sources to search for coherent distortions. Finally, 'microlensing' comes from relatively light lensing objects whose distortions of the luminous object cannot be resolved so that one instead searches for a change in that objects overall luminosity. The most advanced lensing analyses have not only detected dark matter, but have even allowed astrophysicists to construct three dimensional maps of its distribution [83].
- Both X-ray spectroscopy and gravitational lensing converge with the relatively recent observation of the Bullet cluster which was formed by the collision of two large galaxy clusters [84]. By using X-ray spectroscopy to image the hot (luminous) matter and weak gravitational lensing to image mass density, it was seen that the luminous matter lags behind the total mass as one would expect from weakly-interacting dark matter. This observation effectively puts the nail in the coffin for alternative theories to DM, such as modified Newtonian gravity.
- A combination of theoretical and experimental cosmological constraints from the cosmic microwave background (смв) have cemented the so-called 'concordance' or $\Lambda$ CDM (dark energy with cold dark matter) paradigm as an accurate description of our universe $[85,86]$. The measured matter density of the universe $\Omega_{m} \approx 0.04$ does not match its baryonic energy density $\Omega_{b} \approx 0.26$ so that most the matter in the universe must be composed of non-baryonic dark matter. Indirect measurements of $\Omega_{b}$ include analyses of primordial nucleosynthesis of ${ }^{4} \mathrm{He},{ }^{2} \mathrm{H}$ and ${ }^{7} \mathrm{Li}$ [87, 88]; the Sunyaev Zel'dovich effect in which the spectrum of X-ray emission from hot gasses is shifted from inverse scattering off the смв [89], and the Lyman- $\alpha$ forest whose absorption lines indicate the make up of the intergalactic medium [90]. The highlight of observational cosmology, however, was the direct measurement of the CMB spectrum from the cobe [91] and wMAP [92] satellites. The measurement of the acoustic peaks in this spectrum provide the most stringent constraints on dark matter (and dark energy) [93, 94].
- Further evidence comes from the requirement of dark matter in cosmology to generate the density perturbations that led to large scale structure [95, 96] and to account for big bang nucleosynthesis [97].
In addition to these, recent astrophysical observations may be speculatively interpreted as indirect signals of dark matter:
- The excess of cosmic positrons observed by pamela [98] and later confirmed by Fermi [99] and ams-02 [100].
- The 135 GeV line in the Fermi gamma ray spectrum [101-105].
- Observations of scale structure in dwarf galaxies do not match the predictions from $N$-body simulations of collisionless dark matter. These include a difference in the densities of dwarf galaxies [106-109], the inability for massive subhalos to host the brightest observed satellites [110-112], and the discrepancy between the predicted and number of observed satellites in the Milky Way [113, 114]. These observations suggest that the DM sector may include additional self-interactions, see, e.g. [115, 116].
One must be careful to note that these purported signals should be taken with a grain of salt: there are multiple sources of uncertainty and possible explanations within the realm of astrophysics rather than particle physics.

A particularly appealing class of dark matter candidates are weakly-interacting massive particles (wIMPs). The so-called
 'weak scale' annihilation cross section generates the correct relic density $\Omega_{\mathrm{DM}}=0.3$. In the past, this has been presented as evidence for new terascale physics connected to electroweak symmetry breaking and, perhaps, the solution to the Hierarchy problem. However, this should be taken with a grain of salt. First, the wIMP miracle is valid only at the "within a few orders of magnitude" level. Note that a typical weak cross section is $\langle\sigma v\rangle \sim \mathrm{pb}=10^{-36} \mathrm{~cm}^{2}$, so that some amount of tuning is required in the wIMP coupling. In other words, the wIMP miracle is a logarithmic miracle.

A stricter restriction comes from a tension between the correct relic abundance and recent direct detection bounds. As of the writing of this paragraph, the XENON 100 experiment has set an upper limit on the spin-independent elastic wIMP-nucleon cross section on the order of $\sigma_{\text {SI }}=2 \times 10^{-45} \mathrm{~cm}^{2}$ for a 55 GeV wIMP at $90 \%$ confidence [117]. A very naïve assumption is that the annihilation cross section should be roughly of the same order as the direct detection cross section, and so there appears to be significant tuning required to generate a difference on the order of several orders of magnitude between the two processes.

There are ways to generate honest-to-goodness wIMP models, but these appear to be rather special cases in extended models rather than generic phenomena. None-the-less, the existence of dark matter is a strong signal for physics beyond the Standard Model. The characteristic mass scale of this sector and the extent to which it interacts with the Sm, if at all, remain open questions.

### 2.3.5 Other hints

In this section we briefly review additional motivations for physics beyond the Standard Model that, while compelling, will not be a central part of this thesis.

- The Standard Model gauge group can be embedded into a simple group. One can then suppose that the 'grand unified' (GUT) group is broken spontaneously into the SM [118] at a high scale, typically $10^{14}$ to $10^{16} \mathrm{GeV}$. While realistic models are difficult to construct due to limits on, among other things, proton decay, this idea has a strong aesthetic appeal. The requirement that the three gauge couplings unify is non-trivial and puts constraints on the matter content so that $\operatorname{SU}(5)$ unification in supersymmetric extensions of the SM is taken as strong motivation for supersymmetry [119]. Extensions to SO (10) additionally fit the entire SM matter content into a single representation with exactly one additional particle, the right-handed neutrino [ 120,121 ]. One can then speculate that the right-handed neutrino mass exists at a much higher scale, e.g. a scale where $\mathrm{SO}(10) \rightarrow \mathrm{SU}(5)$, allowing a see-saw mechanism as described in Section 2.3.2. A final motivation for unification is that GUT models can explain the observed quantization of charges in the SM.
- The strong charge-parity (CP) phase, $\Theta_{\mathrm{QCD}}$ in (2.3) parameterizes the topological (instanton) vacua of QCD. Each value is an inequivalent vacuum with a different energy density. Note that $\Theta_{\mathrm{QCD}}$ breaks both parity ( P ) and CP . The strong CP problem is the observation that all observations of QCD suggest that this sector of the SM respects CP symmetry so that $\Theta_{\mathrm{QCD}} \ll 1$. This is an apparent fine-tuning of the theory since quantum corrections from the electroweak sector-which does not respect parity-will cause $\Theta_{\mathrm{QCD}}$ to run. One popular solution based on dynamics is the axion [122] [123].
- Baryogenesis is the creation of the asymmetry between baryons and anti-baryons in the early universe which persists today. The Sakharov conditions for such an asymmetry are (1) baryon number violation, (2) charge (c) and cp violation, and (3) non-equilibrium interactions [124]. While these conditions are satisfied in the sm during the early universe, the SM does not appear to be able to generate enough asymmetry to explain the observed universe [125]. In models of leptogenesis, part of this asymmetry is transferred from the lepton sector by sphalerons [126].
- Prior to the Higgs discovery the triviality of the quartic coupling $\lambda$ in (2.8) was a hint for new physics. If the Higgs mass were too heavy, this coupling would be large and would hit a Landau pole below the Planck scale, perhaps suggesting a uv completion by new physics. With the input that $m_{h}=125 \mathrm{GeV}$ and, hence, $\lambda=0.13$, a different picture appears. The negative contributions from top loops dominate the $\beta$-function and drive $\lambda$ negative at high energies [127]. Thus, instead of a Landau pole, it appears that the Standard Model Higgs potential becomes unbounded from below. From the Wilsonian perspective we expect higher order terms such as $\left(H^{\dagger} H\right)^{3}$ stabilize a new vacuum of lower energy and vEv $v^{\prime} \sim M_{\mathrm{Pl}}$. This suggests that the Standard Model Higgs vacuum is not absolutely stable. The analysis in [127] pointed out that in the absence of new physics, the 125 GeV Higgs appears to live in a sliver of metastable parameter space.


### 2.4 CONSTRAINTS ON NEW PHYSICS

Now that we've enumerated reasons our optimism, we present an overview of the types of constraints that a model of new physics must satisfy.

### 2.4.1 The Flavor and cp Problems

Note, further, that the limit of vanishing Yukawas $y \rightarrow 0$ enhances the global symmetries of the Standard Model to $U(3)^{5}$. Indeed, one of the tools of flavor physics is to treat the Yukawas as spurions for flavor symmetry breaking. Since flavor observables are sensitive to new heavy states in loops, the non-observation of deviations in the SM sector seems to imply that the physics of the flavor sector is decoupled or otherwise aligns with this structure. See [128] for a recent application of this 'minimal flavor violation' principle to natural models of supersymmetry.

The flavor structure of the Standard Model is intimately linked to the extent to which it may violate CP. For example, complex phases in the Yukawa matrices are only physical-that is, cannot be rotated away by phase redefinitions in the fields-when there are at least three generations. In this sense flavor and CP observables constrain models of new physics in similar ways. In this section we only briefly address flavor-changing neutral currents in mesons; CP observables are more subtle and will be discussed later in this thesis.

Prospects for the future of flavor physics fall under the banner of 'intensity frontier' physics (to distinguish from the energy and cosmic frontiers) and were recently summarized in [11,129,130]. At the moment, the most stringent constraints come from the quark sector where rare decays of mesons are sensitive to contributions of new particles to FCNCS. The standard approach to apply flavor bounds is to parameterize the new physics contributions in terms of four-fermion
operators [131-133],

$$
\begin{equation*}
\frac{c^{2}}{\Lambda^{2}}\left(\bar{q} \mathcal{O}_{I} q\right)\left(\bar{q} \mathcal{O}_{I} q\right), \tag{2.28}
\end{equation*}
$$

where the $\mathcal{O}$ are products of $\gamma$-matrices and we have suppressed flavor and Lorentz indices for simplicity. One must then match these observables at the new physics scale and run to the meson scale-the appropriate renormalization factors can be taken 'off the shelf' from, for example, $[134,135]$. Measurements $K^{\circ}-\bar{K}^{\circ}$ mixing, for example, constrain $\Lambda / g \gtrsim 10^{4} \mathrm{TeV}$. Even with loop suppression and factors of the CKM matrix, this suggests that signatures TeV-scale new physics should be accessible to flavor machines in the near future. Alternately, they impose strict constraints on the flavor structure of models of new physics.

### 2.4.2 Electroweak precision observables

An alternate approach to probing the radiative effects of new physics is to consider observables which can be measured with enough precision to be sensitive to sensitive to higher loop corrections that can probe heavy states.

The electroweak sector of the Standard Model is described by four parameters: the two gauge couplings $g$ and $g^{\prime}$, the Higgs VEv $v$, and the Higgs mass $m_{h}$. In fact, as described above, only the first three parameters are necessary to describe the nonlinear $\Sigma$ model that describes non-Higgs phenomena at energies below $v$. These, in turn, can be repackaged into three
independent quantities that are directly well measured by experiments: the $Z$ mass, the Fermi constant, and the fine structure constant,

$$
\begin{equation*}
M_{Z}^{2}=\frac{1}{4}\left(g^{2}+g^{\prime 2}\right) v^{2} \quad G_{F}=\frac{1}{\sqrt{2} v^{2}} \quad a_{\mathrm{EM}}=\frac{g^{2} g^{\prime 2}}{4 \pi\left(g^{2}+g^{\prime 2}\right)} \tag{2.29}
\end{equation*}
$$

Assuming that the new physics is heavier than the weak scale, electroweak observables only appear in loops. Following the same type of effective Lagrangian approach,

$$
\begin{equation*}
\mathcal{L}=\mathcal{L}_{S M}+\sum_{i} \frac{c_{i}}{\Lambda^{2}} \mathcal{O}_{i}+\mathcal{O}\left(\Lambda^{-3}\right) \tag{2.30}
\end{equation*}
$$

where the operators $\mathcal{O}$ are assumed to respect $\mathrm{SU}(2)_{L} \times \mathrm{U}(1)_{Y}$. The only dimension-5 operator that does this is the neutrino mass term in (2.26) which we may neglect since the observed neutrino masses are so small that they force the cutoff for that term to be too large to affect electroweak observables. Further, since flavor constraints also force a large cutoff scale, it is sufficient to consider the list of dimension-6 operators that preserve flavor and cP. Such a classification was presented in [136] and leads to a list of 18 independent operators.

This approach, however, is not particularly intuitive. A more practical approach is to focus on combinations of operators that can be measured most sensitively [137-140]. To do this, we rewrite the 18 operators in term of the oblique (vacuum polarization) form factors $\Pi\left(p^{2}\right)$,

$$
\begin{equation*}
\mathcal{L}_{\text {oblique }}=-\frac{1}{2} W_{\mu}^{3} \Pi_{33} W^{3 \mu}-W_{\mu}^{+} \Pi_{W W} W^{-\mu}-\frac{1}{2} B_{\mu} \Pi_{B B} B^{\mu}-B_{\mu} \Pi_{3 B} W^{3 \mu} . \tag{2.31}
\end{equation*}
$$

When expanding the form factors, note that the $\Pi(\mathrm{o})$ terms are masses and the $p^{2} \Pi^{\prime}(\mathrm{o})$ are kinetic terms. Taking up to second order in such an expansion about $p^{2}=\circ$ gives 12 parameters. These are reduced by using the well-measured quantities (2.30) and then further requiring $m_{\gamma}=0$ so that electromagnetism is unbroken. The result is a list of 'oblique parameters', the most famous of which are the Peskin-Takeuchi parameters [141],

$$
\begin{equation*}
\hat{S}=\frac{g}{g^{\prime}} \Pi_{3 B}^{\prime} \quad \hat{T}=\frac{\Pi_{33}-\Pi_{W W}^{2}}{M_{W}^{2}} \quad \hat{U}=\Pi_{W W}^{\prime}-\Pi_{33}^{\prime} \tag{2.32}
\end{equation*}
$$

These quantities provide a simple language to apply precision bounds to models of new physics. Intuitively $\hat{S}$ is sensitive to new chiral species. $\hat{T}$, on the other hand, encodes the deviation from the sm $\rho$ parameter,

$$
\begin{equation*}
\left.\rho\right|_{\text {tree }}=\frac{M_{W}^{2}}{M_{Z}^{2} \cos ^{2} \theta_{W}}=1 \tag{2.33}
\end{equation*}
$$

More generally $\rho=G_{\mathrm{NC}} / G_{\mathrm{CC}}$ is the ratio between the neutral current and the charged current couplings when one integrates out the massive vector bosons,

$$
\begin{equation*}
\mathcal{L}_{\mathrm{eff}}=-\frac{1}{\sqrt{2}}\left(G_{\mathrm{CC}} J_{\mu}^{+} J^{-\mu}+G_{\mathrm{NC}} J_{\mu}^{Z} J^{Z \mu}\right) \tag{2.34}
\end{equation*}
$$

Note that the tree-level relation $\rho=1$ is enforced in the Standard Model through a custodial SO(4) symmetry. Heuristically one may treat $\left\langle H^{\dagger} H\right\rangle$ as a non-zero vector in $\mathbb{R}^{4}$. Alternately, this is the

$$
\mathrm{SO}(4)=\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}
$$

global symmetry of the Higgs sector where $U(1)_{Y} \subset S U(2)_{R}$. Writing $H$ as a bidoublet under $S U(2)_{L} \times \operatorname{SU}(2)_{R}$, we note that

$$
\langle H\rangle=\left(\begin{array}{ll}
v &  \tag{2.35}\\
& \\
& v
\end{array}\right)
$$

so that $\operatorname{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$ is broken to the diagonal subgroup $\mathrm{SU}(2)_{\mathrm{V}}$. This unbroken subgroup protects $\rho=1$ at tree level. This is because the $W^{1,2,3}$ gauge bosons of $S U(2)_{L}$ must transform as triplets under $\operatorname{SU}(2)_{V}$. Since $\operatorname{SU}(2)_{V}$ is unbroken at tree
level, the $W^{i}$ should have the same mass,

$$
\begin{equation*}
\mathcal{L}_{\text {gauge, mass }}=-\frac{1}{2} M_{W}^{2}\left[\left(W^{1}\right)^{2}+\left(W^{2}\right)^{2}\right]-\frac{1}{2} M_{Z}^{2}\left(\cos \theta_{W} W^{3}-\sin \theta_{W} B\right)^{2}, \tag{2.36}
\end{equation*}
$$

so that $M_{W}^{2}=\cos \theta_{W}^{2} M_{Z}^{2}$. This typically sets a very strong constraint on extensions of the Higgs sector that some models of new physics require custodial symmetry to be imposed by hand to avoid these bounds.

### 2.4.3 Dark Matter Constraints

Models of new physics that incorporate dark matter must also pass increasingly stringent bounds from dark matter searches. The foremost bound comes from requiring that the dark matter candidate realize the correct relic density. For thermally produced dark matter this is a bound on the mass and annihilation cross section. One requires that the dark matter abundance that is 'frozen out' when the Hubble expansion rate becomes greater than the DM annihilation rate can account for the observed dark matter density today. As discussed above, one of the most provocative regions of parameter space is when the DM mass and annihilation cross section are both characteristic of the weak scale so that the 'wIMP' dark matter may be associated with electroweak symmetry breaking. See [142] for a review with an emphasis on supersymmetric candidates. It is worth noting that one may also consider non-thermal dark matter production that somewhat circumvents the restrictions of thermal relics, two notable examples are the super-wimp framework [143] and the feebly-interacting massive particle framework [144-146].

Once a model realizes the correct DM relic density, it must also account for the bounds set by direct detection experiments that probe collisions of dark matter with cryogenic bolometers. To reduce background, direct detection experiments are typically designed to select events based on two of the following types of criteria:

- Heat. Phonons from the dm collision are registered as a pulse of thermal energy.
- Charge. The recoil of a nucleus from a dm collision an ionize the detector material. Placing the detector in an electric field allows one to discriminate DM events based on the amount of charge deposited from these ionization events.
- Light. The recoiling nucleus causes the rest of the detector medium to scintillate and emit photons. This signal can be enhanced with photomultiplier tubes and detected.

In this document we neglect recent purported signals of dark matter that are in conflict with bounds from other direct detection experiments. The current bound from the Xenon 100 detector sets an upper limit on the spin-independent wimp-nucleus cross section of $2 \times 10^{-45} \mathrm{~cm}^{2}$ for 55 GeV dM [117].

The early history of the universe imposes cosmological bounds not only on dark matter but new light states in an extension of the Standard Model. Many of these constraints have been examined in context of additional neutrino species [147], though the bounds can be readily translated to more general light particles. The strictest bounds come from the eras of recombination and structure formation. New particles can inject energy into the intergalactic medium during recombination and leave an imprint on the cosmic microwave background. Alternately, these states can smear out early structures. Observations of galaxies today, for example from the Lyman- $\alpha$ forest, thus constrain the types of particles that may have been active during that era. These cosmological bounds are an additional handle for models where dark matter interacts primarily with additional states in a dark sector of the theory. Such models are difficult to probe at colliders but early universe constraints can limit the types of interactions in the dark sector.

### 2.4.4 Collider Bounds

The most prominent bounds on new physics comes from direct production of new particles at the Large Hadron Collider. At the time of this writing, there have yet to be any strong indications of deviations from the Standard Model at the Lhc. If new physics exists, it must either have a characteristic mass scale that is not yet accessible to the 8 TeV run (20/fb) or is otherwise hidden from current searches.

This non-observation sets the strictest bounds on new 'colored' particles charged under $\operatorname{SU}(3)_{c}$ such as squarks and gluinos in supersymmetry. These particles are expected to be produced copiously at the LHC so that their mass scales are
generically bounded from below by approximately a TeV . This appears especially troublesome since the most popular solutions to the Hierarchy problem usually invoke a partner to the top quark to cancel the large top contribution to the Higgs mass.

Over the next few years the LHC is expected to ramp up to its 14 TeV design center of mass energy and will hit luminosities of $\mathcal{O}(100) / \mathrm{fb}$. Collisions at these energies will directly access higher mass scales and the order of magnitude increase in data will illuminate many more search channels for new physics.

### 2.5 DIRECTIONS FOR NEW PHYSICS

We now distinguish between two types of extensions to the Standard Model: those that are weakly coupled and those that are strongly coupled. We explain that while this bifurcation is somewhat artificial, it provides a thematic framework for the context of this work.

### 2.5.1 WEAK COUPLING: SUSY-LIKE

Weakly coupled extensions of the Standard Model are those that are most readily described by additional Feynman rules. In other words, these are additional Lagrangian terms that are manifestly perturbative. The flagship example of this is supersymmetry, where the electroweak sector of the minimal supersymmetric Standard Model (MSSM) inherits the perturbative couplings of the SM.

As discussed above, the 125 GeV Higgs boson implies a perturbative Higgs quartic coupling $\lambda$. This has been interpreted as a triumph of spontaneous symmetry breaking with weak coupling in contrast to the analogous case of the electromagnetic symmetry breaking in the Bardeen-Cooper-Schrieffer (BCS) description of superconductivity.

### 2.5.2 STRONG COUPLING: TECHNICOLOR-LIKE

An alternate possibility is that the new physics sector is not perturbatively described by its fundamental degrees of freedom. This is analogous to low energy QCD where the interactions between quarks and gluons are too strongly coupled to be described by leading order Feynman diagrams.

The prototype for this type of extension of the SM electroweak sector is technicolor, where strong dynamics are used to break electroweak symmetry [48-50]. While the simplest technicolor models successfully give masses to gauge bosons [44, 51 ], an additional extension is required to also give masses to the sm fermions [148, 149]. These models, however, are strongly constrained by the techni-pion couplings to the SM [150, 151] and by constraints on FCNCs [149]. The so-called 'walking technicolor' models attempt to circumvent some of these problems by enhancing the techni-condensate through large quantum effects [152-161].

Stringent constraints from flavor and electroweak precision doomed the simplest technicolor models, and the 125 GeV Higgs is often interpreted as the tombstone for strong dynamics associated with the electroweak symmetry breaking.

### 2.5.3 WEAKLY COUPLED TALES OF STRONG DYNAMICS

The distinction between 'weak' and 'strong' coupling, however, is somewhat artificial. It is often the case that a strongly coupled system offers some weakly coupled perturbative description in terms of effective degrees of freedom. The prototype for this is the nonlinear sigma model (NLEM), also known as chiral perturbation theory, which describes the spectrum of low energy QCD based on the Goldstone bosons the flavor symmetry broken by the $\langle\bar{q} q\rangle$ expectation value. These nonlinear realizations or 'phenomenological Lagrangians' utilize the symmetry breaking structure of the UV theory to describe its IR degrees of freedom. In fact, the particular choice of parameterization is irrelevant [162, 163].

In the case of the NLEM, it is clear that the effective low energy description is less fundamental than the QCD Lagrangian, even though the latter is hopelessly non-perturbative at low energies. In this sense one may argue that QCD, despite its intractability below its strong coupling scale, should be placed on a higher pedestal from the vantage point of microscopic descriptions of nature.

The revolutions in theoretical physics of the 1990 have revised the extent to which an 'effective theory' can be considered a pejorative term. The key development were the discoveries of dualities between quantum field theories (often under the umbrella of techniques from string theory). Two salient examples are Seiberg duality and the AdS/CFT correspondence (more generally, the holographic principle). Seiberg duality relates two supersymmetric gauge theories in their infrared limits while AdS/CFT relates a $(d+1)$-dimensional theory of classical gravity to a 'very quantum' $d$-dimensional conformal theory. Both examples can be used to avoid the non-perturbativity of a gauge theory by constructing an equivalent weakly-coupled physical description. These weakly coupled descriptions look very different from the non-perturbative formulation. At the cost of what appears to be an exotic theory that is completely unrelated to the original, they can offer simple ways to build models that realize the hopes and dreams of Section 2.3 while avoiding the pitfalls of Section 2.4. The key, however, is to appreciate that these simple-yet-unusual features describe the same phenomenon of the very quantum regime of the original theory. In this sense we have a handle for the physical effects of strong dynamics without having to directly deal with the strongly coupled degrees of freedom.

It is this approach to new physics that we explore in this thesis.

Yuhsin: I will only use 10 equations in this talk. This is based on the paper arXiv: 0811.0871 .
Csaba: Does that count as an equation?
Johannes: No, it doesn't even have an equal sign. Don'tyou know what an equation is, Csaba?

27 February 2009


## Warped Extra Dimensions

The idea that the universe may be hiding extra dimensions that happen to be too small to observe may sound like it came straight from an old science fiction novel. Since the late 1990s, however, extra dimensions have been a powerful tool for understanding the behavior of strongly coupled field theories.

### 3.1 Flat extra dimensions

The original proposal for extra dimensions by Kaluza [164], Klein [165], and later Einstein [166] were attempts to unify electromagnetism with gravitation; see [167] for a review. Several decades later the development of string theory-originally as a dual theory to explain the Regge trajectories of hadronic physics-led physicists to revisit the idea of compact extra dimensions [168-170].

In early models, the non-observation of an additional spatial direction was explained by requiring the compactification radius to be too small for macroscopic objects. An alternative explanation now referred to as the 'braneworld' scenario was introduced by Rubakov and Shaposhnikov [171]. They suggested that instead of a very small radius of compactification, it may be that our observed universe is constrained to live in a $(3+1)$-dimensional subspace of a higher dimensional spacetime. This idea was revisited as an explanation for the Hierarchy problem by Arkani-Hamed, Dimopoulos, and Dvali in the ADD or 'large extra dimension' model [172-174]. They observed that in such a set up only gravity is required to propagate in the higher dimensional (bulk) space. In this way, the effective 4D Planck scale is enhanced from the higher dimensional Planck scale by the volume of the extra spatial dimensions. Gravity is much weaker than the electroweak scale because gravitational field lines are diluted through more space.

### 3.2 Warped extra dimensions

Randall and Sundrum proposed an alternate braneworld scenario that explains the hierarchy between the Planck scale and the weak scale while keeping 5 D Planck scale the same order of magnitude as the observed ${ }_{4} \mathrm{D} M_{\mathrm{Pl}}$ [175]. In the rs framework the bulk space is warped along the extra dimension. If the brane containing the SM is placed some distance along this gravitational
well, all of the energy scales are automatically redshifted. When the space is sufficiently warped, even a 'small' extra dimension can produce the many orders of magnitude between $M_{\mathrm{Pl}}$ and the electroweak scale.

In this document we focus on the RS 1 model where the bulk space is bounded by a 4 D brane on either end. The IR (or TeV ) brane is placed at a distance where it is redshifted-and hence is a natural subspace for sm fields-while the uv (or Planck) brane is placed at a position with no redshift. There is also an RS2 model where the UV brane supports the SM and the IR brane is sent to infinity, but we will not explore this idea further in this thesis.

Extra dimensions and the RS scenario in particular are reviewed in several places. Among the author's favorites are [176-180]. In this chapter we will provide a broad overview but will focus on aspects that are not already highlighted in those reviews.

### 3.2.1 The rs set up

The rs framework extends Minkowski space with an extra dimension that is a coset space. The particular coset is $S^{1} / \mathbb{Z}_{2}$ which is called an orbifold, a manifold which in which discrete points are identified. The circle $S^{1}$ of radius $r_{c}$ is parameterized by a coordinate $y \subset\left[-\pi r_{c}, \pi r_{c}\right]$ an the orbifold identification can be taken to be $y \cong-y$. In this way the physical space is the interval between $y=0$ and $y=\pi r_{c}=L$. In this section we follow the treatment of [176].

We want the ${ }_{5} \mathrm{D}$ space to be curved so that we allow it to have a bulk cosmological constant $\Lambda$. The bulk action is

$$
\begin{equation*}
S_{5}=\int d^{4} x \int_{-L}^{+L} d y \sqrt{g}\left(M_{5}^{3} R-\Lambda\right) \tag{3.1}
\end{equation*}
$$

where $g$ is the determinant of the ${ }_{5} \mathrm{D}$ metric, $M_{5}$ is the ${ }_{5} \mathrm{D}$ Planck scale, and $R_{5}$ is the ${ }_{5} \mathrm{D}$ Ricci scalar. As discussed above, there are two 4 D branes in the theory. These are placed on the orbifold fixed points $y=0, L$. Each brane carries its own 4 D contribution to the action with an induced metric:

$$
\begin{align*}
S_{\mathrm{UV}} & =\int d^{4} x \sqrt{g_{\mathrm{UV}}}\left(\mathcal{L}_{\mathrm{UV}}-\Lambda_{\mathrm{UV}}\right)  \tag{3.2}\\
S_{\mathrm{IR}} & =\int d^{4} x \sqrt{g_{\mathrm{IR}}}\left(\mathcal{L}_{\mathrm{IR}}-\Lambda_{\mathrm{IR}}\right) \tag{3.3}
\end{align*}
$$

As shown below, we identify the IR brane as the one which is redshifted so that the 4 D Lagrangian $\mathcal{L}_{\text {IR }}$ is identified with the Standard Model Lagrangian (2.1). Uv brane will play an important role for imposing boundary conditions on fields which propagate in the bulk. For the purposes of the simplest rS models the details of the local Uv physics in $\mathcal{L}_{\text {UV }}$ is irrelevant to the TeV -scale. For non-minimal models, however, one may make use of $\mathcal{L}_{\mathrm{UV}}$ to break symmetries with very controlled mediation to the SM. This is a natural way, for example, to mediate susy breaking [181]). As we discuss below, the holographic principle allows us to interpret such mediation in terms of strong dynamics, e.g. [182].

We posit a non-factorizable metric that is warped in the extra dimension,

$$
\begin{equation*}
d s^{2}=e^{-2 \sigma(\varphi)} \eta_{\mu \nu} d x^{\mu} d x^{\nu}-r_{c}^{2} d \varphi^{2}, \tag{3.4}
\end{equation*}
$$

where it should be clear that $\varphi$ is the angular coordinate for the extra dimension. One can convert this into a metric for the parameter $y=r_{c} \varphi$, but it turns out to be more useful to introduce yet another variable, $z$, such that

$$
\begin{equation*}
d s^{2}=e^{-A(z)}\left(\eta_{\mu \nu} d x^{\mu} d z^{\nu}-d z^{2}\right) \tag{3.5}
\end{equation*}
$$

Recall that the Einstein equation relates the Einstein tensor to the stress energy tensor,

$$
\begin{equation*}
G_{M N}=R_{M N}-\frac{1}{2} R g_{M N}=\frac{1}{4 M_{5}^{3}} T_{M N}=\frac{g_{M N}}{2 M_{5}^{3}} \Lambda+\left.\frac{g_{M N}}{2 M_{5}^{3}}\left(\Lambda_{\mathrm{IR}} \delta(y)+\Lambda_{\mathrm{UV}} \delta(y-L)\right)\right|_{M, N \neq 5} \tag{3.6}
\end{equation*}
$$

This expression is somewhat tedious to calculate using the metric (3.4). Instead, by moving to a conformally flat frame with
coordinate $z$, there is a nice relation between the Einstein tensors of conformally related metrics,

$$
\begin{equation*}
G_{M N}=\tilde{G}_{M N}+\frac{d-2}{2}\left[\frac{1}{2} \tilde{\nabla}_{M} A \tilde{\nabla}_{N} A+\tilde{\nabla}_{M} \tilde{\nabla}_{N} A-\tilde{G}_{M N}\left(\tilde{\nabla}_{K} \tilde{\nabla}^{K} A-\frac{d-3}{4} \tilde{\nabla}_{K} A \tilde{\nabla}^{K} A\right)\right] \tag{3.7}
\end{equation*}
$$

Setting $\tilde{g}_{M N}=\eta_{M N}$, the covariant derivatives become partial derivatives, $\tilde{\nabla}_{M} \rightarrow \partial_{M}$. One can then read off the 55 and $\mu \nu$ components of the Einstein tensor

$$
\begin{equation*}
G_{55}=\frac{3}{2} A^{\prime 2} \quad G_{\mu \nu}=\frac{3}{2} \eta_{\mu \nu}\left(A^{\prime \prime}-\frac{1}{2} A^{\prime 2}\right) \tag{3.8}
\end{equation*}
$$

We can now solve Einstein's equation for the 55 and $\mu \nu$ components separately. The 55 component is independent of the brane tension terms and gives

$$
\begin{equation*}
-\frac{3}{2} A^{\prime 2}=-\frac{1}{4 M_{5}^{3}} \Lambda e^{-A(z)} \tag{3.9}
\end{equation*}
$$

from which we may write

$$
\begin{equation*}
A^{\prime}=e^{-A(z) / 2} \sqrt{-\frac{\Lambda}{6 M_{5}^{3}}} \tag{3.10}
\end{equation*}
$$

The sign inside the square root imposes a negative cosmological constant $\Lambda<0$, and hence the bulk space is five dimensional anti-de Sitter $\left(\mathrm{AdS}_{5}\right)$. We can solve this equation using another trick. Define $f \equiv e^{-A / 2}$ and plug into equation (3.10) to get

$$
\begin{equation*}
-\frac{f^{\prime}}{f^{2}}=\frac{1}{2} \sqrt{-\frac{\Lambda}{6 M_{5}^{3}}} \tag{3.11}
\end{equation*}
$$

The general solution of this differential equation is

$$
\begin{equation*}
e^{-A(z)}=\frac{1}{(k z+1)^{2}} \tag{3.12}
\end{equation*}
$$

where we've defined the curvature $k^{2}=-\Lambda / 12 M_{5}^{3}$. The constant is fixed by imposing $e^{-A(\circ)}=1$, and hence our conformally flat metric takes the form:

$$
\begin{equation*}
d s^{2}=\frac{1}{(k|z|+1)^{2}}\left(\eta_{\mu v} d x^{\mu} d x^{v}-d z^{2}\right) \tag{3.13}
\end{equation*}
$$

We have made the critical replacement of $z \rightarrow|z|$ to maintain the $S^{1} / \mathbf{Z}_{2}$ orbifold symmetry $\varphi \rightarrow-\varphi$, or equivalently $z \rightarrow-z$. Thus the 55 component of the Einstein equation indeed fixes the warp factor.

We proceed to the $\mu v$ components. One can see that (3.8) one requires non-vanishing the brane cosmological constants. (3.13) tells us that $A$ depends on the modulus of $z$ via $A=\ln \left[(k|z|+1)^{2}\right]$. This means that the second derivative terms in (3.8) will generate $\delta$ functions at the orbifold boundaries $z=0, z_{1}$;

$$
\begin{equation*}
A^{\prime \prime}=-\frac{2 k^{2}}{(k|z|+1)^{2}}+\frac{4 k}{k|z|+1}\left(\delta(z)-\delta\left(z-z_{1}\right)\right) \tag{3.14}
\end{equation*}
$$

These $\delta$-functions must be compensated by localized energy densities on the branes, i.e. brane cosmological constants. Physically, these brane cosmological constants compensate the ${ }_{5} \mathrm{D}$ bulk cosmological constant so that the induced ${ }_{4} \mathrm{D}$ brane metric is flat. Inserting equations (3.10) and (3.14) into (3.8),

$$
\begin{equation*}
G_{\mu \nu}=-\frac{3}{2} \eta_{\mu \nu}\left[\frac{4 k^{2}}{(k|z|+1)^{2}}-\frac{4 k\left(\delta\left(z-z_{\mathrm{UV}}\right)-\delta\left(z-z_{\mathrm{IR}}\right)\right.}{k|z|+1}\right] \tag{3.15}
\end{equation*}
$$

Using the definition of $k$ and comparing this to the energy momentum tensor we see that the first term above cancels the bulk
cosmological constant contribution. The remaining $\delta$-function terms must correspond to the brane tensions such that

$$
\begin{equation*}
-\frac{3}{2} \eta_{\mu \nu}\left[-\frac{4 k\left(\delta(z)-\delta\left(z-z_{1}\right)\right)}{k|z|+1}\right]=\frac{\eta_{\mu v}}{4 M_{5}^{3}}\left[\frac{\left.\Lambda_{\mathrm{UV}} \delta(z)-\Lambda_{\mathrm{IR}} \delta\left(z-z_{1}\right)\right)}{k|z|+1}\right] \tag{3.16}
\end{equation*}
$$

Finally, we have a relation between the brane tensions,

$$
\begin{equation*}
\Lambda_{\mathrm{IR}}=-\Lambda_{\mathrm{UV}}=\sqrt{\frac{-\Lambda}{24 M_{5}^{3}}} \tag{3.17}
\end{equation*}
$$

This expresses that we have 'unloaded' the 4 D curvature into the bulk. While this is a different perspective on the cosmological constant, we should note that this is a tuning in the values of $\Lambda_{I R, U V}$.

### 3.2.2 GENERATING THE HIERARCHY

In order to understand how this framework generates the weak-Planck hierarchy, we first explore the low energy theory that is generated by the RS scenario. We are especially interested in writing the 4 D Planck mass $M_{\mathrm{Pl}}$ and the Standard Model masses in terms of the remaining unconstrained ${ }_{5} \mathrm{D}$ parameters $M_{5}, k$ (or $\Lambda$ ), and $r_{c}$. For convenience, we return to the non-conformal frame,

$$
\begin{equation*}
d s^{2}=e^{-2 k|y|} \eta_{\mu \nu} d x^{\mu} d x^{\nu}-d y^{2} \tag{3.18}
\end{equation*}
$$

Note that we have identified

$$
\begin{equation*}
\frac{1}{(k|z|+1)^{2}}=e^{-2 k|y|} \tag{3.19}
\end{equation*}
$$

for convenience here. After this chapter it is more useful to define the $z$ coordinates such that the left-hand side of this expression is $1 / k|z|^{2}$ or $(R / z)^{2}$, where $R$ is the radius of curvature. In particular,

$$
\begin{equation*}
z=R e^{k y} \quad k=1 / R \tag{3.20}
\end{equation*}
$$

We now derive the 4 D effective Planck mass, $M_{\mathrm{Pl}}$. We must assume that the radius $r_{c}$ is fixed at some constant value. In the following section we will motivate a mechanism by which the $r_{c}$ modulus is stabilized. 4 D graviton excitations $h_{\mu v}(x)$ can be inserted into the metric on top of the flat 4 D metric as follows

$$
\begin{equation*}
d s^{2}=e^{-2 k|y|}\left(\eta_{\mu \nu}+h_{\mu v}(x)\right) d x^{\mu} d x^{\nu}-d y^{2} \tag{3.21}
\end{equation*}
$$

Taking the curavture of the $h_{\mu v}(x)$ perturbation into account, we obtain an additional contribution to the bulk gravitational action

$$
\begin{equation*}
\Delta S_{g}=M_{5}^{3} \int d^{4} x \int_{-L}^{L} d y e^{-4 k|y|} \sqrt{\tilde{g}} e^{2 k|y|} \tilde{R} \tag{3.22}
\end{equation*}
$$

Where $\tilde{g}_{\mu \nu}=\eta_{\mu \nu}+h_{\mu \nu}(x)$ and $\tilde{R}$ is the 4 D Ricci tensor formed by $\tilde{g}_{\mu \nu}$. By performing the $y$ integral we get a contribution to the 4 D effective action whose coefficient is the 4 D effective Planck mass $M_{\mathrm{Pl}}$. Explicitly,

$$
\begin{equation*}
M_{\mathrm{Pl}}^{2}=M_{5}^{3} \int_{-L}^{L} d y e^{-2 k|y|}=\frac{M_{5}^{3}}{k}\left(1-e^{-k L}\right) \tag{3.23}
\end{equation*}
$$

The key feature in the above equation is that it is insensitive to the size of the extra dimension $r_{c}$. This is in contrast to other braneworld scenarios. Further, if we let the 5 D parameters take natural values near the fundamental Planck scale, then the 4 D Planck mass is the same order as the ${ }_{5} \mathrm{D}$ Planck mass, $M_{\mathrm{Pl}} \sim M_{5}$.

We now consider the generation of the electroweak scale, $v$. In the rs model, the Standard Model Lagrangian is part of $\mathcal{L}_{\text {IR }}$. The presence of the warp factor in $\sqrt{g_{\text {IR }}}$ and implicitly in the contraction of vector indices will force us to rescale our fields to
maintain canonical normalization. This rescaling will be the source of the exponential suppression of the weak scale relative to the Planck scale. Consider the Higgs sector on the IR brane with a ${ }_{5}$ D parameter $v_{0}$ characterizing the Higgs vev,

$$
\begin{align*}
S_{\mathrm{H}} & =\left.\int d^{4} x \sqrt{g_{\mathrm{IR}}}\left[g_{\mathrm{IR}}^{\mu \nu} D_{\mu} H\left(D_{v} H\right)^{\dagger}-\lambda\left(|H|^{2}-v_{\mathrm{o}}^{2}\right)^{2}\right]\right|_{y=L}  \tag{3.24}\\
& =\int d^{4} x e^{-4 k L} \sqrt{\tilde{g}}\left[e^{2 k L} \tilde{g}^{\mu \nu} D_{\mu} H\left(D_{v} H\right)^{\dagger}-\lambda\left(|H|^{2}-v_{\mathrm{o}}^{2}\right)^{2}\right] \tag{3.25}
\end{align*}
$$

Recall that $\tilde{g}_{\mu \nu}$ is just the Minkowski metric with a 4D gravtion perturbation. Now watch carefully, this is the magical part. In order to work in an effective 4D low-energy theory, we need to canonically normalize our Higgs field $H \rightarrow e^{k L} H$ and so we write this above line as

$$
\begin{equation*}
S_{\mathrm{H}}=\int d^{4} x \sqrt{\tilde{g}}\left[\tilde{g}^{\mu \nu} D_{\mu} H\left(D_{v} H\right)^{\dagger}-\lambda\left(|H|^{2}-e^{-2 k L} v_{\mathrm{o}}^{2}\right)^{2}\right] . \tag{3.26}
\end{equation*}
$$

This tells us that the effective Higgs action takes its usual 4 D form with the vacuum expectation value given by $v=e^{-k L} v_{0}$. Since masses are generated by the Yukawa terms after electroweak symmetry breaking, we see that mass terms $m_{0}$ also become rescaled by the same factor,

$$
\begin{equation*}
m=e^{-k L} m_{0} . \tag{3.27}
\end{equation*}
$$

The wondeful result states that dimensionful quantities on the brane have been warped while leaving dimensionless parameters such as the Yukawa coupling of the Higgs coupling $\lambda$ unchanged.

Unlike the 4D effective Planck mass $M_{\mathrm{Pl}} \sim M_{5}$, the masses of the electroweak Standard Model particles are exponentially sensitive to the product $k L$. To avoid fine-tuning and a 'hierarchy', we expect the 'fundamental' dimension 1 parameters $M_{5}, k$ (or alternately $R$ or $\Lambda$ ), and $v_{0}$ take natural values on the order of the Planck scale. We see from (3.27) that the 15 orders of magnitude between the Planck and weak scale can be successfully generated with natural values of $k r_{c} \approx 30$. We've thus elminated the need for excessive fine-tuning and have removed the 'problem' from the hierarchy.

### 3.2.3 Radius stabilization

Solving the hierarchy problem by introducing of a metric which is exponentially sensitive to the extra dimension appears to be slight of hand. The statement that $k r_{c} \sim 30$ does not by itself solve the fine tuning problem in the Standard Model. Specifically, one might wonder if this is a relic of a nonlinear choice of coordinates. In that case, the 'miracle' that $k r_{c}$ is $\mathcal{O}(10)$ would actually hide an exponential sensitivity to the precise value of $k r_{c}$. In other words, one might expect that the natural scale at which $r_{c}$ is stabilized would be $r_{c} \sim 1 / k$ so that $k r_{c} \sim 30$ appears tuned.

Fortunately, this is not the case. The radius of the extra dimension is a modulus in our theory and should be treated as a dynamical degree of freedom, $r_{c}=r(x, y)$, known as the radion. It is associated with the 4 D scalar component arising form the decomposition of the ${ }_{5} \mathrm{D}$ metric. Because the radion has no potential in our theory it is a massless particle whose phenomenology would violate the equivalence principle and Newton's law. Thus there must be a mechanism to stablize the radion moduli to dynamically fix radius of our extra dimension to our desired value. This radius stabilization is the key ingredient for understanding why the exponential hierarchy is physical: the spacetime is warped between the two branes. By providing a mechanism by which the two branes can be separated in the gravitational well of the extra dimension, fields propagating in this space are actually redshifted as they 'fall' towards the IR brane.

A standard solution in the rS model is the Goldberger-Wise mechanism [183, 184], where radion kinetic and potential terms conspire against one another to create a radion potential with a desirable vacuum. We briefly outline the general procedure following [185]. We remark that the Goldberger-Wise mechanism is one simple option to stabilize the size of the extra dimension, but turns out to be close to what actually happens in string compactifications [186].

Physically we will stabilize the radius of the extra dimension by allowing $r_{c}$ to be generated dynamically. The radion kinetic term energetically prefers a large radius where derivatives are small. One may balance this by a non-trivial bulk potential for the radius that can be generated by adding brane-localized potentials.

The approach we take is sometimes referred to as the 'superpotential' method, though this is related to the superpotential
of supersymmetry only in the most general sense that it produces first order equations of motion. The main idea is that we search for a solution to a bulk scalar field which includes the effect of gravitational backreaction. That is, we seek a solution to the Einstein equations for a bulk scalar field $\Phi$ including, in principle, brane-localized terms. Fortunately, there are a class of potentials $V(\Phi)$ for which a closed solution exist. These potentials are written in terms of an arbitrary superpotential $W(\Phi)$ as

$$
\begin{equation*}
V(\Phi)=\frac{1}{8}\left(\frac{\partial W}{\partial \Phi}\right)^{2}-\frac{1}{6 M_{5}^{3}} W(\Phi)^{2} . \tag{3.28}
\end{equation*}
$$

Assuming that the vev of $\Phi$ respects 4D Lorentz invariance, $\langle\Phi\rangle=\varphi(y)$, the second order bulk equations can be reduced to first order equations of motion by quadrature to

$$
\begin{equation*}
\varphi^{\prime}(y)=\frac{1}{2} \frac{\partial W(\varphi)}{\partial \varphi} \quad A^{\prime}(y)=\frac{1}{6 M_{5}^{4}} W(\varphi) . \tag{3.29}
\end{equation*}
$$

We now include the effect of brane-localized terms for $\Phi$ coming from $\mathcal{L}_{\mathrm{IR}, \mathrm{UV}}$. These can be constrained by requiring that the surface terms generated from integration by parts vanish and yield the above equations of motion. The result is that

$$
\begin{equation*}
\pm \mathcal{L}_{\mathrm{IR}, \mathrm{UV}}=\frac{1}{2} W^{\prime}(\varphi(y=\mathrm{o}, L))[1+\Phi-\varphi(y=\mathrm{o}, L)] \pm \tilde{\mathcal{L}}_{\mathrm{IR}, \mathrm{UV}} . \tag{3.30}
\end{equation*}
$$

Here $\tilde{\mathcal{L}}_{\mathrm{IR}, \mathrm{UV}}$ are terms which fix the scalar boundary values $\varphi(y=L, \mathrm{o})$. This generates a non-trivial profile for $\Phi$ which can be used to tune the size of $r_{c}$. As an aside, the radion couples to the IR brane-localized Standard Model fields according to the trace of the 4 D energy-momentum tensor. In this way it carries couplings that are very similar to that of the Higgs boson. See [187] for a recent exploration of this idea in light of the 125 Gev Higgs.

### 3.3 MODERN RS MODELS

Over the last decade, however, the rS model has evolved to address challenges and take advantage of opportunities. Here we briefly summarize some of the key developments leading up to the 'modern' rS scenario which is what is now implied when theorists refer to the Randall-Sundrum framework. This is important phenomenologically since the signatures of these models have evolved accordingly.

- Original rs [175]. As described above, the SM is completely localized on the IR brane. The Planck-Tev hierarchy is generated by warping the space between the branes, $M_{P 1}$ is warped down to TeV at the IR brane. The main signature are graviton KK modes since gravity is the only thing to propagate in the bulk.
- RS with Bulk Fields [188-190]. It was quickly realized that by pulling the Standard Model fields into the extra dimension one could solve problems with electroweak precision observables (specifically the $S$-parameter) and flavor-changing neutral currents. In order to maintain the solution to the Hierarchy problem, the Higgs remained localized on the IR brane or highly peaked toward it. An added benefit of this framework is that one can naturally explain the hierarchy in fermion masses with anarchic Yukawa matrices.
- Custodial rs [191]. Even with bulk fields, the 'realistic' rs models suffer from generically large contributions to the $T$-parameter. One way to solve this is to impose a custodial symmetry on the model. The bulk gauge symmetry is $S U(2)_{L} \times S U(2)_{R} \times U(1)_{X}$, and the model has additional heavy matter and gauge states.
- Variants and extensions of rs. The above models still have a 'little hierarchy problem' owing to the discrepancy between the $\mathcal{O}(1-10 \mathrm{Tev})$ IR brane scale and the electroweak scale. Ways to avoid this include Higgsless models [192], the gaugephobic Higgs [193], and embedding the rs model within a little Higgs framework [194]. Closely related to the little Higgs are the [holographic] composite pseudo-Goldstone Higgs models [195]. In both cases the Higgs is a pseudo-Goldstone related to some global symmetry breaking. However, the composite Higgs models have a percent level tuning since the Higgs potential is generated at one-loop, whereas the symmetry structure of the little Higgs allows a natural separation between $v$ and the global symmetry breaking scale $f$.

Alternately, one can ignore the electroweak Hierarchy Problem and use the rs framework as a solution to the little Hierarchy problem (see [196] and references therein). These 'littlens' models can be used for flavor and electroweak precision while invoking some other solution to the Hierarchy problem. In this sense they are the 'opposite' of the RS + little Higgs models where the warped geometry solves most of the Hierarchy and the little Higgs solves the remaining little Hierarchy.

### 3.4 Holographic interpretation

Finally, we address the strongly coupled interpretation of RS scenario. Through the AdS/CFT correspondence (more generally, the holographic principle) [197-199], one may understand the warped extra dimension as the renormalization group flow of a strongly coupled 4D gauge theory. This idea is explored in a non-stringy context in [180,200-203]. Rather than presenting a detailed derivation, we will be heuristic and refer the reader to the above references. For an intuitive motivation for why a higher dimensional theory could plausibly describe the behavior of a strongly coupled 4 D theory, consider the following hypothetical dialogue of a theorist trying to explain his theory of an extra dimension to an experimentalist:

Theorist: I have this great new theory.
Experimentalist: Neat. What does it predict?
Theorist: Well, you have a series of evenly spaced resonances...
Experimentalist: We already discovered that! They're called hadrons.
The point of the story is that extra dimensions predicts Kaluza-Klein excitations of each bulk field. These can be identified with bound states of a strongly coupled theory (e.g. QCD). While 'extra dimensions' may sound exotic, strong coupling is something that we know exists in nature. Models of warped extra dimensions give us a handle for the phenomenology of the low energy behavior of models of new strongly coupled physics.

In the Randall-Sundrum scenario, it is most convenient to work with the standard conformal AdS metric,

$$
\begin{equation*}
d s^{2}=\left(\frac{R}{z}\right)^{2}\left(\eta_{\mu \nu} d x^{\mu} d x^{v}-d z^{2}\right) \tag{3.31}
\end{equation*}
$$

where $z$ takes values between the UV brane $R \sim M_{\mathrm{Pl}}^{-1}$ and the IR brane $R^{\prime} \sim \mathrm{TeV}^{-1}$. In the absence of these branes the isometries of the space are equivalent to the conformal group $\operatorname{SO}(2,4)$ in four dimensions. In the explicit supersymmetric realization of $\mathrm{AdS} / \mathrm{CFT}$ correspondence, the isometries of $\mathrm{AdS}_{5} \times S^{5}$ are dual to the $\mathcal{N}=4$ superconformal group. As a cartoon, one can imagine 4D Minkowski slices of AdS. Moving along the interval corresponds to scale transformations since shifts in $z$ correspond to a rescaling of the induced metric on each slice.

The UV and IR branes thus break the conformal invariance. The UV brane can be understood as a cutoff dependence on the uv physics associated with the nearly-conformal 4D theory. The excitations near the uv brane correspond to elementary 'preon' degrees of freedom: these are the quark-like objects in the gauge theory which are confined at energies much lower than $1 / R$. On the other hand, the IR brane corresponds to a spontaneous breaking of conformal invariance. Note that this brane carries the length/energy scale of the extra dimension: $1 / R^{\prime}$ is the scale at which Kaluza-Klein excitations appear. These are identified with bound states analogous to mesons in QCD.

The profiles of fields in the extra dimension interpolate between the 'elementary' 4 D degrees of freedom near the uv brane and the 'composite' 4D degrees of freedom near the IR brane. In this sense they determine the extent to which a field is an admixture of elementary and composite states. This is a key insight for understanding the dynamics of bulk rs fields. Fields which are peaked near the IR brane like the Higgs are mostly composite. As we show below, one may generate the flavor hierarchies of the Standard Model in the RS framework by forcing the zero modes of light fields to get exponentially small masses due to exponentially suppressed couplings with the Higgs due to the warping of the zero mode profiles. In this sense the electron is very light because it carries very little wavefunction overlap with the Higgs. From the point of view of the conformal 4D theory, we say that the electron is mostly a fundamental field which is not sensitive to the interactions of the composite Higgs. The bulk profiles of fields are controlled by their spin and bulk mass parameter. This latter quantity is identified with the anomalous dimension of the 4 D operator.

The gauge symmetries of the bulk theory are identified with global symmetries of the conformal theory. When the bulk gauge field has a zero mode, there exists a massless 4D gauge field in the strongly coupled theory so that we say that the global
symmetry is gauged. If, on the other hand, boundary conditions are chosen so that the gauge field has no zero mode, the 4 D gauge symmetry is broken. When the ${ }_{5} \mathrm{D}$ gauge symmetry is broken on the uv brane, the 4 D zero mode picks up a Planck scale mass and completely decouples. We thus interpret this as a purely global symmetry. On the other hand, when the 5 D gauge symmetry is broken ont eIR brane, the 4 D zero mode picks up a mass on the order of $1 / R^{\prime}$. We say that the spontaneous breaking of conformal invariance also spontaneously breaks the weakly gauged symmetry. This picture is summarized in Table 3.1. Note that in the the application of the holographic principle to RS models carries some degree of agnosticism since it is not necessarily true that operators of the elementary degrees of freedom generate the particular anomalous dimensions that are assumed in the holographic interpretation.

| Bulk of AdS | $\leftrightarrow$ | CFT |
| :---: | :---: | :---: |
| Coordinate (z) along AdS | $\leftrightarrow$ | Energy scale in CFT |
| Appearance of UV brane | $\leftrightarrow$ | CFT has a cutoff |
| Appearance of IR brane | $\leftrightarrow$ | conformal symmetry broken spontaneously by CFT |
| KK modes localized on IR brane | $\leftrightarrow$ | composites of CFT |
| Modes on the UV brane | $\leftrightarrow$ | Elementary fields coupled to CFT |
| Gauge fields in bulk | $\leftrightarrow$ | CFT has a global symmetry |
| Bulk gauge symmetry broken on Uv brane | $\leftrightarrow$ | Global symmetry not gauged |
| Bulk gauge symmetry unbroken on Uv brane | $\leftrightarrow$ | Global symmetry weakly gauged |
| Higgs on IR brane | $\leftrightarrow$ | CFT produces composite Higgs |
| Bulk gauge symmetry broken on IR brane by boundary conditions | $\leftrightarrow$ | Strong dynamics that breaks CFT also breaks gauge symmetry |

Table 3.1: Relevant rules for model building using the AdS/CFT correspondence, table from [177].

As a final remark on holography, we present a brief history of the development of explicitly realized dualities that are derived from string theory.

- Maldacena [197]. The original AdS/CFT correspondence related a gravitational theory on $\operatorname{AdS}_{5} \times S^{5}$ to $\mathcal{N}=4$ superconformal field theory. This is way too symmetric. Because we know that the isometry group of the gravity theory is related to the internal symmetry of the gauge theory, we would like to find ways to modify the $S^{5}$ into a less symmetric space where still have a handle on the gauge theory.
- KW [204]. Klebanov and Witten found that if the extra dimensions form a conifold, AdS ${ }^{5} \times \mathrm{T}^{1,1}$, then one can break most of the supersymmetries. This turns out to be dual to $\mathcal{N}=1$ superconformal field theory. We'll say a few more words (but not that much more) about $\mathrm{T}_{1}, 1$ below. This will be as far as we will break the supersymmetry. As long as the theory is superconformal, however, there will be no RG running and we will not have anything Rs-like (where we recall that the dual picture of RS associates the warping with RG flow)
- KT [205]. Klebanov and Tseytlin found that adding fluxes into the mix (from wrapped $D_{5}$ branes) generates the
necessary back reaction to produce the desired 'warped throat'. We are thus reduced to $\mathcal{N}=1$ super Yang-Mills. This is pretty good, but the geometry contains a naked curvature singularity at the tip of the cone. In rs language we would say that there is no IR brane.
- Ks [206]. Klebanov and Strassler then smoothed out the tip of the conifold by blowing up the $S^{3}$ at the tip to produce the deformed conifold. The point is that this gets rid of the naked singularity and provides the desired structure for the IR brane and has a remarkable description in terms of a 'cascading' super QCD theory. This duality cascade also offers a way to understand the holographic duality in terms of the Seiberg duality of supersymmetric gauge theory. One remaining item for a "realistic" string realization is that the UV end of the conifold must be attached to a compact manifold. (It's much easier to work with noncompact conifolds as the tips of compactified conifolds.)
- GKP [207]. Finally, Giddings, Polchinski, and Kachru included the deformed conifold as an appendage to a Calabi-Yau flux compactification. As we shall see, GKP construction also provides a way to generate the hierarchy in scales between the IR (Ks tip) and the UV (compact manifold) branes.


### 3.5 Anarchic Flavor in Randall-Sundrum models

We summarize here the relevant aspects of flavor physics and the RS scenario. For a review of the general framework see e.g. $[177,180,201,208,209]$. We consider a ${ }_{5} \mathrm{D}$ warped interval $z \in\left[R, R^{\prime}\right]$ with an infrared (IR) brane at $z=R^{\prime} \sim(\mathrm{TeV})^{-1}$ and an ultraviolet (UV) brane at $z=R \sim M_{\mathrm{Pl}}$, the AdS curvature scale. In conformal coordinates the metric is

$$
\begin{equation*}
d s^{2}=\left(\frac{R}{z}\right)^{2}\left(d x_{\mu} d x_{v} \eta^{\mu \nu}-d z^{2}\right) \tag{3.32}
\end{equation*}
$$

One may recover the classic Rs conventions with the identifications $z=R \exp (k y)$ and $k=1 / R, k \exp (-k L)=1 / R^{\prime}$.
Fermions are Dirac fields that propagate in the bulk and can be written in terms of left- and right-handed Weyl spinors $\chi$ and $\bar{\psi}$ via

$$
\begin{equation*}
\Psi(x, z)=\binom{\chi(x, z)}{\bar{\psi}(x, z)} \tag{3.33}
\end{equation*}
$$

In order to obtain a spectrum with chiral zero modes, fermions must have chiral (orbifold) boundary conditions,

$$
\begin{equation*}
\psi_{L}\left(x^{\mu}, R\right)=\psi_{L}\left(x^{\mu}, R^{\prime}\right)=0 \quad \text { and } \quad \chi_{R}\left(x^{\mu}, R\right)=\chi_{R}\left(x^{\mu}, R^{\prime}\right)=0 \tag{3.34}
\end{equation*}
$$

where the subscripts $L$ and $R$ denote the $S U(2)_{L}$ doublet $(L)$ and singlet $(R)$ representations, i.e. the chirality of the zero mode (SM fermion). The localization of the normalized zero mode profile is controlled by the dimensionless parameter $c$,

$$
\begin{equation*}
\chi_{c}^{(\circ)}(x, z)=\frac{1}{\sqrt{R^{\prime}}}\left(\frac{z}{R}\right)^{2}\left(\frac{z}{R^{\prime}}\right)^{-c} f_{c} \chi_{c}^{(\circ)}(x) \quad \text { and } \quad \psi_{c}^{(\circ)}(x, z)=\chi_{-c}^{(\circ)}(x, z) \tag{3.35}
\end{equation*}
$$

where $c / R$ is the fermion bulk mass. Here we have defined the RS flavor function characterizing the fermion profile on the IR brane,

$$
\begin{equation*}
f_{c}=\sqrt{\frac{1-2 c}{1-\left(R / R^{\prime}\right)^{1-2 c}}} \tag{3.36}
\end{equation*}
$$

We assume that the Higgs is localized on the IR brane. The Yukawa coupling is

$$
\begin{equation*}
S_{\mathrm{Yuk}}=\int d^{4} x\left(\frac{R}{R^{\prime}}\right)^{4}\left[-\frac{1}{\sqrt{2}}\left(\bar{Q}_{i} \cdot \tilde{H} R Y_{u, i j} U_{j}+\bar{Q}_{i} \cdot H R Y_{d, i j} D_{j}+\bar{E}_{i}\left(R Y_{i j}\right) L_{j} \cdot H+\text { h.c. }\right)\right] \tag{3.37}
\end{equation*}
$$

where $Y_{i j}$ are dimensionless $3 \times 3$ matrices such that $\left(Y_{5}\right)_{i j}=R Y_{i j}$ is the dimensionful parameter appearing in the ${ }_{5} \mathrm{D}$ Lagrangian with $Y$ assumed to be a random 'anarchic' matrix with average elements of order $Y_{*}$. After including warp factors
and canonically normalizing fields, the effective 4D Yukawa and zero mode mass matrices are

$$
\begin{equation*}
y_{i j}^{S M}=f_{\mathcal{L}_{i}} Y_{i j} f_{-c_{R_{j}}} \quad m_{i j}=\frac{v}{\sqrt{2}} y_{i j}^{S M}, \tag{3.38}
\end{equation*}
$$

so that the fermion mass hierarchy is set by the $f_{1} \ll f_{2} \ll f_{3}$ structure for both left- and right-handed zero modes. At the same time, the hierarchical pattern of the скм matrix is also generated naturally. In other words, the choice of $c$ for each fermion family introduces additional flavor structure into the theory that generates the zero mode spectrum while allowing the fundamental Yukawa parameters to be anarchic.

In this document we work in the gauge basis where the bulk mass matrices and the interactions of the neutral gauge bosons are flavor diagonal but not flavor universal. The Yukawa couplings are non-diagonal in this basis and cause the resulting fermion mass matrices to be non-diagonal. Since these off-diagonal entries are governed by the small parameter $v R^{\prime}$, we will treat them as a perturbative correction in the mass insertion approximation.

In the Standard Model the diagonalization of the fermion masses transmits the flavor structure of the Yukawa sector to the kinetic terms via the скм matrix where it is manifested in the flavor-changing charged current through the $W^{ \pm}$boson. We shall use the analogous mass basis in Chapter 5.3 for our calculation of the Yukawa constraints from $\mu \rightarrow 3 e$ and $\mu \rightarrow e$ conversion operators. The key point is that in the gauge basis the interaction of the neutral gauge bosons is flavor diagonal but not flavor universal. The different fermion wave functions cause the overlap integrals to depend on the bulk mass parameters. Once we rotate into the mass eigenbasis we obtain flavor changing couplings for the neutral kr gauge bosons. This is shown heuristically in Figure 3.5.1.

In the lepton sector this does not occur for the zero mode photon since its wavefunction remains flat after electroweak symmetry breaking and hence $\mu \rightarrow e \gamma$ remains a loop-level process. Thus for the primary analysis of this paper we choose a basis where the ${ }_{5} \mathrm{D}$ fields are diagonal with respect to the bulk masses while the Yukawas are completely general. In this basis all of the relevant flavor-changing effects occur due to the Yukawa structure of the theory with no contributions from $W$ loops. In the Standard Model, this corresponds to the basis before diagonalizing the fermion masses so that all flavor-changing effects occur through off-diagonal elements in the Yukawa matrix manifested as mass insertions or Higgs interactions. This basis is particularly helpful in the ${ }_{5} \mathrm{D}$ mixed position-momentum space framework since the Higgs is attached to the IR brane, which simplifies loop integrals.

Realistic RS models typically require a mechanism to suppress generically large contributions to the Peskin-Takeuchi $T$ parameter and the $Z \bar{b} \bar{b}$ coupling; a common technique is to extend the bulk gauge symmetry to [191, 195, 210-214]

$$
\begin{equation*}
S U(3)_{c} \times S U(2)_{L} \times S U(2)_{R} \times U(1)_{X} \times P_{L R} . \tag{3.39}
\end{equation*}
$$

Here $P_{L R}$ is a discrete symmetry exchanging the $S U(2)_{L}$ and $S U(2)_{R}$ factors; in order to protect the left-handed $Z b \bar{b}$ coupling from anomalously large corrections, the left-handed down type quarks have to be eigenstates under $P_{L R}$. This in turn requires enlarged fermion representations with respect to the minimal model. Specifically the quark representations containing the SM zero modes are ( $i=1,2,3$ ):

$$
\begin{align*}
& \xi_{1 L}^{i}=\left(\begin{array}{cc}
\chi_{L}^{u_{i}}(-+)_{5 / 3} & q_{L}^{u_{i}}(++)_{2 / 3} \\
\chi_{L}^{d_{i}}(-+)_{2 / 3} & q_{L}^{d_{i}}(++)_{-1 / 3}
\end{array}\right)_{2 / 3},  \tag{3.40}\\
& \xi_{2 R}^{i}=u_{R}^{i}(++)_{2 / 3},  \tag{3.41}\\
& \xi_{3 R}^{i}=T_{3 R}^{i} \oplus T_{4 R}^{i}=\left(\begin{array}{c}
\psi_{R}^{\prime i}(-+)_{5 / 3} \\
U_{R}^{i}(-+)_{2 / 3} \\
D_{R}^{\prime i}(-+)_{-1 / 3}
\end{array}\right)_{2 / 3} \oplus\left(\begin{array}{c}
\psi_{R}^{\prime \prime i}(-+)_{5 / 3} \\
U_{R}^{\prime \prime i}(-+)_{2 / 3} \\
D_{R}^{i}(++)_{-1 / 3}
\end{array}\right)_{2 / 3} . \tag{3.42}
\end{align*}
$$

Here $\xi_{1 L}^{i}$ is an $S U(2)_{L} \times S U(2)_{R}$ bidoublet, $\xi_{2 R}^{i}$ is singlet under both $S U(2)$ s, and $T_{3 R}^{i}$ and $T_{4 R}^{i}$ are triplets under $S U(2)_{L}$ and $S U(2)_{R}$ respectively, with all of them carrying $U(1)_{X}$ charge $+2 / 3$. The corresponding states of opposite chirality are obtained by reversing the boundary conditions.

As we will see later, while the additional gauge bosons present in the custodial model do not have a significant impact on the $b \rightarrow q \gamma$ and $b \rightarrow q g(q=d, s)$ amplitudes, the additional fermion modes contribute and generally enhance the effect.


Figure 3.5.1: Heuristic representation of the rs model with bulk fields. The Higgs boson is completely localized on the IR brane and breaks electroweak symmetry. Fermions that are peaked toward the IR brane pick up large zero mode masses while fermions peaked away from the IR brane have nearly massless zero modes. The gauge boson zero modes are nearly flat so that the overlap integral of the gauge-fermion interaction picks up the orthonormality of the fermion profiles. Those which couple to the Higgs vev become kinked near the IR brane leading to couplings between zero mode fermions and kк fermions.

Derivation of mixed space propagators

The usual treatment of particles in extra dimensions is to treat them analogously to vibrational modes of a violin string whose endpoints are attached. In that picture the additional particles predicted by theory are identified with higher harmonics of the string. In this chapter we present an alternate framework in which the entire tower of possible harmonics are treated in a unified way.

### 4.1 Overview

In this chapter we explain how to derive the mixed position-momentum space propagators for a bulk fermion and a gauge boson in the rs scenario. This mixed formalism carries more technical baggage than the usual Kaluza-Klein decomposition but often clarifies ambiguities associated with infinite sums in the KK formalism. General formulae for the scalar function associated with bulk propagators of arbitrary-spin fields in RS can be found in [215]. Here we focus on spin-1/2 and spin-1 fields, highlighting non-trivial aspects of their Lorentz representation.

### 4.2 THE GEOMETRY OF RS FERMIONS

We begin by reviewing geometrical objects in a warped geometry that may be unfamiliar to those who are used to quantum field theory on a flat background. Our treatment will be physical rather than mathematical. For an excellent review of more formal topics, we refer the reader to [20, 21,23]. For more details from the physics perspective, see [179, 216].

### 4.2.1 Vielbeins

The familiar $\gamma$ matrices which obey the Clifford algebra are only defined for flat spaces. Specifically, they live on the tangent space of our spacetime. In order to define curved-space generalizations of objects like the Dirac operator $i \gamma^{\mu} \partial_{\mu}$, we need a way to convert spacetime indices $M$ to tangent space indices $a$. Vielbeins, $e_{\mu}^{a}(x)$, are the geometric objects which do this. The completeness relations associated with vielbeins allow them to be interpreted as a sort of "square root" of the metric in the
sense that

$$
\begin{equation*}
g_{M N}(x)=e_{M}^{a}(x) e_{M}^{b}(x) \eta_{a b} \tag{4.1}
\end{equation*}
$$

where $\eta_{a b}=\operatorname{diag}(+,-, \cdots,-)$ is the Minkowski metric on the tangent space. For our particular purposes we need the inverse vielbein, $E_{a}^{M}(x)$, defined such that

$$
\begin{equation*}
E_{a}^{M}(x) e_{N}^{a}(x)=\delta^{M}{ }_{N} \quad E_{a}^{M}(x) e_{N}^{b}(x)=\delta_{a}{ }^{b} \tag{4.2}
\end{equation*}
$$

The capital ' $E$ ' for the inverse vielbein is a pedantic notation that helps distinguish $e_{\mu}^{a}$ from its inverse. In practice (and later in this document) we will write $E_{a}^{M}$ as $e_{a}^{M}$. Spacetime indices are raised and lowered using the spacetime metric $g_{M N}(x)$ while tangent space indices are raised and lowered using the flat metric $\eta_{a b}(x)$.

Physically we may think of the vielbein is in terms of reference frames. The equivalence principle states that at any point one can always set up a coordinate system such that the metric is flat (Minkowski) at that point. Thus for each point $x$ in space there exists a family of coordinate systems that are flat at $x$. For each point we may choose one such coordinate system, which we call a frame. By general covariance one may define a map that transforms to this flat coordinate system at each point. This is the vielbein. One can see that it is a kind of local gauge transformation, and indeed this is the basis for treating gravity as a gauge theory built upon diffeomorphism invariance. Mathematically, the vielbein represents the frame bundle on the spacetime.

### 4.2.2 SpIN COVARIANT DERIVATIVE

We are familiar that the covariant derivative is composed of a partial derivative term plus connection terms which depend on the particular object being differentiated. For example, the covariant derivative on a spacetime vector $V^{\mu}$ is

$$
\begin{equation*}
D_{M} V^{N}=\partial_{M} V^{N}+\Gamma_{M L}^{N} V^{L} . \tag{4.3}
\end{equation*}
$$

The vielbein allows us to work with objects with a tangent space index, $a$, instead of just spacetime indices, $\mu$. The $\gamma$ matrices allow us to further convert tangent space indices to spinor indices. We would then define a covariant derivative acting on the tangent space vector $V^{a}$,

$$
\begin{equation*}
D_{M} V^{a}=\partial_{M} V^{a}+\omega_{M b}^{a} V^{b}, \tag{4.4}
\end{equation*}
$$

where the quantity $\omega_{M b}^{a}$ is called the spin covariant derivative. Consistency of the two equations implies

$$
\begin{equation*}
D_{M} V^{a}=e_{N}^{a} D_{M} V^{N} . \tag{4.5}
\end{equation*}
$$

This is sufficient to determine the spin connection. It is a fact from differential geometry that the spin connection is expressed in terms of the veilbeins via [216]

$$
\begin{align*}
\omega_{M}^{a b} & =\frac{1}{2} g^{R P} e_{R}^{[a} \partial_{[M} e_{P]}^{b]}+\frac{1}{4} g^{R P} g^{T S} e_{R}^{[a} e_{T}^{b]} \partial_{[S} e_{P]}^{c} e_{M}^{d} \eta_{c d}  \tag{4.6}\\
& =\frac{1}{2} e^{N a}\left(\partial_{M} e_{N}^{b}-\partial_{N} e_{M}^{b}\right)-\frac{1}{2} e^{N b}\left(\partial_{M} e_{N}^{a}-\partial_{N} e_{M}^{a}\right)-\frac{1}{2} e^{P a} e^{R b}\left(\partial_{P} e_{R c}-\partial_{R} e_{R c}\right) e_{M}^{c} \tag{4.7}
\end{align*}
$$

When acting on spinors one needs the appropriate structure to convert the $a, b$ tangent space indices into spinor indices. This is provided by

$$
\begin{equation*}
\sigma_{a b}=\frac{1}{4}\left[\gamma_{a}, \gamma_{b}\right] \tag{4.8}
\end{equation*}
$$

so that the appropriate spin covariant derivative is

$$
\begin{equation*}
D_{M}=\partial_{M}+\frac{1}{2} \omega_{M}^{a b} \sigma_{a b} . \tag{4.9}
\end{equation*}
$$

### 4.2.3 Antisymmetrization and Hermiticity

The fermionic action on a curved background is

$$
\begin{equation*}
S=\int d^{d} x \sqrt{\left|g_{d}\right|} \bar{\Psi}\left(i E_{a}^{M} \gamma^{a} \overleftrightarrow{D_{M}}-m\right) \Psi \tag{4.10}
\end{equation*}
$$

where the antisymmetrized covariant derivative is defined by a difference of right- and left-acting derivatives

$$
\begin{equation*}
\overleftrightarrow{D_{M}}=\frac{1}{2} D_{M}-\frac{1}{2} \overleftarrow{D_{M}} \tag{4.11}
\end{equation*}
$$

This is somewhat subtle. The canonical form of the fermionic action must be antisymmetric in this derivative in order for the operator to be Hermitian and thus for the action to be real. In flat space we are free to integrate by parts in order to write a the action in exclusively terms of a right-acting Dirac operator.

Hermiticity is defined with respect to an inner product. The inner product in this case is given by

$$
\begin{equation*}
\left\langle\Psi_{1} \mid \mathcal{O} \Psi_{2}\right\rangle=\int d^{5} x \sqrt{g} \overline{\Psi_{1}} \mathcal{O} \Psi_{2} \tag{4.12}
\end{equation*}
$$

A manifestly Hermitian operator is $\mathcal{O}_{H}=\frac{1}{2}\left(\mathcal{O}+\mathcal{O}^{\dagger}\right)$, where we recall that

$$
\begin{equation*}
\left\langle\Psi_{1} \mid \mathcal{O}^{\dagger} \Psi_{2}\right\rangle=\left\langle\mathcal{O} \Psi_{1} \mid \Psi_{2}\right\rangle=\int d^{5} x \sqrt{g} \overline{\mathcal{O} \Psi_{1}} \Psi_{2} \tag{4.13}
\end{equation*}
$$

The definition of an inner product on the phase space of a quantum field theory is a nontrivial matter on curved spacetimes. Since our spacetime is not warped in the time direction there is no ambiguity in picking a canonical Cauchy surface to quantize our fields and we may follow the usual procedure of Minkowski space quantization with the usual Minkowski spinor inner product.

As a sanity-check, consider the case of the partial derivative operator $\partial_{\mu}$ on flat space time. The Hermitian conjugate of the operator is the left-acting derivative, $\overleftarrow{\partial_{\mu}}$, by which we really mean

$$
\int d^{d} x \overline{\Psi_{1}} \partial^{\dagger} \Psi_{2}=\left\langle\Psi_{1} \mid \partial_{\mu}^{\dagger} \Psi_{2}\right\rangle=\left\langle\partial_{\mu} \Psi_{1} \mid \Psi_{2}\right\rangle=\int d^{d} x \overline{\partial_{\mu} \Psi_{1}} \Psi_{2}=\int d^{d} x \overline{\Psi_{1}} \overleftarrow{\partial_{\mu}} \Psi_{2}=\int d^{d} x \overline{\Psi_{1}}\left(-\partial_{\mu}\right) \Psi_{2}
$$

In the last step we've integrated by parts and dropped the boundary term. We see that the Hermitian conjugate of the partial derivative is negative itself. Thus the partial derivative is not a Hermitian operator. This is why the momentum operator is given by $\hat{P}_{\mu}=i \partial_{\mu}$, since the above analysis then yields $\hat{P}_{\mu}^{\dagger}=\hat{P}_{\mu}$, where we again drop the boundary term and recall that the $i$ flips sign under the bar.

Now we can be explicit in what we mean by the left-acting derivative in (4.10). The operator $i E_{a}^{M} \gamma^{a} D_{M}$ is not Hermitian and needs to be made Hermitian by writing it in the form $\mathcal{O}_{H}=\frac{1}{2}\left(\mathcal{O}+\mathcal{O}^{\dagger}\right)$. Thus we may write a manifestly Hermitian Dirac operator as,

$$
\begin{align*}
\bar{\Psi}(\text { Dirac }) \Psi & =\bar{\Psi}\left[\frac{1}{2}\left(i E_{a}^{M} \gamma^{a} D_{M}\right)+\frac{1}{2}\left(i E_{a}^{M} \gamma^{a} D_{M}\right)^{\dagger}\right] \Psi  \tag{4.14}\\
& =\bar{\Psi}_{-}^{i} E_{a}^{M} \gamma^{a} D_{M} \Psi+\frac{i}{2} E_{a}^{M} \gamma^{a} D_{M} \Psi \Psi  \tag{4.15}\\
& =\bar{\Psi} \frac{i}{2} E_{a}^{M} \gamma^{a} D_{M} \Psi-\frac{i}{2} E_{a}^{M} \overline{\gamma^{a} D_{M} \Psi} \Psi \tag{4.16}
\end{align*}
$$

where we've used the fact that $E_{a}^{M}$ is a real function with no spinor indices. The second term on the right-hand side can be massaged further,

$$
\begin{equation*}
\overline{\gamma^{a} D_{M} \Psi} \Psi=\Psi^{\dagger} \overleftarrow{D_{M}}{ }^{\dagger} \gamma^{a \dagger} \gamma^{\circ} \Psi=\Psi^{\dagger}\left(\overleftarrow{\partial_{M}}+\omega_{M}^{b c} \sigma^{b c \dagger}\right) \gamma^{\circ} \gamma^{a} \Psi=\bar{\Psi} \overleftarrow{D_{M}} \gamma^{a} \Psi=\bar{\Psi} \gamma^{a} \overleftarrow{D_{M}} \Psi \tag{4.17}
\end{equation*}
$$

Note that we have used that $\gamma^{M \dagger}=\gamma^{\circ} \gamma^{M} \gamma^{\circ}$ and, in the last line, that $\left[\sigma^{b c}, \gamma^{a}\right]=0$. Putting this all together, we can write down our manifestly real fermion action as in (4.10),

$$
\begin{align*}
S & =\int d^{d} x \sqrt{|g|} \bar{\Psi}\left(i E_{a}^{M} \gamma^{a} \overleftrightarrow{D_{M}}-m\right) \Psi  \tag{4.18}\\
& =\int d^{d} x \sqrt{|g|}\left(\frac{i}{2} \bar{\Psi} E_{a}^{M} \gamma^{a} D_{M} \Psi-\frac{i}{2} \overline{D_{M} \Psi} E_{a}^{M} \gamma^{a} \Psi-m \bar{\Psi} \Psi\right) \tag{4.19}
\end{align*}
$$

All of this may seem overly pedantic since integration by parts allows one to go back and forth between the 'canonical' form and the usual 'right-acting only' form of the fermion kinetic operator. Our interest, however, is to apply this to the Randall-Sundrum background where integration by parts introduces boundary terms and so it is crucial to take the canonical form of the Dirac operator as the starting point.

This is sometimes under appreciated in the phenomenological literature on thsi topic. For example, [189] refers to [21] for the fermionic action on a warped space. The latter, however, assumes a spacetime without boundaries so that integration by parts may be performed to convert to a right-acting Dirac operator. As shown above, this generates boundary terms on the RS spacetime. Fortunately, the chiral boundary conditions imposed in [189] cancel the incorrect boundary terms. More generally-for example, for models with vector matter-this point must be treated with care.

### 4.3 The Randall-Sundrum Bulk Fermion Action

We now specialize the above analysis to the case of the Randall-Sudrum background.

### 4.3.1 THE RS SPIN CONNECTION

In Appendix 4.A we explicitly derive the spin connection on the rs background. The result is

$$
\begin{equation*}
\frac{1}{2} \omega_{M}^{a b} \sigma_{a b}=\frac{1}{4 z}\left(\gamma_{M} \gamma_{S}+\delta_{M}^{s}\right) \tag{4.20}
\end{equation*}
$$

so that the spin covariant derivative is

$$
D_{M}= \begin{cases}\partial_{\mu}+\frac{1}{4 z} \gamma_{\mu} \gamma_{5} & \text { if } M=\mu  \tag{4.21}\\ \partial_{S} & \text { if } M=5\end{cases}
$$

For all the geometric background above, we are led to something rather anticlimactic: the spin connection drops out of the action.

$$
\begin{equation*}
S=\int d^{5} x \frac{i}{2}\left(\frac{R}{z}\right)^{4}\left(\bar{\Psi} \gamma^{M} \overleftrightarrow{\partial_{M}} \Psi+\frac{1}{4 z} \bar{\Psi} \gamma_{\mu} \gamma_{5} \gamma^{\mu} \Psi-\frac{1}{4 z} \overline{\gamma_{\mu} \gamma_{5} \gamma^{\mu} \Psi} \Psi\right) \tag{4.22}
\end{equation*}
$$

The two spin connection terms cancel since $\overline{\gamma_{\mu} \gamma^{5} \gamma^{\mu} \Psi} \Psi=\bar{\Psi} \gamma_{\mu} \gamma_{S} \gamma^{\mu} \Psi$, so that upon including a bulk mass term,

$$
S=\int d^{5} x \frac{i}{2}\left(\frac{R}{z}\right)^{4} \bar{\Psi} \gamma^{M} \overleftrightarrow{\partial_{M}} \Psi-\int d^{5} x \frac{i}{2}\left(\frac{R}{z}\right)^{S} m \bar{\Psi} \Psi=\int d^{5} x \frac{i}{2}\left(\frac{R}{z}\right)^{4} \bar{\Psi}\left(\gamma^{M} \overleftrightarrow{\partial_{M}}-\frac{c}{z}\right) \Psi
$$

where $c=m R=m / k$ is a dimensionless parameter that is the ratio of the bulk mass to the curvature. We show below that the bulk mass does not contribute directly to the 4 D Kaluza-Klein mass spectrum of the model. Instead, $c$ determines the localization of the ${ }_{5} \mathrm{D}$ wavefunction. This, in turn, determines the overlap with the Higgs field and the contribution to masses from electroweak symmetry breaking. More comprehensive discussions can be found in the original paper by Grossman and Neubert [189] or the review by Gherghetta [202].

### 4.3.2 Right-acting RS Fermionic Action

When deriving the Dirac equation from the variational principle we set all of our operators to be right-acting, i.e. acting on $\Psi$, so that we can vary with respect to $\bar{\Psi}$ to get an operator equation for $\Psi$. Obtaining this is from (4.23) is now a straightforward matter of integration by parts of the left-acting derivative term. Note that it is crucially important that we pick up a derivative acting on the metric/vielbein factor $(R / z)^{4}$. We would have missed this term if he had mistakenly written our original 'canonical action,' (4.10), as being right-acting only.

The integration by parts for the $M=\mu=0, \cdots, 4$ terms proceeds trivially since these directions have no boundary and the metric/vielbein factor is independent of them. Performing the $M=5$ integration by parts we find

$$
\begin{equation*}
S=\int d^{4} x \int_{R^{\prime}}^{R} d z\left(\frac{R}{z}\right)^{4} \bar{\Psi}\left(i \not \partial+i \gamma^{5} \partial_{5}-i \frac{2}{z} \gamma^{5}-\frac{c}{z}\right) \Psi+\left.(\text { boundary term })\right|_{R^{\prime}} ^{R} . \tag{4.24}
\end{equation*}
$$

The term in the parenthesis can be identified with the Dirac operator for the Randall-Sundrum model with bulk fermions. This 'definition' is up to conventions regarding the inclusion of the mass term and factors of $i$. The boundary term is

$$
\begin{equation*}
\text { (boundary) }=\left.(R / z)^{4}(\psi \chi-\bar{\chi} \bar{\psi})\right|_{R^{\prime}} ^{R} \tag{4.25}
\end{equation*}
$$

where we've written out the Dirac spinor $\Psi$ in terms of two-component Weyl spinors $\chi$ and $\psi$. This term vanishes when we impose chiral boundary conditions, which we review in the next section. The final form of the rs fermion action is

$$
\begin{equation*}
S=\int d^{4} x \int_{R^{\prime}}^{R} d z\left(\frac{R}{z}\right)^{4} \bar{\Psi}\left(i \not \partial+i \gamma^{5} \partial_{5}-i \frac{2}{z} \gamma^{5}-\frac{c}{z}\right) \Psi \tag{4.26}
\end{equation*}
$$

In terms of Weyl spinors this is

$$
S=\int d^{4} x \int_{R^{\prime}}^{R} d z\left(\frac{R}{z}\right)^{4}\left(\begin{array}{ll}
\psi & \bar{\chi}
\end{array}\right)\left(\begin{array}{cc}
-\partial_{5}+\frac{2-c}{z} & i \not \partial  \tag{4.27}\\
i \not \partial & \partial_{5}-\frac{2+c}{z}
\end{array}\right)\binom{\chi}{\psi},
$$

where we use the two-component slash convention $\vec{\psi}=v_{\mu} \bar{\sigma}^{\mu}, \gamma=v_{\mu} \sigma^{\mu}$.

### 4.3.3 CHIRAL BOUNDARY CONDITIONS

Recall that 5 D theories are vector-like, meaning that the fundamental spinor representation is a Dirac spinor (containing both left- and right-handed components) rather than a chiral Weyl spinor. This is understood straightforwardly by considering $\gamma^{5}$. In four dimensions, $\gamma_{4 \mathrm{D}}^{5}=i \gamma^{\circ} \gamma^{1} \gamma^{2} \gamma^{3}$ is a special operator that can be used to identify chiralities via $P_{L, R}=\frac{1}{2}\left(1 \pm \gamma_{4 \mathrm{D}}^{5}\right)$. In ${ }_{5 \mathrm{D}}$, however, $\gamma^{5}$ is just the $\gamma$ matrix corresponding to the $z$ direction and there is no analogous 'special' matrix associated with chirality. Note, further, that the Clifford algebra, $\left\{\gamma^{M}, \gamma^{N}\right\}=\eta^{M N}$ forces $\gamma^{5}=-i \gamma_{4 D}^{5}$. The $\gamma^{\circ}, \cdots, \gamma^{5}$ form a basis for the four component spinor representation of the ${ }_{5} \mathrm{D}$ Clifford algebra. For details of representations of the Clifford algebra in general dimensionality, see the appendices in the second volume of Polchinksi [217].

The vector nature of ${ }_{5} \mathrm{D}$ spinors is an immediate problem for model-building since the Standard Model is manifestly chiral and there appears to be no way to write down a chiral fermion without immediately introducing a partner fermion of opposite chirality and the same couplings.To get around this problem, we can require that only the zero modes of the ${ }_{5} \mathrm{D}$ fermions-those which are identified with Standard Model states-to be chiral. We show that one chirality of zero modes can indeed be projected out, while the heavier Kaluza-Klein excitations are vectorlike but massive.

We can project out the zero modes of the wrong-chirality components of a bulk Dirac ${ }_{5} \mathrm{D}$ fermion by imposing chiral boundary conditions that these states vanish on the branes. Since zero modes have trivial profiles, these boundary conditions force the mode to be identically zero everywhere. For left-chiral boundary conditions, $\psi=0$ on the branes, while for right-chiral boundary conditions $\chi=\mathrm{o}$ on the branes. Thus we are guaranteed that both terms in (4.25) vanish at $z=R, R^{\prime}$ for either chirality.

Imposing these chiral boundary conditions is equivalent to the statement that the compactified extra dimension is an orbifold. From a phenomenological point of view, the language of boundary conditions is preferred since it avoids potential
ambiguities with the sign of the fermion mass term. Further, the language of boundary conditions best connects to the actual process of solving partial differential equations that we follow. This treatment of boundary conditions for compact spaces was first discussed from this viewpoint in [218].

### 4.4 Bulk fermion Fields

Bulk fermions in the RS scenarios were explored in a series of papers [188-190,219-222], where [189] in particular is noted for its systematic discussion of the bulk action in a non-supersymmetric context. We follow a slightly different approach and now derive the mixed space propagators for these fields.

The Green's function equation for the general RS fermion propagator can be solved directly from the Strum-Liouville equation (see, e.g. [215]), though this can obscure some of the intuition of the results. Here we provide a pedagogical derivation of the 5 D bulk fermion propagator in a flat and warped interval extra dimension.

### 4.4.1 FLAT 5D FERMION PROPAGATOR

First we derive the chiral fermion propagator in a flat interval extra dimension $z \in(0, L)$ as a model calculation for the warped fermion propagator which is presented in Section 4.4.2. A complete set of propagators for a flat 5 D interval was derived in [223] using finite temperature field theory techniques.

We derive these results by directly solving the Green's function equations. The propagator from a given point $x^{\prime}$ to another point $x$ is given by the two-point Green's function of the ${ }_{5} \mathrm{D}$ Dirac operator,

$$
\begin{equation*}
\mathcal{D} \Delta\left(x, x^{\prime}\right) \equiv\left(i \gamma^{M} \partial_{M}-m\right) \Delta\left(x, x^{\prime}\right)=i \delta^{(5)}\left(x-x^{\prime}\right) \tag{4.28}
\end{equation*}
$$

where $M$ runs over 5 D indices. We shall treat the noncompact dimensions in momentum space and the finite dimension is in position space. In this formalism, the Green's function equation is

$$
\begin{equation*}
\left(\not p+i \partial_{5} \gamma^{5}-m\right) \Delta\left(p, z, z^{\prime}\right)=i \delta\left(z-z^{\prime}\right) \tag{4.29}
\end{equation*}
$$

where we use $\gamma^{5}=\operatorname{diag}\left(i \mathbb{1}_{2},-i \mathbb{1}_{2}\right)$.
This is a first-order differential equation with nontrivial Dirac structure. To solve this equation we define a pseudo-conjugate Dirac operator (which is neither a complex nor Hermitian conjugate),

$$
\begin{equation*}
\overline{\mathcal{D}}=i \gamma^{M} \partial_{M}+m \tag{4.30}
\end{equation*}
$$

Using this to "square" the Dirac operator, we can swap the Dirac equation for a simpler Klein-Gordon equation that is second order and diagonal on the space of Weyl spinors,

$$
\mathcal{D} \overline{\mathcal{D}}=\left(\begin{array}{ll}
\partial_{5}^{2}-\partial^{2}-m^{2} &  \tag{4.31}\\
& \partial_{5}^{2}-\partial^{2}-m^{2}
\end{array}\right)
$$

It is straightforward to solve for the Green's functions $F\left(p, z, z^{\prime}\right)$ of the $\mathcal{D} \overline{\mathcal{D}}$ operator in mixed position-momentum space,

$$
\mathcal{D} \overline{\mathcal{D}} F\left(p, z, z^{\prime}\right)=\left(\begin{array}{ll}
\partial_{5}^{2}+p^{2}-m^{2} &  \tag{4.32}\\
& \partial_{5}^{2}+p^{2}-m^{2}
\end{array}\right)\left(\begin{array}{ll}
F_{-} & \\
& F_{+}
\end{array}\right)=i \delta\left(z-z^{\prime}\right)
$$

From these we can trivially construct a solution for the Green's function of (4.28),

$$
\Delta\left(p, z, z^{\prime}\right) \equiv \overline{\mathcal{D}} F\left(p, z, z^{\prime}\right)=\left(\begin{array}{cc}
\left(-\partial_{5}+m\right) F_{-} & \sigma^{\mu} p_{\mu} F_{+}  \tag{4.33}\\
\bar{\sigma}^{\mu} p_{\mu} F_{-} & \left(\partial_{5}+m\right) F_{+}
\end{array}\right)
$$

| $A_{+}^{L<}=\mathrm{s}_{p} \Delta z^{\prime} \mathrm{s}_{p} L$ | $A_{+}^{L>}=\mathrm{s}_{p} z^{\prime} \mathrm{c}_{p} L$ | $A_{+}^{R<}=\mathrm{o}$ | $A_{+}^{R>}=-\mathrm{c}_{p} z^{\prime} \mathrm{s}_{p} L$ |  |
| :--- | :--- | :--- | :--- | :--- |
| $B_{+}^{L<}=\mathrm{o}$ | $B_{+}^{L>}=\mathrm{s}_{p} z^{\prime} \mathrm{s}_{p} L$ | $B_{+}^{R<}=-\mathrm{c}_{p} \Delta z^{\prime}$ | $B_{+}^{R>}=-\mathrm{c}_{p} z^{\prime} \mathrm{c}_{p} L$ |  |
| $A_{-}^{L<}=$ | o | $A_{-}^{L>}=-\mathrm{c}_{p} z^{\prime} \mathrm{s}_{p} L$ | $A_{-}^{R<}=\mathrm{s}_{p} \Delta z^{\prime}$ | $A_{-}^{R>}=-\mathrm{s}_{p} z^{\prime} \mathrm{c}_{p} L$ |
| $B_{-}^{L<}=\mathrm{c}_{p} \Delta z^{\prime}$ | $B_{-}^{L>}=-\mathrm{c}_{p} z^{\prime} \mathrm{c}_{p} L$ | $B_{-}^{R<}=\mathrm{o}$ | $B_{-}^{R>}=\mathrm{s}_{p} z^{\prime} \mathrm{s}_{p} L$ |  |

Table 4.1: Flat case coefficients in (4.35) upon solving with the boundary conditions (4.37-4.40). We have used the notation $\mathrm{c}_{p} x=\cos \chi_{p} x, \mathrm{~s}_{p} x=\sin \chi_{p} x$, and $\Delta z^{\prime}=\left(L-z^{\prime}\right)$.

We solve this by separating $F_{ \pm}(z)$ into pieces

$$
F_{ \pm}\left(p, z, z^{\prime}\right)= \begin{cases}F_{ \pm}^{<}\left(p, z, z^{\prime}\right) & \text { if } z<z^{\prime}  \tag{4.34}\\ F_{ \pm}^{>}\left(p, z, z^{\prime}\right) & \text { if } z>z^{\prime}\end{cases}
$$

and then solving the homogeneous Klein-Gordon equations for each $F^{<}$and $F^{>}$. The general solution is

$$
\begin{equation*}
F_{ \pm}^{<,>}\left(p, z, z^{\prime}\right)=A_{ \pm}^{<,>} \cos \left(\chi_{p} z\right)+B_{ \pm}^{<,>} \sin \left(\chi_{p} z\right), \tag{4.35}
\end{equation*}
$$

where the eight coefficients $A_{ \pm}^{<,>}$and $B_{ \pm}^{<,>}$are determined by the boundary conditions at $\mathrm{o}, L$ and $z^{\prime}$. The factor $\chi_{p}$ is the magnitude of $p_{5}$ and is defined by

$$
\begin{equation*}
\chi_{p}=\sqrt{p^{2}-m^{2}} \tag{4.36}
\end{equation*}
$$

We impose matching boundary conditions at $z=z^{\prime}$. By integrating the Green's function equation (4.32) over a sliver $z \in\left[z^{\prime}-\varepsilon, z^{\prime}+\varepsilon\right]$ we obtain the conditions

$$
\begin{align*}
\partial_{5} F_{ \pm}^{>}\left(z^{\prime}\right)-\partial_{5} F_{ \pm}^{<}\left(z^{\prime}\right) & =i,  \tag{4.37}\\
F_{ \pm}^{>}\left(z^{\prime}\right)-F_{ \pm}^{<}\left(z^{\prime}\right) & =0 . \tag{4.38}
\end{align*}
$$

These are a total of four equations. The remaining four equations imposed at the branes impose the chirality of the fermion zero mode and are equivalent to treating the interval as an orbifold. We denote the propagator for the ${ }_{5} \mathrm{D}$ fermion with a left-chiral (right-chiral) zero mode by $\Delta^{L}\left(\Delta^{R}\right)$. We impose that the Green's function vanishes if a "wrong-chirality" state propagates to either brane,

$$
\begin{align*}
& \left.P_{R} \Delta^{L}\left(p, z, z^{\prime}\right)\right|_{z=0, L}=\left.P_{R} \overline{\mathcal{D}} F^{L}\left(p, z, z^{\prime}\right)\right|_{z=0, L}=0,  \tag{4.39}\\
& \left.P_{L} \Delta^{R}\left(p, z, z^{\prime}\right)\right|_{z=0, L}=\left.P_{L} \overline{\mathcal{D}} F^{R}\left(p, z, z^{\prime}\right)\right|_{z=0, L}=0, \tag{4.40}
\end{align*}
$$

where $P_{L, R}=\frac{1}{2}\left(1 \mp i \gamma^{S}\right)$ are the usual 4 D chiral projection operators. Note from (4.33) that each of these equations is actually a set of two boundary conditions on each brane. For example, the left-handed boundary conditions may be written explicitly as

$$
\begin{align*}
\left.F_{-}^{L}\left(p, z, z^{\prime}\right)\right|_{z=0, L} & =\mathrm{o},  \tag{4.41}\\
\left.\left(\partial_{5}+m\right) F_{+}^{L}\left(p, z, z^{\prime}\right)\right|_{z=0, L} & =\mathrm{o}, \tag{4.42}
\end{align*}
$$

where we have used that $p_{\mu}$ is arbitrary. It is well-known that only one boundary condition for a Dirac fermion needs to be imposed in order not to overconstrain the first-order Dirac equation since the bulk equations of motion convert boundary conditions for $\chi$ into boundary conditions for $\psi$ [218]. In this case, however, we work with a second-order Klein-Gordon equation that does not mix $\chi$ and $\psi$. Thus the appearance and necessity of two boundary conditions per brane for a chiral fermion is not surprising; we are only converting the single boundary condition on $\Delta\left(p, z, z^{\prime}\right)$ into two boundary conditions
for $F\left(p, z, z^{\prime}\right)$.
Solving for the coefficients $A_{ \pm}^{<,>}(p, z)$ and $B_{ \pm}^{<,>}(p, z)$ for each type of fermion (left- or right-chiral zero modes) one finds the results in Table 4.1. Using trigonometric identities one may combine the $z<z^{\prime}$ and $z>z^{\prime}$ results to obtain

$$
\begin{equation*}
F_{ \pm}^{X}=\frac{-i \cos \chi_{p}\left(L-\left|z-z^{\prime}\right|\right)+\gamma^{5} \wp_{X} \cos \chi_{p}\left(L-\left(z+z^{\prime}\right)\right)}{2 \chi_{p} \sin \chi_{p} L} \tag{4.43}
\end{equation*}
$$

where $X=\{L, R\}$ with $\wp_{L}=+1$ and $\wp_{R}=-1$. This result differs from that of [223] by a factor of 2 since that paper treats the compactified space as an orbifold over the entire $S^{1}$ rather than just an interval $[0, \pi R]$. The fermion Green's function can then be obtained trivially from (4.33).

### 4.4.2 WARPED 5D FERMION PROPAGATOR

We now derive the chiral fermion propagator in a warped interval extra dimension following the same strategy as Appendix 4.4.1. The Dirac operator is obtained from the variation of the Randall-Sundrum free fermion action,

$$
\begin{equation*}
S_{\mathrm{RS}}(\text { fermion })=\int d x \int_{R}^{R^{\prime}} d z\left(\frac{R}{z}\right)^{4} \bar{\Psi}\left(i \gamma^{M} \partial_{M}-i \frac{2}{z} \gamma^{s}-\frac{c}{z}\right) \Psi, \tag{4.44}
\end{equation*}
$$

where $c=m R$ and we have integrated the left-acting derivatives by parts. The Dirac operator is a product of the $(R / z)^{4}$ prefactor coming from the $\operatorname{AdS}$ geometry and an operator $\mathcal{D}$ given by

$$
\begin{equation*}
\mathcal{D}=i \gamma^{M} \partial_{M}-i \frac{2}{z} \gamma^{5}-\frac{c}{z} \tag{4.45}
\end{equation*}
$$

We would like to find the mixed position/momentum space two-point Green's function satisfying

$$
\begin{equation*}
(R / z)^{4} \mathcal{D} \Delta\left(p, z, z^{\prime}\right)=i \delta\left(z-z^{\prime}\right) \tag{4.46}
\end{equation*}
$$

Following (4.30) we define a pseudo-conjugate Dirac operator

$$
\begin{equation*}
\overline{\mathcal{D}}=i \gamma^{M} \partial_{M}-i \frac{2}{z} \gamma^{5}+\frac{c}{z} \tag{4.47}
\end{equation*}
$$

and 'square' $\mathcal{D}$ into a diagonal second-order operator,

$$
\mathcal{D} \overline{\mathcal{D}}=\left(\begin{array}{cc}
\mathcal{D} \overline{\mathcal{D}}- & \circ  \tag{4.48}\\
\circ & \mathcal{D} \overline{\mathcal{D}}{ }_{+}
\end{array}\right) \quad \mathcal{D} \overline{\mathcal{D}}_{ \pm}=\partial^{2}-\partial_{5}^{2}+\frac{4}{z} \partial_{5}+\frac{c^{2} \pm c-6}{z^{2}}
$$

Next we follow (4.32) and solve for the Green's function of this squared operator in mixed position/momentum space where $\partial^{2} \rightarrow-p^{2}$,

$$
-(R / z)^{4} \mathcal{D} \overline{\mathcal{D}} F\left(p, z, z^{\prime}\right)=-\binom{R}{z}^{4}\left(\begin{array}{ll}
\mathcal{D} \overline{\mathcal{D}}- &  \tag{4.49}\\
& \mathcal{D} \overline{\mathcal{D}}_{+}
\end{array}\right)\left(\begin{array}{ll}
F_{-} & \\
& F_{+}
\end{array}\right)=i \delta\left(z-z^{\prime}\right)
$$

The solution to the Dirac Green's function equation (4.46) is then given by $\Delta\left(p, z, z^{\prime}\right)=\overline{\mathcal{D}} F\left(p, z, z^{\prime}\right)$. We shall separate $F\left(p, z, z^{\prime}\right)$ into solutions for the cases $z>z^{\prime}$ and $z<z^{\prime}$ following (4.34). The general solution to the homogeneous equation (4.49) with $z \neq z^{\prime}$ is

$$
\begin{equation*}
F_{ \pm}^{<,>}\left(p, z, z^{\prime}\right)=A_{ \pm}^{<,>} z^{\frac{5}{2}} J_{c \pm \frac{1}{2}}(p z)+B_{ \pm}^{<,>} z^{\frac{s}{2}} Y_{c \pm \frac{1}{2}}(p z), \tag{4.50}
\end{equation*}
$$

where $J_{n}$ and $Y_{n}$ are Bessel functions of the first and second kinds, $A_{ \pm}^{<,>}$and $B_{ \pm}^{<,>}$are coefficients to be determined by boundary conditions, and $p$ is the analog of $\chi_{p}$ defined by $p=\sqrt{p_{\mu} p^{\mu}}$. Note that this differs from (4.36) since there is no explicit bulk mass dependence. In (4.50) the bulk masses enter only in the order of the Bessel functions as $\left(c \pm \frac{1}{2}\right)$.

| $A_{+}^{L<}=-a_{L} z^{\prime \frac{5}{2}} Y_{c-\frac{1}{2}}(p R) \tilde{S}_{c}^{+}\left(p z^{\prime}, p R^{\prime}\right)$ | $A_{+}^{R<}=-\alpha_{R} z^{\frac{5}{2}} Y_{c+\frac{1}{2}}(p R) S_{c}^{+}\left(p z^{\prime}, p R^{\prime}\right)$ |
| :--- | :--- |
| $B_{+}^{L<}=a_{L} z^{\prime \frac{5}{2}} J_{c-\frac{1}{2}}(p R) \tilde{S}_{c}^{+}\left(p z^{\prime}, p R^{\prime}\right)$ | $B_{+}^{R<}=a_{R} z^{\prime \frac{5}{2}} J_{c+\frac{1}{2}}(p R) S_{c}^{+}\left(p z^{\prime}, p R^{\prime}\right)$ |
| $A_{-}^{L<}=-a_{L} z^{\prime \frac{5}{2}} Y_{c-\frac{1}{2}}(p R) S_{c}^{-}\left(p z^{\prime}, p R^{\prime}\right)$ | $A_{-}^{R<}=-a_{R} z^{\frac{s}{2}} Y_{c+\frac{1}{2}}(p R) \tilde{S}_{c}^{-}\left(p z^{\prime}, p R^{\prime}\right)$ |
| $B_{-}^{L<}=a_{L} z^{\prime \frac{5}{2}} J_{c-\frac{1}{2}}(p R) S_{c}^{-}\left(p z^{\prime}, p R^{\prime}\right)$ | $B_{-}^{R<}=\alpha_{R} z^{\prime \frac{5}{2}} J_{c+\frac{1}{2}}(p R) \tilde{S}_{c}^{-}\left(p z^{\prime}, p R^{\prime}\right)$ |

Table 4.2: Left-handed rs fermion propagator coefficients: the $z>z^{\prime}$ coefficients are obtained by swapping $R \leftrightarrow R^{\prime}$ in the arguments of the functions, leaving the $\alpha_{L, R}$ constant.

The matching boundary conditions at $z=z^{\prime}$ are given by (4.37) and (4.38) modified by a factor of $\left(R / z^{\prime}\right)^{4}$ from (4.49),

$$
\begin{align*}
\partial_{5} F_{ \pm}^{>}\left(z^{\prime}\right)-\partial_{5} F_{ \pm}^{<}\left(z^{\prime}\right) & =i\left(R / z^{\prime}\right)^{-4}  \tag{4.51}\\
F_{ \pm}^{>}\left(z^{\prime}\right)-F_{ \pm}^{<}\left(z^{\prime}\right) & =0 . \tag{4.52}
\end{align*}
$$

The chiral boundary conditions are the same as in the flat case, (4.39) and (4.40) with the appropriate insertion of (4.47).

We may now solve for the $A$ and $B$ coefficients. It is useful to write these in terms of common factors that appear in their expressions. To this end, let us define the prefactors

$$
\begin{equation*}
a_{L}=\frac{i \pi}{2 R^{4}} \frac{1}{S_{c}^{-}\left(p R, p R^{\prime}\right)} \quad a_{R}=\frac{i \pi}{2 R^{4}} \frac{1}{S_{c}^{+}\left(p R, p R^{\prime}\right)} \tag{4.53}
\end{equation*}
$$

and a set of antisymmetric functions

$$
\begin{align*}
& S_{c}^{ \pm}(x, y)=J_{c \pm \frac{1}{2}}(x) Y_{c \pm \frac{1}{2}}(y)-J_{c \pm \frac{1}{2}}(y) Y_{c \pm \frac{1}{2}}(x)  \tag{4.54}\\
& \tilde{S}_{c}^{ \pm}(x, y)=J_{c \pm \frac{1}{2}}(x) Y_{c \mp \frac{1}{2}}(y)-J_{c \mp \frac{1}{2}}(y) Y_{c \pm \frac{1}{2}}(x) \tag{4.55}
\end{align*}
$$

With these definitions the coefficients for the left- and right-handed $F$ functions are given in Table 4.2. The $F_{ \pm}^{L, R}$ functions may thus be written out succinctly for $z \leq z^{\prime}$ as

$$
\begin{align*}
& F_{+}^{L<}=\alpha_{L}\left(z z^{\prime}\right)^{5 / 2} \tilde{S}_{c}^{+}\left(p z^{\prime}, p R^{\prime}\right) \tilde{S}_{c}^{-}(p R, p z)  \tag{4.56}\\
& F_{-}^{L<}=a_{L}\left(z z^{\prime}\right)^{5 / 2} S_{c}^{-}\left(p z^{\prime}, p R^{\prime}\right) S_{c}^{-}(p R, p z)  \tag{4.57}\\
& F_{+}^{R<}=a_{R}\left(z z^{\prime}\right)^{5 / 2} S_{c}^{+}\left(p z^{\prime}, p R^{\prime}\right) S_{c}^{+}(p R, p z)  \tag{4.58}\\
& F_{-}^{R<}=\alpha_{R}\left(z z^{\prime}\right)^{5 / 2} \tilde{S}_{c}^{-}\left(p z^{\prime}, p R^{\prime}\right) \tilde{S}_{c}^{+}(p R, p z) \tag{4.59}
\end{align*}
$$

The expressions for $z>z^{\prime}$ are obtained by making the replacement $\left\{R \leftrightarrow R^{\prime}\right\}$ in the arguments of the $S_{c}$ functions. We now use the notation in (4.34) and drop the $<,>$ superscripts. From these the fermion Green's function can be obtained trivially from the analog of (4.33),

$$
\Delta\left(p, z, z^{\prime}\right) \equiv \overline{\mathcal{D}} F\left(p, z, z^{\prime}\right)=\left(\begin{array}{cc}
D_{-} F_{-} & \sigma^{\mu} p_{\mu} F_{+}  \tag{4.60}\\
\bar{\sigma}^{\mu} p_{\mu} F_{-} & D_{+} F_{+}
\end{array}\right), \quad D_{ \pm} \equiv \pm\left(\partial_{5}-\frac{2}{z}\right)+\frac{c}{z}
$$

Note that in the UV limit $\left(\chi_{p} \gg 1 / R\right)$ the Bessel functions reduce to phase-shifted trigonometric functions so that we indeed recover the flat 5 D propagators.

### 4.4.3 EUCLIDEAN WARPED 5 D FERMION PROPAGATOR

Finally, it is convenient to write the Wick-rotated form of the fermion propagators since these will provide the relevant Feynman rules in loop diagrams such as $\mu \rightarrow e \gamma$. We shall write out the scalar $F$ functions in a convenient form that we use throughout the rest of this document. The derivation is identical to that outlined above with the replacement $p^{2}=-p_{E}^{2}$ (i.e. $\left.\partial=i \partial_{E}\right)$ in the Green's function equation so that we shall simply state the results. The Euclidean scalar functions are written in terms of the modified Bessel functions $I$ and $K$ which behave like exponentials in the UV. Let us define the auxiliary functions

$$
\begin{align*}
& S_{c}\left(x_{ \pm}, x_{ \pm}^{\prime}\right)=I_{c \pm 1 / 2}(x) K_{c \pm 1 / 2}\left(x^{\prime}\right)-I_{c \pm 1 / 2}\left(x^{\prime}\right) K_{c \pm 1 / 2}(x)  \tag{4.61}\\
& S_{c}\left(x_{ \pm}, x_{\mp}^{\prime}\right)=I_{c \pm 1 / 2}(x) K_{c \mp 1 / 2}\left(x^{\prime}\right)-I_{c \mp 1 / 2}\left(x^{\prime}\right) K_{c \pm 1 / 2}(x)  \tag{4.62}\\
& T_{c}\left(x_{ \pm}, x_{\mp}^{\prime}\right)=I_{c \pm 1 / 2}(x) K_{c \mp 1 / 2}\left(x^{\prime}\right)+I_{c \mp 1 / 2}\left(x^{\prime}\right) K_{c \pm 1 / 2}(x) . \tag{4.63}
\end{align*}
$$

Since we would like to write dimensionless loop integrals, let us define the dimensionless variables $y \equiv k_{E} R^{\prime}$ and $x=k_{E} z$, which are the natural quantities which appear as arguments of the Bessel functions. We write the warp factor as $w=\left(R / R^{\prime}\right)$. It is convenient to pull out overall factors to write the $F$ functions as

$$
\begin{equation*}
F_{ \pm}\left(k_{E}, z, z^{\prime}\right)=i w^{-4} R^{\prime} \tilde{F}_{ \pm, y}^{x x^{\prime}} . \tag{4.64}
\end{equation*}
$$

The Euclidean scalar functions for $x>x^{\prime}$ (i.e. $z>z^{\prime}$ ) are given by

$$
\begin{array}{ll}
\tilde{F}_{-}^{L}=\frac{\left(x x^{\prime}\right)^{5 / 2}}{y^{5}} \frac{S_{c_{L}}\left(x_{-}, y_{-}\right) S_{c_{L}}\left(x_{-}^{\prime}, w y_{-}\right)}{S_{c_{L}}\left(y_{-}, w y_{-}\right)} & \tilde{F}_{+}^{L}=-\frac{\left(x x^{\prime}\right)^{5 / 2}}{y^{5}} \frac{T_{c_{L}}\left(x_{+}, y_{-}\right) T_{c_{L}}\left(x_{+}^{\prime}, w y_{-}\right)}{S_{c_{L}}\left(y_{-}, w y_{-}\right)} \\
\tilde{F}_{-}^{R}=-\frac{\left(x x^{\prime}\right)^{5 / 2}}{y^{5}} \frac{T_{c_{R}}\left(x_{-}, y_{+}\right) T_{c_{R}}\left(x_{-}^{\prime}, w y_{+}\right)}{S_{c_{R}}\left(y_{+}, w y_{+}\right)} & \tilde{F}_{+}^{R}=\frac{\left(x x^{\prime}\right)^{5 / 2}}{y^{5}} \frac{S_{c_{R}}\left(x_{+}, y_{+}\right) S_{c_{R}}\left(x_{+}^{\prime}, w y_{+}\right)}{S_{c_{R}}\left(y_{+}, w y_{+}\right)} . \tag{4.66}
\end{array}
$$

The functions for $x<x^{\prime}$ are given by replacing $x \leftrightarrow x^{\prime}$ in the above formulas. With these definitions the Euclidean fermion propagator given by the analog of (4.60),

$$
\Delta\left(k_{E}, x, x^{\prime}\right) \equiv i \frac{R^{\prime}}{w^{4}} \overline{\mathcal{D}} \tilde{F}_{y}^{x x^{\prime}}=\left(\begin{array}{ll}
y \tilde{D}_{+}+\tilde{F}_{-} & \sigma^{\mu} y_{\mu} \tilde{F}_{+}  \tag{4.67}\\
\bar{\sigma}^{\mu} y_{\mu} \tilde{F}_{-} & y \tilde{D}_{-} \tilde{F}_{+}
\end{array}\right), \quad \quad \tilde{D}_{ \pm} \equiv \pm\left(\partial_{x}-\frac{2}{x}\right)+\frac{c}{x}
$$

### 4.5 BULK GAUGE FIELDS

We now move on to the case of bulk gauge fields. We follow the approach of [215], though we adapt it to follow the same type of derivation espoused above for the fermion propagator. The bulk action is

$$
\begin{equation*}
S_{5}=\int d^{4} x d z \sqrt{g}\left[-\frac{1}{4} F_{M N} F^{M N}+(\text { brane })+(\text { gauge fixing })\right] \tag{4.68}
\end{equation*}
$$

### 4.5.1 INVERTING THE QUADRATIC TERM

To derive the propagator, we would like to write the kinetic term in the form $A_{M} \mathcal{O}^{M N} A_{N}$ so that we may invert the quadratic differential operator $\mathcal{O}^{M N}$. This require judicious integration by parts including the $(R / z)$ factors from the metric and the measure, $\sqrt{g}$. The relevant integration is

$$
\begin{equation*}
\frac{R}{4 z} F^{M N} F_{M N}=-\frac{R}{2} A^{N} \partial^{M}\left(\frac{1}{z} \partial_{M}\right) A_{N}+\frac{R}{2} A^{N} \partial^{M}\left(\frac{1}{z} \partial_{N}\right) A_{N}+\frac{R}{2} \partial^{M}\left(\frac{1}{z} A^{N} \partial_{[M} A_{N]}\right), \tag{4.69}
\end{equation*}
$$

where the last term integrates to a boundary term. Observe that this boundary term vanishes for both Dirichlet and Neumann boundary conditions so that it vanishes for $\mu \rightarrow v$ and 5 th component scalar propagators. It does not vanish, however, for the case of vector-scalar mixing. For simplicity, we will drop the term here in anticipation that it will be removed by gauge fixing. With this caveat, the above integration becomes

$$
\begin{equation*}
\frac{R}{4 z} F^{M N} F_{M N}=A_{\mu}\left[\frac{R}{2 z} \partial^{2} \eta^{\mu \nu}-\frac{R}{2} \partial_{z}\left(\frac{1}{z} \partial_{z}\right) \eta^{\mu \nu}-\frac{R}{2 z} \partial^{\mu} \partial^{\nu}\right] A_{\nu}+A_{5} \frac{R}{z} \partial_{z} \partial^{\mu} A_{\mu}-A_{5} \frac{R}{2 z} \partial^{2} A_{5} \tag{4.70}
\end{equation*}
$$

This is now in the desired form: we can read off the quadratic differential operators which encode the propagation of the ${ }_{5} \mathrm{D}$ gauge bosons. Observe that we have a term that connects the 4 D vector $A_{\mu}$ to the 4 D scalar $A_{5}$. In our mixed position-momentum space formalism, we prefer to leave these as separate fields. As shown below, this term is removed by a judicious choice of gauge fixing.

### 4.5.2 GaUge fixing

Before proceeding, we must now gauge fix to remove the gauge redundancy which otherwise appears as unphysical states in the propagator. Ideally we would like to pick a gauge where the scalar vanishes $A_{5}=\mathrm{o}$ and the vector has a convenient gauge, say, Lorenz gauge $\partial_{\mu} A^{\mu}=0$. Unfortunately, these gauges are incompatible. Intuitively this is because we only have a single gauge fixing functional to work with in the path integral so that we are allowed to set at most one expression to vanish. Instead, motivated by the potential for vector-scalar mixing from the boundary term of (4.70), we choose a gauge fixing functional which cancels this mixing term,

$$
\begin{equation*}
\mathcal{L}_{\text {gauge fix }}=-\left(\frac{R}{z}\right) \frac{1}{2 \xi}\left[\partial_{\mu} A^{\mu}-\xi z \partial_{z}\left(\frac{1}{z} A_{5}\right)\right]^{2} \tag{4.71}
\end{equation*}
$$

We have introduced a gauge fixing parameter $\xi$ which will play the role of the ordinary $R_{\xi}$ gauge fixing parameter in 4 D . We can integrate by parts to convert this to the form $A_{M} \mathcal{O}_{\text {gauge fix }}^{M N} A_{N}$,

$$
\begin{equation*}
\mathcal{L}_{\text {gauge fix }}=A_{\mu} \frac{1}{2 \xi} \frac{R}{z} \partial^{\mu} \partial_{v} A_{v}-A_{5} \frac{R}{z} \partial_{z} \partial^{\mu} A_{\mu}+A_{5} \frac{\xi}{2} \frac{R}{z} \partial_{z}\left[z \partial_{z}\left(\frac{1}{z} A_{5}\right)\right] . \tag{4.72}
\end{equation*}
$$

Observe that the second term here cancels the unwanted mixing term in (4.70). Summing this together with the gauge kinetic term gives a clean separation for the kinetic terms for the gauge vector and scalar:

$$
\begin{align*}
\mathcal{L}_{\text {gauge }}+\mathcal{L}_{\text {gauge fix }}= & A_{\mu}\left[\frac{R}{2 z} \partial^{2} \eta^{\mu \nu}-\frac{R}{2} \partial_{z}\left(\frac{1}{z} \partial_{z}\right) \eta^{\mu \nu}-\left(1-\frac{1}{\xi}\right) \frac{R}{2 z} \partial^{\mu} \partial^{\nu}\right] A_{\nu}  \tag{4.73}\\
& +A_{5} \frac{R}{2 z}\left[-\partial^{2}+\xi\left(\frac{1}{z^{2}}-\frac{1}{z} \partial_{z}+\partial_{z}^{2}\right)\right] A_{5}  \tag{4.74}\\
\equiv & A_{\mu} \mathcal{O}^{\mu \nu} A_{\nu}+A_{5} \mathcal{O}_{5} A_{5} . \tag{4.75}
\end{align*}
$$

### 4.5.3 Mixed space propagators

Now that we've written out the quadratic part of the gauge Lagrangian in the desired form, we may now invert the $\mathcal{O}^{\mu \nu}$ and $\mathcal{O}_{5}$ operators to obtain the propagators. We work in momentum space for the Minkowski directions and so identify $p_{\mu}=i \partial_{\mu}$. Starting with $\Delta^{\mu \nu}$,

$$
\begin{equation*}
\mathcal{O}^{\mu \nu}=-\frac{R}{2 z} p^{2}\left[\eta^{\mu \nu}-\frac{p^{\mu} p^{\nu}}{p^{2}}+\frac{1}{p^{2}} z \partial_{z}\left(\frac{1}{z} \partial_{z}\right) \eta^{\mu \nu}+\frac{1}{\xi} \frac{p^{\mu} p^{\nu}}{p^{2}}\right] \tag{4.76}
\end{equation*}
$$

where it is useful to note that $z \partial_{z}\left(\frac{1}{z} \partial_{z}\right)=\partial_{z}^{2}-\frac{1}{z} \partial_{z}$. We now solve the Green's function equation,

$$
\begin{equation*}
\mathcal{O}^{\mu \rho} \Delta_{\rho v}=i \delta\left(z-z^{\prime}\right) \delta_{v}^{\mu} . \tag{4.77}
\end{equation*}
$$

At this point, one may wonder whether the right-hand side of this equation should be modified by additional warp factors coming from either $\sqrt{g}$ or $\sqrt{g_{55}}$. The answer is no and can be seen by looking at the gauge boson partition function,

$$
\begin{equation*}
Z[A] \sim e^{i S+\int d^{4} x d z \sqrt{8} J^{M} A_{M}} . \tag{4.78}
\end{equation*}
$$

Here each component $A^{\hat{N}}$ of the gauge field has a source of the form $J^{M} \sim \delta^{(5)}\left(x-x^{\prime}\right) \delta_{\hat{N}}^{M}$. Note, however, that this is flat space $\delta$-function is no covariant with respect to the warped ${ }_{5} \mathrm{D}$ metric. The correct form is, in fact,

$$
\begin{equation*}
J^{M} \sim \frac{\delta^{(s)}\left(x-x^{\prime}\right)}{\sqrt{g}} \delta_{\hat{\mathrm{N}}}^{M} \tag{4.79}
\end{equation*}
$$

Note that the factor of $\sqrt{g}$ in $J^{M}$ cancels that in the measure so that indeed the Green's function equation (4.77) has the unwarped $\delta$-function on the right-hand side.

Let us make the ansatz that the Green's function may be written in terms of two functions $G_{p}\left(z, z^{\prime}\right)$ and $H_{p}\left(z, z^{\prime}\right)$,

$$
\begin{equation*}
-i \Delta_{\mu \nu}=\eta_{\mu \nu} G_{p}+\frac{p_{\mu} p_{v}}{p^{2}} H_{p} . \tag{4.80}
\end{equation*}
$$

The point will be that this separation will allow us to simplify (4.77) with the complicated differential operator (4.76). In particular, we show below that $F$ and $G$ will be Green's functions of simpler operators. Plugging in our ansatz gives

$$
\begin{equation*}
-i\left[\eta^{\mu \rho}-\frac{p^{\mu} p^{\rho}}{p^{2}}\left(1-\frac{1}{\xi}\right)+\frac{1}{p^{2}} \eta^{\mu \rho}\left(\partial_{z}^{2}-\frac{1}{z} \partial_{z}\right)\right]\left(\eta_{\rho v} G_{p}+\frac{p_{\rho} p_{v}}{p^{2}} H_{p}\right)=-i \frac{z}{R} \delta\left(z-z^{\prime}\right) \delta_{v}^{\mu} \frac{1}{p^{2}} . \tag{4.81}
\end{equation*}
$$

We now note that the only product on the left-hand side that can generate the $\delta_{v}^{\mu}$ on the right-hand side is the product $\eta^{\mu \rho} \eta_{\rho v}$. Identifying this piece on the left-hand side with the $\delta$-function source indeed yields a simpler Green's function equation,

$$
\begin{equation*}
\left(p^{2}+\partial_{z}^{2}-\frac{1}{z} \partial_{z}\right) G_{p}\left(z, z^{\prime}\right)=\frac{z}{R} \delta\left(z-z^{\prime}\right) \tag{4.82}
\end{equation*}
$$

This completely specifies the function $F\left(z, z^{\prime}\right)$ up to boundary conditions. What about the rest of (4.81)? The remaining terms contain both $F$ and $G$ and must sum to zero:

$$
\begin{equation*}
-(\xi-1) G_{p}+H_{p}+\frac{\xi}{p^{2}}\left(\partial_{z}^{2}-\frac{1}{z} \partial_{z}\right) G_{p}=0 . \tag{4.83}
\end{equation*}
$$

We now apply a trick: we redefine $G$ in such a way that we can remove the $F$ dependence above. We have a hint since the last term already contains the differential operator for which $F$ is a Green's function. Let us thus define

$$
\begin{equation*}
H_{p}=-G_{p}+\tilde{H}_{p} . \tag{4.84}
\end{equation*}
$$

Using (4.82) to convert $F$ into $\delta\left(z-z^{\prime}\right)$, we end up with a Green's function equation for $\tilde{G}$,

$$
\begin{equation*}
\left[\frac{p^{2}}{\xi}+\left(\partial_{z}^{2}-\frac{1}{z} \partial_{z}\right)\right] \tilde{H}=\frac{z}{R} \delta\left(z-z^{\prime}\right) . \tag{4.85}
\end{equation*}
$$

Now a useful shortcut presents itself. This is precisely the same Green's function equation as (4.82) except with a rescaled momentum, $p \rightarrow p / \sqrt{ } \xi$. Thus, we find that

$$
\begin{equation*}
\tilde{H}_{p}\left(z, z^{\prime}\right)=G_{p / \sqrt{\xi}}\left(z, z^{\prime}\right) . \tag{4.86}
\end{equation*}
$$

And thus both $H$ and $G$ are determined up to boundary conditions. The resulting expression is

$$
\begin{equation*}
\Delta_{\mu v}\left(z, z^{\prime}\right)=-i\left(\eta_{\mu \nu}-\frac{p_{\mu} p_{v}}{p^{2}}\right) G_{p}\left(z, z^{\prime}\right)-i \frac{p_{\mu} p_{v}}{p^{2}} G_{p / \sqrt{\xi}}\left(z, z^{\prime}\right) . \tag{4.87}
\end{equation*}
$$

### 4.5.4 BULK MASS DEFORMATION

At this point, it is useful to follow through the above derivations in the case where the gauge boson is perturbed by a bulk mass, $m$. In the RS models of interest for this document the Higgs is localized on the IR brane and does not generate such a bulk mass. We show below that instead this exercise will be precisely what is required to solve the propagator for the $A_{5}$ mode with unbroken gauge symmetry. The mass perturbation takes the form

$$
\begin{equation*}
\Delta S_{\mathrm{mass}}=\int d^{4} x, d z \sqrt{g} \frac{1}{2}\left(\frac{m}{R}\right)^{2} A_{M} g^{M N} A_{N}=\int d^{4} x, d z \sqrt{g} \frac{1}{2} \frac{R}{z}\left(\frac{m}{z}\right)^{2} A_{M} \eta^{M N} A_{N} \tag{4.88}
\end{equation*}
$$

For our purposes we may ignore the $M, N=5$ components. The modified version of the quadratic operator (4.76) for the 4 D vector is

$$
\begin{equation*}
\mathcal{O}^{\mu \nu}=-\frac{R}{2 z} p^{2}\left[\eta^{\mu \nu}-\frac{p^{\mu} p^{\nu}}{p^{2}}+\frac{1}{p^{2}} z \partial_{z}\left(\frac{1}{z} \partial_{z}\right) \eta^{\mu \nu}+\frac{1}{\xi} \frac{p^{\mu} p^{\nu}}{p^{2}}-\frac{m^{2}}{z^{2}} \eta^{\mu \nu} \frac{1}{p^{2}}\right] \tag{4.89}
\end{equation*}
$$

The Green's function equation for $G_{p}$ is modified from (4.82) to

$$
\begin{equation*}
\left(p^{2}+\partial_{z}^{2}-\frac{1}{z} \partial_{z}-\frac{m^{2}}{z^{2}}\right) G_{p}^{m}\left(z, z^{\prime}\right)=\frac{z}{R} \delta\left(z-z^{\prime}\right) \tag{4.90}
\end{equation*}
$$

This is another Green's function equation which completely specifies $G_{p}^{m}$ up to boundary conditions.

### 4.5.5 GAUGE SCALAR PROPAGATOR

We now have to tools to determine the propagator for the $A_{5}$ scalar. The ${ }_{5} \mathrm{D}$ operator $\mathcal{O}_{5}$ in $(4.75)$ is

$$
\begin{equation*}
\mathcal{O}_{5}=\frac{R}{2 z}\left[p^{2}+\xi\left(\frac{1}{z^{2}}-\frac{1}{z} \partial_{z}+\partial_{z}^{2}\right)\right] \tag{4.91}
\end{equation*}
$$

The Green's function equation is

$$
\begin{equation*}
-i \mathcal{O}_{s} \Delta_{p}^{(s)}=i \frac{z}{R} \delta\left(z-z^{\prime}\right) \tag{4.92}
\end{equation*}
$$

Observe the relative sign to the 4 D vector Green's function equation, (4.81), since $z$ points in a spacelike direction. Dividing both sides by $\xi$, we may write this in a suggestive form,

$$
\begin{equation*}
\left(\frac{p^{2}}{\xi}+\partial_{z}^{2}-\frac{1}{z} \partial_{z}+\frac{1}{z^{2}}\right) \Delta_{p}^{(s)}=-\frac{1}{\xi} \frac{z}{R} \delta\left(z-z^{\prime}\right) \tag{4.93}
\end{equation*}
$$

Observe that this takes the same form as (4.90) except the right-hand side has an additional factor of $-\xi^{-1}$ and the $p^{2} / \xi$ term on the left-hand side can be interpreted as a mass term with mass $m=i$. In other words, we may use the Section 4.5 .4 analysis to directly solve for the propagator,

$$
\begin{equation*}
\Delta_{p}^{(s)}=\frac{i}{\xi} G_{\frac{p}{\sqrt{\xi}}}^{i} \tag{4.94}
\end{equation*}
$$

In principle, one should also calculate the ghost propagators required for gauge invariance. However, since we will not calculate pure gauge loops, we will avoid this derivation and refer the reader to [215].

### 4.5.6 Solving the Green's functions

The solution to the homogeneous version of (4.82), where the right-hand side is set to zero, is

$$
\begin{equation*}
G_{p}\left(z, z^{\prime}\right)=A\left(z^{\prime}\right) z J_{1}(k z)+B\left(z^{\prime}\right) z Y_{1}(k z) \tag{4.95}
\end{equation*}
$$

We fix the original position $z^{\prime}$ and separately consider the cases where the propagation position $z$ is greater or less than $z^{\prime}$ :

$$
G_{p}\left(z, z^{\prime}\right)=\left\{\begin{array}{lll}
G_{k}^{>}\left(z, z^{\prime}\right) & \text { if } & z>z^{\prime}  \tag{4.96}\\
G_{k}^{<}\left(z, z^{\prime}\right) & \text { if } & z<z^{\prime}
\end{array}\right.
$$

Now we must impose boundary conditions. Since matching the Sm spectrum requires zero modes for the 4 D vectors, we impose Neumann conditions on both branes for these modes. Note, however, that this choice combined with boundary term in the action (4.70) implies that the natural boundary condition for the 4 D singlet is Dirichlet [177]. This is because any other boundary condition would not permit arbitrary variation of the $A^{5}$ field on the boundaries. We thus have the boundary conditions

$$
\begin{equation*}
\left.\partial_{z} G_{k}^{>}\left(z, z^{\prime}\right)\right|_{z=R^{\prime}}=\left.0 \quad \partial_{z} G_{k}^{<}\left(z, z^{\prime}\right)\right|_{z=R}=0 \tag{4.97}
\end{equation*}
$$

As a useful trick, we may use the expression

$$
\begin{equation*}
y_{p}^{\prime}(\alpha x)=\alpha y_{p-1}(\alpha x)-\frac{p}{x} y_{p}(\alpha x), \tag{4.98}
\end{equation*}
$$

for a Bessel function of the first or secondkind, $y=J, Y$. This gives a simple expression for the derivative of $G_{p}$,

$$
\begin{equation*}
\partial_{z} G_{p}\left(z, z^{\prime}\right)=z^{\prime} A\left(z^{\prime}\right) p J_{\circ}(p z)+z^{\prime} B\left(z^{\prime}\right) p Y_{\circ}(p z) . \tag{4.99}
\end{equation*}
$$

The boundary conditions thus give expressions for the coefficients in terms of functions $f$ and $g$ that depend on both $z^{\prime}$ and $p$ :

$$
\begin{align*}
& \partial_{z} G_{p}^{>}\left(R^{\prime}, z^{\prime}\right)=0 \Rightarrow\left\{\begin{array}{rlr}
A^{>}\left(z^{\prime}\right) & =-Y_{0}\left(p R^{\prime}\right) f\left(z^{\prime}\right) \\
B^{>}\left(z^{\prime}\right) & = & J_{0}\left(p R^{\prime}\right) f\left(z^{\prime}\right)
\end{array}\right.  \tag{4.100}\\
& \partial_{z} G_{p}^{<}\left(R, z^{\prime}\right)=0 \Rightarrow\left\{\begin{array}{rrr}
A^{<}\left(z^{\prime}\right) & =-Y_{0}(p R) g\left(z^{\prime}\right) \\
B^{<}\left(z^{\prime}\right) & = & J_{0}(p R) g\left(z^{\prime}\right)
\end{array}\right. \tag{4.101}
\end{align*} .
$$

We next impose matching conditions at $z=z^{\prime}$ by integrating the defining Green's function equation over a sliver of width $\varepsilon$ around $z=z^{\prime}$, (4.82):

$$
\begin{equation*}
\int_{z^{\prime}-\varepsilon}^{z^{\prime}+\varepsilon} d z\left(p^{2}+\partial_{z}^{2}-\frac{1}{z} \partial_{z}\right) G_{p}\left(z, z^{\prime}\right)=\frac{z^{\prime}}{R} \tag{4.102}
\end{equation*}
$$

The first term vanishes in the $\varepsilon \rightarrow \circ$ limit. The last term may be written as

$$
\begin{equation*}
\int \frac{d z}{z} \partial_{z} G=\left.\frac{1}{z} G\right|_{z=z^{\prime}}+\int \frac{d z}{z^{2}} G . \tag{4.103}
\end{equation*}
$$

This vanishes since $G$ must be continuous across $z^{\prime}$ or else $\partial_{z} G$ would have a $\delta$ function; $G^{>}\left(z^{\prime}, z^{\prime}\right)=G^{<}\left(z^{\prime}, z^{\prime}\right)$. We are thus left with the expression

$$
\begin{equation*}
\partial_{z} G_{p}^{>}\left(z^{\prime}, z^{\prime}\right)-\partial_{z} G_{p}^{<}\left(z^{\prime}, z^{\prime}\right)=\frac{z^{\prime}}{R} . \tag{4.104}
\end{equation*}
$$

For simplicity, let us introduce a useful shorthand for the Green's functions that propagate to the same position in the extra dimension, $G^{>}\left(z^{\prime}, z^{\prime}\right)$ and $G^{<}\left(z^{\prime}, z^{\prime}\right)$,

$$
\begin{align*}
& G^{>}\left(z^{\prime}, z^{\prime}\right)=z^{\prime}(a Y-b J) f  \tag{4.105}\\
& G^{<}\left(z^{\prime}, z^{\prime}\right)=z^{\prime}(\bar{a} Y-\bar{b} J) g \tag{4.106}
\end{align*}
$$

where $Y, J$ refer to $Y_{1}\left(p z^{\prime}\right), J_{1}\left(p z^{\prime}\right), f$ and $g$ are evaluated at $z^{\prime}$, and the $(a, b, \bar{a}, \bar{b})$ coefficients correspond to $\left(A^{>}, B^{>}, A^{<}, B^{<}\right)$ in $(4.100-4.101)$. With this notation, the solution of $(4.104)$ for $f$ is

$$
\begin{equation*}
f=\frac{1}{k R} \frac{1}{a y-b j}+g \frac{\bar{a} y-\bar{b} j}{a y-b_{j}} . \tag{4.107}
\end{equation*}
$$

To obtain the expression for $g$, we use the continuity of $G, G^{>}\left(z^{\prime}, z^{\prime}\right)=G^{<}\left(z^{\prime}, z^{\prime}\right)$. The algebra is somewhat tedious but is straightforward using a computer algebra system such as Mathematica; the key identity is

$$
\begin{equation*}
Y_{\circ}\left(k z^{\prime}\right) J_{1}\left(k z^{\prime}\right)-J_{\circ}\left(k z^{\prime}\right) Y_{1}\left(k z^{\prime}\right)=\frac{2}{k \pi z^{\prime}} . \tag{4.108}
\end{equation*}
$$

The resulting expressions for $g$ and $f$ (after plugging in the complete solution for $g$ ) are

$$
\begin{equation*}
g=\frac{\pi z^{\prime}}{2 R} \frac{a Y-b J}{\bar{a} b-a \bar{b}} \quad f=\frac{\pi z^{\prime}}{2 R} \frac{\bar{a} Y-\bar{b} J}{\bar{a} b-a \bar{b}} . \tag{4.109}
\end{equation*}
$$

We thus have the final expression for the Green's function,

$$
\begin{equation*}
G_{p}^{>}\left(z, z^{\prime}\right)=\frac{\pi}{2} \frac{z z^{\prime}}{R} \frac{\left[-Y_{0}(p R) J_{1}\left(p z^{\prime}\right)+J_{0}(k R) Y_{1}\left(k z^{\prime}\right)\right]\left[-Y_{\circ}\left(p R^{\prime}\right) J_{1}(p z)-J_{0}\left(k R^{\prime}\right) Y_{1}(k z)\right]}{J_{0}(k R) Y_{\circ}\left(k R^{\prime}\right)-Y_{\circ}(k R) J_{0}\left(k R^{\prime}\right)} \tag{4.110}
\end{equation*}
$$

The $G_{p}^{<}\left(z, z^{\prime}\right)$ solution is given by swapping $R \leftrightarrow R^{\prime}$ in the numerator. Observe that the second factor in the numerator is obtained from the first from $\left(R \rightarrow R^{\prime}\right)$ and $\left(z \leftrightarrow z^{\prime}\right)$. This completely specifies the 4 D vector propagator $\Delta_{\mu \nu}$ through (4.87).

One may apply similar manipulations to determine the general massive Green's function $G_{p}^{m}\left(z, z^{\prime}\right)$ in (4.90). Recall that this is necessary for determining the 4 D scalar $A_{5}$ propagator. There are two caveats to remember:

- The key modification in the differential operator for this Green's function is a term $-m^{2} / z^{2}$. The homogeneous solution (4.95) is generalized to $J_{1} \rightarrow J_{\sqrt{1+m^{2}}}$ and $Y_{1} \rightarrow Y_{\sqrt{1+m^{2}}}$.
- As explained above and in [177], the correct boundary condition for the 4D scalars in the 5 D gauge field is Dirichlet. This is required if one wants to allow arbitrary variation of the gauge fields on the branes when varying the action and is consistent with the non-observation of adjoint scalars in the low-energy 4D theory.

Specializing to the case $m=i$ which is relevant for the scalar propagator,

$$
\begin{equation*}
G_{p}^{i>}\left(z, z^{\prime}\right)=\frac{\pi}{2} \frac{z z^{\prime}}{R} \frac{\left[-Y_{\circ}(p R) J_{0}\left(p z^{\prime}\right)+J_{0}(p R) Y_{\circ}\left(p z^{\prime}\right)\right]\left[-Y_{\circ}\left(p R^{\prime}\right) J_{\circ}(p z)+J_{\circ}\left(p R^{\prime}\right) Y_{\circ}(p z)\right]}{-Y_{\circ}(p R) J_{\circ}\left(p R^{\prime}\right)-Y_{\circ}\left(p R^{\prime}\right) J_{\circ}(p R)} \tag{4.111}
\end{equation*}
$$

where again $G_{p}^{i<}\left(z, z^{\prime}\right)$ is obtained from $G^{i>}$ by the substitution $z \leftrightarrow z^{\prime}$. The $A_{5}$ propagator is obtained from this expression via (4.94).

### 4.5.7 Wick rotation to Euclidean space

The Euclidean space propagators are related by a wick rotation, $p=i p_{E}$. Thus one may re-solve the Green's function equation with $p^{2} \rightarrow-p_{E}^{2}$. The result changes the Bessel functions $J \rightarrow K$ and $Y \rightarrow I$ along with several signs. Appropriate formulae are presented below.

### 4.6 Bulk Feynman Rules

Here we summarize the ${ }_{5}$ D position-momentum space Feynman rules used to derive the amplitudes in Chapters 5 and 7. All couplings are written in terms of ${ }_{5} \mathrm{D}$ quantities. The brane-localized Higgs field is drawn as a dashed line and the fifth component of a bulk gauge boson is drawn as a dotted line.


$$
=i g_{5}\left(\frac{R}{z}\right)^{4} \gamma^{\mu}
$$

$$
\longrightarrow=\Delta_{k}\left(z, z^{\prime}\right)
$$

$$
\sum_{,-r_{1}}
$$

$$
\leadsto \sim \sim=-i \eta^{\mu v} G_{k}\left(z, z^{\prime}\right)
$$

$$
=i e_{5}\left(p_{+}-p_{-}\right)_{\mu}
$$

$$
\ldots \ldots \ldots=i \bar{G}_{k}\left(z, z^{\prime}\right)
$$



$$
\text { ~~ }=\varepsilon^{\mu}(q) f_{A}^{(o)}
$$

$$
=\frac{i}{2} e_{5} g_{5} v \eta^{\mu v}
$$



$$
=i\left(\frac{R}{R^{\prime}}\right)^{3} Y_{5}
$$

The ${ }_{5} \mathrm{D}$ Lagrangian parameters are related to the usual Standard Model parameters by

$$
\begin{align*}
g_{5}^{2} & =g_{S M}^{2} R \ln R^{\prime} / R  \tag{4.112}\\
e_{5} f_{A}^{(\circ)} & =e_{S M}  \tag{4.113}\\
Y_{5} & =R Y
\end{align*}
$$

$$
(4.114)
$$

where $Y$ represents an anarchic 4D Yukawa matrix that is related to the Standard Model Yukawa by (??). The $f_{c}$ fermion flavor functions are defined in (??). The vector propagator functions $G_{k}\left(z, z^{\prime}\right)$ and $\bar{G}_{k}\left(z, z^{\prime}\right)$ are explicitly derived in [215], which also contains generic formulae for analogous functions for fields of general spin and additional gauge boson vertices. Using the dimensionless $x$ and $y$ variables defined in (5.44) and assuming $z>z^{\prime}$, the Euclidean space vector Green's functions are

$$
\begin{align*}
& G_{k}\left(z, z^{\prime}\right)=\frac{\left(R^{\prime}\right)^{2}}{R} G_{y}\left(x, x^{\prime}\right)=\frac{\left(R^{\prime}\right)^{2}}{R} \frac{x x^{\prime}}{y} \frac{T_{10}(x, y) T_{1 \circ}\left(x^{\prime}, w y\right)}{S_{\circ \circ}(w y, y)}  \tag{4.115}\\
& \bar{G}_{k}\left(z, z^{\prime}\right)=\frac{\left(R^{\prime}\right)^{2}}{R} \bar{G}_{y}\left(x, x^{\prime}\right)=\frac{\left(R^{\prime}\right)^{2}}{R} \frac{x x^{\prime}}{y} \frac{S_{\circ \circ}(x, y) S_{\circ \circ}\left(x^{\prime}, w y\right)}{S_{\circ \circ}(w y, y)} \tag{4.116}
\end{align*}
$$

where

$$
\begin{align*}
T_{i j}(x, y) & =I_{i}(x) K_{j}(y)+I_{j}(y) K_{i}(x)  \tag{4.117}\\
S_{i j}(x, y) & =I_{i}(x) K_{j}(y)-I_{j}(y) K_{i}(x) \tag{4.118}
\end{align*}
$$

and $w=R / R^{\prime}$. For $z<z^{\prime}$ the above formula is modified by $x \leftrightarrow x^{\prime}$. The three gauge boson couplings are given by

$=i e_{5} \frac{R}{z}\left[\left(k-k^{+}\right)^{\rho} \eta^{\mu \nu}+\left(k^{-}-k\right)^{v} \eta^{\mu \rho}+\left(k^{+}-k^{-}\right)^{\mu} \eta^{\nu \rho}\right]$


Here we have used the convention where all momenta are labeled by the charge of the particle and are flowing into the vertex. The $A_{\mu} W_{5}^{+} W_{v}^{-}$vertex is given by $e_{5}(R / z) \eta^{\mu \nu}\left(\partial_{z}^{\mu}-\partial_{z}\right)$. The Euclidan space fermion propagator $\Delta_{k}\left(z, z^{\prime}\right)$ is given in (4.67).

## 4.A Derivation of the rs spin connection

In the rs background we may write the vielbein and inverse vielbein as

$$
\begin{equation*}
e_{M}^{a}(z)=\frac{R}{z} \delta_{M}^{a} \quad E_{a}^{M}(z)=\frac{z}{R} \delta_{a}^{M} \tag{4.119}
\end{equation*}
$$

We may write out the spin connection term of the covariant derivative as

$$
\begin{equation*}
\omega_{M}^{a b}=\underbrace{\frac{1}{2} g^{R P} e_{R}^{[a} \partial_{[M} e_{P]}^{b]}}_{\omega_{M}^{a b}(1)}+\underbrace{\frac{1}{4} g^{R P} g^{T S} e_{R}^{[a} e_{T}^{b]} \partial_{[S} e_{P]}^{a} e_{M}^{d} \eta_{c d}}_{\omega_{M}^{a b}(2)} \tag{4.120}
\end{equation*}
$$

This can be simplified using the fact that the vielbein only depends on $z$. The first part is

$$
\begin{align*}
\omega_{M}^{a b}(1) & =\frac{1}{2} g^{R P} e_{R}^{a} \partial_{[M} e_{P]}^{b}-\frac{1}{2} g^{R P} e_{R}^{b} \partial_{[M} e_{P]}^{a}  \tag{4.121}\\
& =\frac{1}{2} g^{R P} e_{R}^{a} \partial_{M} e_{P}^{b}-\frac{1}{2} g^{R P} e_{R}^{a} \partial_{P} e_{M}^{b}-\frac{1}{2} g^{R P} e_{R}^{b} \partial_{M} e_{P}^{a}+\frac{1}{2} g^{R P} e_{R}^{b} \partial_{P} e_{M}^{a}  \tag{4.122}\\
& =-\frac{1}{2 z} g^{R P} e_{R}^{a} e_{P}^{b} \delta_{M}^{5}+\frac{1}{2 z} g^{R P} e_{R}^{a} e_{M}^{b} \delta_{P}^{5}+\frac{1}{2 z} g^{R P} e_{R}^{b} e_{P}^{a} \delta_{M}^{5}-\frac{1}{2 z} g^{R P} e_{R}^{b} e_{M}^{a} \delta_{P}^{5}  \tag{4.123}\\
& =-\frac{1}{2 z} \eta^{a b} \delta_{M}^{5}+\frac{1}{2 z} g^{R 5} e_{R}^{a} e_{M}^{b}+\frac{1}{2 z} \eta^{b a} \delta_{M}^{5}-\frac{1}{2 z} g^{R s} e_{R}^{b} e_{M}^{a}  \tag{4.124}\\
& =-\frac{1}{2 z} \eta^{a b}\left(\delta_{M}^{5}-\delta_{M}^{5}\right)+\frac{1}{2 z} g^{R 5}\left(e_{R}^{a} e_{M}^{b}-e_{R}^{b} e_{M}^{a}\right)  \tag{4.125}\\
& =-\frac{1}{2 z} \delta_{5}^{R}\left(\delta_{R}^{a} \delta_{M}^{b}-\delta_{R}^{b} \delta_{M}^{a}\right)  \tag{4.126}\\
& =\frac{1}{2 z} \delta_{M}^{[a} \delta_{5}^{b]},
\end{align*}
$$

(4.127)
where we've used $\partial_{M} e_{P}^{b}=-\frac{1}{z} e_{P}^{b} \delta_{M}^{5}$ and the completeness relation $g^{M N} e_{M}^{a} e_{M}^{b}=\eta^{a b}$. The second part is given by

$$
\begin{align*}
& \omega_{M}^{a b}(2)=\frac{1}{4} g^{R P} g^{T S} e_{R}^{a} e_{T}^{b} \partial_{[S} e_{P]}^{c} e_{M}^{d} \eta_{c d}-\frac{1}{4} g^{R P} g^{T S} e_{R}^{b} e_{T}^{a} \partial_{[S} e_{P]}^{c} e_{M}^{d} \eta_{c d}  \tag{4.128}\\
& =\frac{1}{4} g^{R P} g^{T S} e_{R}^{a} e_{T}^{b} \partial_{s} e_{P}^{c} e_{M}^{d} \eta_{c d}-\frac{1}{4} g^{R P} g^{T S} e_{R}^{b} e_{T}^{a} \partial_{s} e_{P}^{c} e_{M}^{d} \eta_{c d} \\
& -\frac{1}{4} g^{R P} g^{T S} e_{R}^{a} e_{T}^{b} \partial_{P} e_{S}^{c} e_{M}^{d} \eta_{c d}+\frac{1}{4} g^{R P} g^{T S} e_{R}^{b} e_{T}^{a} \partial_{P} e_{S}^{c} e_{M}^{d} \eta_{c d}  \tag{4.129}\\
& =-\frac{1}{4 z} g^{R P} g^{T S} e_{R}^{a} e_{T}^{b} \delta_{S}^{s} e_{P}^{c} e_{M}^{d} \eta_{c d}+\frac{1}{4 z} g^{R P} g^{T S} e_{R}^{b} e_{T}^{a} \delta_{S}^{S} e_{P}^{c} e_{M}^{d} \eta_{c d} \\
& +\frac{1}{4 z} g^{R P} g^{T S} e_{R}^{a} e_{T}^{b} \delta_{P}^{s} e_{S}^{c} e_{M}^{d} \eta_{c d}-\frac{1}{4 z} g^{R P} g^{T S} e_{R}^{b} e_{T}^{a} \delta_{P}^{S} e_{S}^{c} e_{M}^{d} \eta_{c d}  \tag{4.130}\\
& =\frac{1}{4 z}\left(-\eta^{a c} g^{T S} e_{T}^{b} e_{M}^{d} \eta_{c d}+\eta^{b c} g^{T 5} e_{T}^{a} e_{M}^{d} \eta_{c d}+g^{R S} \eta^{b c} e_{R}^{a} e_{M}^{d} \eta_{c d}-g^{R S} \eta^{a c} e_{R}^{b} e_{M}^{d} \eta_{c d}\right)  \tag{4.131}\\
& =\frac{1}{4 z}\left(\delta_{5}^{T} \delta_{T}^{b} \delta_{M}^{d} \delta_{d}^{a}-\delta_{5}^{T} \delta_{T}^{a} e_{M}^{d} \delta_{d}^{b}-\delta_{5}^{R} \delta_{R}^{a} \delta_{M}^{d} \delta_{d}^{b}+\delta_{5}^{R} \delta_{R}^{b} \delta_{M}^{d} \delta_{d}^{a}\right)  \tag{4.132}\\
& =\frac{1}{2 z}\left(\delta_{5}^{b} \delta_{M}^{a}-\delta_{5}^{a} \delta_{M}^{b}-\delta_{5}^{a} \delta_{M}^{b}+\delta_{5}^{b} \delta_{M}^{a}\right)  \tag{4.133}\\
& =\frac{1}{2 z} \delta_{M}^{[a} \delta_{s}^{b]} . \tag{4.134}
\end{align*}
$$

Note that these vanish identically for $M=5$. We can now write out the spin-connection part of the covariant derivative,

$$
\begin{align*}
\frac{1}{2} \omega_{M}^{a b} \sigma_{a b} & =\frac{1}{2}\left(\frac{1}{z} \delta_{M}^{[a} \delta_{S}^{b]}\right)_{M \neq s} \frac{1}{4}\left[\gamma_{a}, \gamma_{b}\right]  \tag{4.135}\\
& =\frac{1}{4 z}\left(\gamma_{M} \gamma_{5}+\delta_{M}^{5}\right) \tag{4.136}
\end{align*}
$$

where we've inserted a factor of $\delta_{M}^{5}$ to cancel the $\left(\gamma_{s}\right)^{2}$ when $M=5$. (Note that the natural convention is that $\left(\gamma^{s}\right)^{2}=-\mathbb{1}$ since this is what satisfies the ${ }_{5} \mathrm{D}$ Clifford algebra.)

Finally, the spin connection part of the covariant derivative is

$$
\begin{equation*}
\frac{1}{2} \omega_{M}^{a b} \sigma_{a b}=\frac{1}{4 z}\left(\gamma_{M} \gamma_{s}+\delta_{M}^{s}\right) \tag{4.137}
\end{equation*}
$$

so that the spin covariant derivative is

$$
D_{M}= \begin{cases}\partial_{\mu}+\frac{1}{4 z} \gamma_{\mu} \gamma_{5} & \text { if } M=\mu  \tag{4.138}\\ \partial_{5} & \text { if } M=5\end{cases}
$$



Warped Penguins

Penguin diagrams encode exotic quantum processes where particles of one type are transformed into particles of another type. These processes are especially sensitive to deviations from the Standard Model and are a natural place to search for new physics. In theories of extra dimensions, however, it is not always clear whether one may calculate the rates for these penguins predictively. This chapter uses the formalism of the previous chapter to perform an explicit calculation and show that it is indeed insensitive to quantum corrections.

### 5.1 Overview

We present an analysis of the loop-induced magnetic dipole operator in the Randall-Sundrum model of a warped extra dimension with anarchic bulk fermions and an IR brane-localized Higgs. These operators are finite at one-loop order and we explicitly calculate the branching ratio for $\mu \rightarrow e \gamma$ using the mixed position/momentum space formalism. The particular bound on the anarchic Yukawa and Kaluza-Klein (кк) scales can depend on the flavor structure of the anarchic matrices. This effect encapsulates the misalignment between the bulk mass parameters and the Yukawa matrices in flavor space. We quantify how these models realize this misalignment. We also review tree-level lepton flavor bounds in these models and show that these are are in mild tension with the $\mu \rightarrow e \gamma$ bounds from typical models with a 3 Tev Kaluza-Klein scale. Further, we illuminate the nature of the one-loop finiteness of these diagrams and show how to accurately determine the degree of divergence of a five-dimensional loop diagram using both the five-dimensional and kк formalism. This power counting can be obfuscated in the four-dimensional Kaluza-Klein formalism and we explicitly point out subtleties that ensure that the two formalisms agree. Finally, we remark on the existence of a perturbative regime in which these one-loop results give the dominant contribution.

### 5.2 InTRODUCTION

The Randall-Sundrum (RS) set up for a warped extra dimension is a novel framework for models of electroweak symmetry breaking [175]. When fermion and gauge fields are allowed to propagate in the bulk, these models can also explain the fermion mass spectrum through the split fermion proposal [189, 190, 224]. In these anarchic flavor models each element of
the Yukawa matrices can take natural $\mathcal{O}(1)$ values because the hierarchy of the fermion masses is generated by the exponential localization of the fermion wave functions away from the Higgs field [225, 226].

The same small wavefunction overlap that yields the fermion mass spectrum also gives hierarchical mixing angles [225, 227-229] and suppresses tree-level flavor-changing neutral currents (FCNCs) by the RS-GIM mechanism [225,226]. This built-in protection, however, may not always be sufficient to completely protect against the most dangerous types of experimental FCNC constraints. In the quark sector, for example, the exchange of Kaluza-Klein (кк) gluons induces left-right operators that contribute to CP violation in kaons and result in generic bounds of $\mathcal{O}(10-20 \mathrm{Tev})$ for the KK gluon mass [209,230-234]. To reduce this bound one must either introduce additional structure (such as horizontal symmetries [235, 236] or flavor alignment [237,238]) or alternately gain several $\mathcal{O}(1)$ factors [239] by promoting the Higgs to a bulk field, inducing loop-level QCD matching, etc. This latter approach is limited by tension with loop-induced flavor-violating effects [240].

The leptonic sector of the anarchic model is similarly bounded by FCNCs. Agashe, Blechman and Petriello recently studied the two dominant constraints in the lepton sector: the loop-induced $\mu \rightarrow e \gamma$ photon penguin from Higgs exchange and the tree-level contribution to $\mu \rightarrow 3 e$ and $\mu \rightarrow e$ conversion from the exchange of the $Z$ boson KK tower [241]. These processes set complementary bounds due to their complementary dependence on the overall magnitude of the anarchic Yukawa coupling, $Y_{*}$. While $\mu \rightarrow e \gamma$ is proportional to $Y_{*}^{3}$ due to two Yukawa couplings and a chirality-flipping mass insertion, the dominant contribution to $\mu \rightarrow 3 e$ and $\mu \rightarrow e$ conversion comes from the nonuniversality of the $Z$ boson near the IR brane. In order to maintain the observed mass spectrum, increasing the Yukawa coupling pushes the bulk fermion profiles away from the IR brane and hence away from the flavor-changing part of the $Z$. This reduces the effective four-dimensional (4D) FCNC coupling so that these processes are proportional to $Y_{*}^{-1}$. For a given KK gauge boson mass, these processes then set an upper and lower bound on the Yukawa coupling which are usually mutually exclusive.

A key feature of the lepton sector is that one expects large mixing angles rather than the hierarchical angles in the Cabbibo-Kobayashi-Maskawa (скм) matrix. One way to obtain this is by using a global flavor symmetry for the lepton sector [242] (see also [243, 244]). Including these additional global symmetries can relax the tension between the two bounds. For example, imposing an $\mathrm{A}_{4}$ symmetry on the leptonic sector completely removes the tree-level constraints [242]. Another interesting possibility for obtaining large lepton mixing angles is to have the wavefunction overlap for the neutrino Yukawa peak near the uv brane [245]. For generic models with anarchic fermions, however, [241] found that the tension between $\mu \rightarrow e \gamma$ and tree-level processes ( $\mu \rightarrow 3 e$ and $\mu \rightarrow e$ conversion) push the gauge boson KK scale to be on the order of 5-10 Tev.

The main goal of this paper is to present a detailed one-loop calculation of the $\mu \rightarrow e \gamma$ penguin in the rs model with a brane-localized Higgs and to show that this amplitude is finite.

To perform the calculation and obtain a numerical result we choose to work in the five-dimensional ( ${ }_{5} \mathrm{D}$ ) mixed position/momentum space formalism [223,246]. This setup is natural for calculating processes on an interval with brane-localized terms, as shown in Figure 5.2.1. In particular, there are no sums over KK modes, the chiral boundary conditions are fully incorporated in the 5 D propagators, and the Uv behavior is clear upon Wick rotation where the basis of Bessel functions becomes exponentials in the 4 D loop momentum. The physical result is, of course, independent of whether the calculation was done in 5 D or in 4 D via a кK decomposition. We show explicit one-loop finiteness in the кK decomposed theory and remark upon the importance of taking into account the correct number of KK modes relative to the momentum cutoff when calculating finite ${ }_{5} \mathrm{D}$ loops.

The chapter is organized as follows: We begin in Section 5.3 by summarizing tree-level constraints on the anarchic Yukawa scale. We refer to Chapter 3.5 for a review of the conventions of the anarchic RS model. We then proceed the analysis of $\mu \rightarrow e \gamma$. The dipole operators involved in this process are discussed in Section 5.4 and the relevant coefficient is calculated using ${ }_{5} \mathrm{D}$ methods in Section 5.5 . In Section 5.6 we discuss the origin of finiteness in these operators in both the ${ }_{5} \mathrm{D}$ and 4 D frameworks. We remark on subtleties in counting the superficial degree of divergence, the matching of the number of кк modes with any effective 4D momentum cutoff, and remark on the expected two-loop degree of divergence. We conclude with an outlook for further directions in Section 5.7. In Appendix 5.A we highlight the matching of local 4D effective operators to nonlocal 5 D amplitudes. Next in Appendices 5.B and 5.C we give estimates for the size of each diagram and analytic expressions for the (next-to)leading $\mu \rightarrow e \gamma$ diagrams. Appendix 5 .D focuses on the formalism of quantum field theory in mixed position/momentum space and power counting. Finally, in Appendix 5 .E we explicitly demonstrate a subtle


Figure 5.2.1: A contribution to $\mu \rightarrow e \gamma$ from a brane-localized Higgs. The dashed line represents the Higgs while the cross represents a Yukawa coupling with a Higgs vev.
cancellation in the single-mass insertion neutral Higgs diagram that is referenced in Section 5.6.

### 5.3 Tree-Level constraints from $\mu \rightarrow 3 e$ and $\mu \rightarrow e$ CONVERSION

For a fixed кк gauge boson mass $M_{\mathrm{KK}}$, limits on $\mu \rightarrow 3^{e}$ and $\mu \rightarrow e$ conversion in nuclei provide the strongest lower bounds on the anarchic Yukawa scale $Y_{*}$. These tree-level processes are parameterized by Fermi operators generated by $Z$ and $Z^{\prime}$ exchange, where the prime indicates the кк mode in the mass basis. The effective Lagrangian for these lepton flavor-violating Fermi operators are traditionally parameterized as [247]

$$
\begin{align*}
\mathcal{L}= & \frac{4 G_{F}}{\sqrt{2}}\left[g_{3}\left(\bar{e}_{R} \gamma^{\mu} \mu_{R}\right)\left(\bar{e}_{R} \gamma_{\mu} e_{R}\right)+g_{4}\left(\bar{e}_{L} \gamma^{\mu} \mu_{L}\right)\left(\bar{e}_{L} \gamma^{\mu} e_{L}\right)+g_{5}\left(\bar{e}_{R} \gamma^{\mu} \mu_{R}\right)\left(\bar{e}_{L} \gamma_{\mu} e_{L}\right)\right. \\
& \left.+g_{6}\left(\bar{e}_{L} \gamma^{\mu} \mu_{L}\right)\left(\bar{e}_{R} \gamma_{\mu} e_{R}\right)\right]+\frac{G_{F}}{\sqrt{2}} \bar{e} \gamma^{\mu}\left(v-a \gamma_{5}\right) \mu \sum_{q} \bar{q} \gamma_{\mu}\left(v^{q}-a^{q} \gamma_{5}\right) q, \tag{5.1}
\end{align*}
$$

where we have only introduced the terms that are non-vanishing in the RS set up, and use the normalization where $v^{q}=T_{3}^{q}-2 Q^{q} \sin ^{2} \theta$. The axial coupling to quarks, $a^{q}$, vanishes in the dominant contribution coming from coherent scattering off the nucleus. The $g_{3,4,5,6}$ are responsible for $\mu \rightarrow 3 e$ decay, while the $v, a$ are responsible for $\mu \rightarrow e$ conversion in nuclei. The rates are given by (with the conversion rate normalized to the muon capture rate):

$$
\begin{align*}
\operatorname{Br}(\mu \rightarrow 3 e) & =2\left(g_{3}^{2}+g_{4}^{2}\right)+g_{5}^{2}+g_{6}^{2},  \tag{5.2}\\
\operatorname{Br}(\mu \rightarrow e) & =\frac{p_{e} E_{e} G_{F}^{2} F_{p}^{2} m_{\mu}^{3} a^{3} Z_{e f f}^{4}}{\pi^{2} Z \Gamma_{\text {capt }}} Q_{N}^{2}\left(v^{2}+a^{2}\right), \tag{5.3}
\end{align*}
$$

where the parameters for the conversion depend on the nucleus and are calculated in the Feinberg-Weinberg approximation [248] and we write the charge for a nucleus with atomic number $Z$ and neutron number $N$ as

$$
\begin{equation*}
Q_{N}=v^{u}(2 Z+N)+v^{d}(2 N+Z) \tag{5.4}
\end{equation*}
$$

. The most sensitive experimental constraint comes from muon conversion in ${ }_{22}^{48} \mathrm{Ti}$, for which

$$
\begin{equation*}
E_{e} \sim p_{e} \sim m_{\mu}, \quad F_{p} \sim 0.55, \quad \quad Z_{\text {eff }} \sim 17.61, \quad \quad \Gamma_{\text {capt }} \sim 2.6 \cdot \frac{10^{6}}{\mathrm{~s}} \tag{5.5}
\end{equation*}
$$

We now consider these constraints for a minimal model (where $f_{e_{L}}=f_{e_{R}}, f_{\mu_{L}}=f_{\mu_{R}}$ ) and for a model with custodial protection.

### 5.3.1 MINIMAL RS MODEL

In order to calculate the coefficients in the effective Lagrangian (5.1), we need to estimate the flavor-violating couplings of the neutral gauge bosons in the theory. In the basis of physical KK states all lepton flavor-violating couplings are the consequence of the non-uniformity of the gauge boson wave functions. Let us first consider the effect of the ordinary $Z$ boson, whose wave function is approximately (we use the approximation (2.19) of [249] with a prefactor for canonical normalization)

$$
\begin{equation*}
h^{(\circ)}(z)=\frac{1}{\sqrt{R \log \frac{R^{\prime}}{R}}}\left[1+\frac{M_{Z}^{2}}{4} z^{2}\left(1-2 \log \frac{z}{R}\right)\right] \tag{5.6}
\end{equation*}
$$

The coupling of the $Z$ to fermions can be calculated by performing the overlap integral with the fermion profiles in (??) and is found to be

$$
\begin{equation*}
g^{Z f f}=g_{\mathrm{SM}}^{Z}\left(1+\frac{\left(M_{Z} R^{\prime}\right)^{2} \log \frac{R^{\prime}}{R}}{2(3-2 c)} f_{c}^{2}\right) \tag{5.7}
\end{equation*}
$$

After rotating the fields to the mass eigenbasis we find that the off-diagonal coupling of the $Z$ boson to charged leptons is given by the nonuniversal term and is approximately

$$
\begin{equation*}
g_{L, R}^{Z e \mu} \approx\left(g_{S M}^{Z}\right)^{L, R} \Delta_{e \mu}^{(\circ)} \equiv\left(g_{S M}^{Z}\right)^{L, R} \frac{\left(M_{Z} R^{\prime}\right)^{2} \log \frac{R^{\prime}}{R}}{2(3-2 c)} f_{e_{L, R}} f_{\mu_{L, R}} \tag{5.8}
\end{equation*}
$$

Using these couplings one can estimate the coefficients of the 4-Fermi operators in (5.1),

$$
\begin{equation*}
g_{3,4}=2 g_{L, R}^{2} \Delta_{e \mu}^{(\circ)} \quad g_{5,6}=2 g_{L} g_{R} \Delta_{e \mu}^{(\circ)} \quad(v \pm a)=2 g_{L, R} \Delta_{e \mu}^{(\circ)} \tag{5.9}
\end{equation*}
$$

where the $g_{L, R}$ are proportional to the left- and right-handed charged lepton couplings to the $Z$ in the Standard Model, $g_{L}=-\frac{1}{2}+s_{W}^{2}$ and $g_{R}=s_{W}^{2}$. The $Z^{\prime}$ exchange contribution to $\mu \rightarrow 3 e(\mu \rightarrow e)$ is a $15 \%(5 \%)$ correction and the $\gamma^{\prime}$ exchange diagram is an additional $5 \%(1 \%)$ correction; we shall ignore both here. We make the simplifying assumption that $f_{e_{L}}=f_{e_{R}}$ and $f_{\mu_{L}}=f_{\mu_{R}}$ and then express these in terms of the Standard Model Yukawa couplings as $f=\sqrt{\lambda / Y_{*}}$. The expressions for the lepton flavor-violating processes are then

$$
\begin{align*}
\operatorname{Br}(\mu \rightarrow 3 e) & =10^{-13}\left(\frac{3 \mathrm{TeV}}{M_{\mathrm{KK}}}\right)^{4}\left(\frac{2}{Y_{*}}\right)^{2}  \tag{5.10}\\
\operatorname{Br}(\mu \rightarrow e)_{\mathrm{Ti}} & =2 \cdot 10^{-12}\left(\frac{3 \mathrm{TeV}}{M_{\mathrm{KK}}}\right)^{4}\left(\frac{2}{Y_{*}}\right)^{2} \tag{5.11}
\end{align*}
$$

The current experimental bounds are $\operatorname{Br}(\mu \rightarrow 3 e)<10^{-12}[250]$ and $\operatorname{Br}(\mu \rightarrow e)_{\mathrm{Ti}}<6.1 \cdot 10^{-13}$ [251] so that $\mu \rightarrow e$ conversion provides the most stringent constraint,

$$
\begin{equation*}
\left(\frac{3 \mathrm{TeV}}{M_{\mathrm{KK}}}\right)^{2}\left(\frac{2}{Y_{*}}\right)<0.5 \tag{5.12}
\end{equation*}
$$

For a $3 \mathrm{Tev} Z^{\prime}$, the anarchic Yukawa scale must satisfy $Y_{*} \gtrsim 3.7$, which agrees with [241].

## 5．3．2 CUSTODIALLY PROTECTED MODEL

Since the bound in（5．12）is model dependent，one might consider weakening this constraint by having the leptons transform under the custodial group

$$
\begin{equation*}
\operatorname{SU}(2)_{L} \times \operatorname{SU}(2)_{R} \times \mathrm{U}(1)_{X} \times \mathrm{P}_{L R}, \tag{5.13}
\end{equation*}
$$

where $\mathrm{P}_{L R}$ is a discrete $L \leftrightarrow R$ exchange symmetry．Such a custodial protection was introduced in［211］to eliminate large corrections to the $Z b \bar{b}$ vertex in the quark sector．It was later found that this symmetry also eliminates some of the FCNCS in the $Z$ sector［209］so that one might also expect it to alleviate the lepton flavor violation bounds．We shall now estimate the extent to which custodial symmetry can relax the bound on $Y_{*}$ ．Further discussion including neutrino mixing can be found in［252］．

To custodially protect the charged leptons one choses the $(L, R)_{X}$ representation $(2,2)_{\text {o }}$ for the left－handed leptons， $(3,1)$ 。 $\oplus(1,3)$ 。 for the charged right－handed leptons，and $(1,1)$ 。 for the right－handed neutrinos．There are two neutral zero mode gauge bosons，the Standard Model $Z$ and $\gamma$ ，and three neutral кк excitations，$\gamma^{\prime}, Z^{\prime}$ and $Z_{H}$ ，where the latter two are linear combinations of the $Z$ and $Z_{X}$ boson modes．The coupling of the left handed leptons to the ordinary $Z$ and the $Z^{\prime}$ are protected since those couplings are exactly flavor universal in the limit where $P_{L R}$ is exact．The breaking of $P_{L R}$ on the UV brane leads to small residual contributions which we neglect．The remaining flavor－violating couplings for the left－handed leptons come from the exchange of $Z_{H}$ and the $\gamma^{\prime}$ ，while the right－handed leptons are unprotected．

Since $(v-a)$ couples to right－handed leptons its coupling is unprotected and is the same as in（5．9）．For $(v+a)$ ，on the other hand，the leading－order effect comes from the $Z^{(1)}$ component of the $Z_{H}$ ，whose composition in terms of gauge кK states is［209］

$$
\begin{equation*}
Z_{H}=\cos \xi Z^{(1)}+\sin \xi Z_{X}^{(1)}+\beta Z^{(\circ)} \tag{5.14}
\end{equation*}
$$

where $Z^{(\circ)}$ is the flat zero mode $Z$－boson which does not contribute to FCNCs， $\cos \xi \approx \sqrt{\frac{1}{2}-s_{W}^{2}} / c_{W}$ ，and $\beta$ is a small correction of order $\mathcal{O}\left(v^{2} / M_{\mathrm{KK}}^{2}\right)$ ．The flavor－changing coupling of the кK gauge bosons is analogous to that of кК gluons in［230］，

$$
\begin{equation*}
g_{L, R}^{Z^{(1)} e \mu} \approx\left(g_{S M}^{Z}\right)^{L, R} \Delta_{e \mu}^{L, R(1)} \equiv\left(g_{S M}^{Z}\right)^{L, R} \sqrt{\log \frac{R^{\prime}}{R}} \gamma_{c} f_{e_{L, R}} f_{\mu_{L, R}} \tag{5.15}
\end{equation*}
$$

where

$$
\begin{equation*}
\gamma_{c}=\frac{\sqrt{2}}{J_{1}\left(x_{1}\right)} \int_{0}^{1} d x x^{1-2 c} J_{1}\left(x_{1} x\right) \approx \frac{\sqrt{2}}{J_{1}\left(x_{1}\right)} \frac{0.7 x_{1}}{2(3-2 c)} \tag{5.16}
\end{equation*}
$$

and $x_{1}=M_{\mathrm{KK}} R^{\prime}$ is the first zero of $J_{\mathrm{O}}(x)$ ．The analogous $\gamma^{(1)}$ coupling is given by $g_{S \mathrm{SM}}^{Z} \rightarrow e$ ．Taking into account the $Z_{\mathrm{H}}$ and $\gamma^{(1)}$ ，the $(v+a)$ effective coupling to left－handed leptons is

$$
\begin{equation*}
(v+a)=2 g_{L} g_{\mathrm{KK}} \frac{M_{Z}^{2}}{M_{\mathrm{KK}}^{2}}\left(\cos ^{2} \xi+\frac{Q_{\mathrm{N}}^{Z_{\mathrm{X}}}}{Q_{N}} \cos \xi \sin \xi\right) \Delta_{e \mu}^{L(1)}+2 s_{W}^{2} c_{W}^{2} g_{\mathrm{KK}} \frac{M_{Z}^{2}}{M_{\mathrm{KK}}^{2}} \frac{Q_{N}^{\gamma}}{Q_{N}} \Delta_{e \mu}^{L(1)} . \tag{5.17}
\end{equation*}
$$

The $\cos \xi \sin \xi$ term in the parenthesis represents the $Z_{X}^{(1)}$ component of the $Z_{H}$ which couples to the quarks in the nucleus via

$$
\begin{equation*}
Q_{N}^{Z_{X}}=-\frac{1}{\sqrt{2}} c_{W} \cos \xi\left({ }_{5} Z+{ }_{7} N\right)-\frac{2 \sqrt{2}}{\cos \xi} s_{W} \frac{g^{\prime}}{g}(Z+N), \quad g_{\mathrm{KK}}=\frac{1}{\sqrt{\log R^{\prime} / R}} \tag{5.18}
\end{equation*}
$$

The $g_{\text {KK }}$ factor gives the universal（flavor－conserving）coupling of кK gauge bosons to zero mode fermions．$Q_{N}^{\gamma}$ is the electric charge of the nucleus normalized according to（5．3），$Q_{N}^{\gamma}=2 Z$ ．

Minimizing over the flavor factors $f_{e_{L, R}}$ and $f_{\mu_{L, R}}$ subject to the zero mode fermion mass spectrum and comparing to the experimental bound listed above（5．12），we find that the conversion rate must satisfy

$$
\begin{equation*}
\left(\frac{3 \mathrm{TeV}}{M_{\mathrm{KK}}}\right)^{2}\left(\frac{2}{Y_{*}}\right)<1.6 . \tag{5.19}
\end{equation*}
$$

lowering the bound to $Y_{*} \gtrsim 1$ for a 3 TeV KK gauge boson scale.

### 5.4 Operator analysis of $\mu \rightarrow e \gamma$

We work in 't Hooft-Feynman gauge $(\xi=1)$ and a flavor basis where all bulk masses $c_{i}$ are diagonal. The ${ }_{5} \mathrm{D}$ amplitude for $\mu \rightarrow e \gamma$ takes the form

$$
\begin{equation*}
C H \cdot \bar{L}_{i} \sigma^{M N} E_{j} F_{M N} \tag{5.20}
\end{equation*}
$$

where it is understood that the 5 D fields should be replaced by the appropriate external states which each carry an independent $z$ position in the mixed position/momentum space formalism. These positions must be separately integrated over when matching to an effective 4 D operator so that $(5.20)$ can be thought of as a dimension-8 ${ }_{5} \mathrm{D}$ scattering amplitude whose prefactor $C$ is a function of the external state positions, as explained in Appendix 5.A. When calculating this amplitude in the mixed position/momentum space formalism, the physical external state fields have definite kK number, which we take to be zero modes. The external field profiles and internal propagators depend on 4 D momenta and $z$-positions so that vertex $z$-positions are integrated from $z=R$ to $z=R^{\prime}$ while loop momenta are integrated as usual.

After plugging in the wave functions for the fermion and photon zero modes, including all warp factors, matching the gauge coupling, and expanding in Higgs-induced mass insertions, the leading order 4D operator and coefficients for $\mu \rightarrow e \gamma$ are

$$
\begin{equation*}
R^{\prime 2} \frac{e}{16 \pi^{2}} \frac{v}{\sqrt{2}} f_{L_{i}}\left(a_{k \ell} Y_{i k} Y_{k \ell}^{\dagger} Y_{\ell j}+b_{i j} Y_{i j}\right) f_{-E_{j}} \bar{L}_{i}^{(\circ)} \sigma^{\mu v} E_{j}^{(\circ)} F_{\mu v}^{(\circ)}+\text { h.c. } \tag{5.21}
\end{equation*}
$$

The term proportional to three Yukawa matrices comes from the diagrams shown in Figs. 5.5.1 and 5.5.2, while the single-Yukawa term comes from those in Figure 5.5.3. In the limit where the bulk masses are universal, we may treat the Yukawas as spurions of the $\mathrm{U}(3)^{3}$ lepton flavor symmetry and note that these are the products of Yukawas required for a chirality-flipping, flavor-changing operator.

In anarchic flavor models, however, the bulk masses for each fermion species is independent and introduce an additional flavor structure into the theory so that the $\mathrm{U}(3)^{3}$ lepton flavor symmetry is not restored even in the limit $Y \rightarrow 0$. The indices on the dimensionless $a_{k \ell}$ and $b_{i j}$ coefficients encode this flavor structure as carried by the internal fermions of each diagram. Because the lepton hierarchy does not require very different bulk masses, both $a_{k \ell}$ and $b_{i j}$ are nearly universal.

Next note that the zero-mode mass matrix (??) introduces a preferred direction in flavor space which defines the mass basis. In fact, up to the non-universality of $b_{i j}$, the single-Yukawa term in ( 5.21 ) is proportional to-or aligned—with (??). Hence upon rotation to the mass basis, the off-diagonal elements of this term are typically much smaller than its value in the flavor basis [253,254] and would be identically zero if the bulk masses were universal. Given a set of bulk mass parameters, the extent to which a specific off-diagonal element of the $b_{i j}$ term is suppressed depends on the particular structure of the anarchic ${ }_{5} \mathrm{D}$ Yukawa matrix. This is a novel feature since the structure of the underlying anarchic Yukawa is usually washed out in observables by the hierarchies in the $f_{c}$ flavor functions.

On the other hand, a product of anarchic matrices typically indicates a very different direction in flavor space from the original matrix so that the $a_{i j}$ term is not aligned and we may simplify the product to

$$
\begin{equation*}
\sum_{k, \ell} a_{k \ell} Y_{i k} Y_{k \ell}^{\dagger} Y_{\ell j}=a Y_{*}^{3} \tag{5.22}
\end{equation*}
$$

for each $i$ and $j$. Here we have defined the prefactor $a$; different definitions can include an overall $\mathcal{O}(1)$ factor from the sum over anarchic matrix elements. We have used the anarchic limit and the assumption that neither $a_{k \ell}$ nor $b_{i j}$ vary greatly over realistic bulk mass values. This assumption is justified in Section 5.5 where we explicitly calculate these coefficients to leading order. Further, we have assumed that the scales of the anarchic electron and neutrino Yukawa matrices are the same so that $\left(Y_{E}\right)_{i j} \sim\left(Y_{N}\right)_{i j} \sim Y_{*}$.

To determine the physical $\mu \rightarrow e \gamma$ amplitude from this expression we must go to the standard 4 D mass eigenbasis by
performing a bi-unitary transformation to diagonalize the Standard Model Yukawa,

$$
\begin{equation*}
\lambda^{\mathrm{SM}}=U_{L} \lambda^{(\mathrm{diag})} U_{R}^{\dagger} \tag{5.23}
\end{equation*}
$$

where the magnitudes of the elements of the unitary matrices $U_{L, R}$ are set, in the anarchic scenario, by the hierarchies in the flavor constants

$$
\begin{equation*}
\left(U_{L}\right)_{i j} \sim \frac{f_{L_{i}}}{f_{L_{j}}} \text { for } f_{L_{i}}<f_{L_{j}} \tag{5.24}
\end{equation*}
$$

For future simplicity, let us define the relevant part of the $b_{i j} Y_{i j}$ matrix after this rotation,

$$
\begin{equation*}
b Y_{*}=\sum_{k, \ell}\left(U_{L}\right)_{2 k} b_{k \ell} Y_{k \ell}\left(U_{R}^{\dagger}\right)_{\ell_{1}} \tag{5.25}
\end{equation*}
$$

The traditional parameterization for the $\mu \rightarrow e \gamma$ amplitude is written as [241]

$$
\begin{equation*}
\frac{-i C_{L, R}}{2 m_{\mu}} \bar{u}_{L, R} \sigma^{\mu v} u_{R, L} F_{\mu v} \tag{5.26}
\end{equation*}
$$

where $u_{L, R}$ are the left- and right-handed Dirac spinors for the leptons. Comparing ( 5.21 ) with ( 5.26 ) and using the magnitudes of the off-diagonal terms in the $U_{L}$ rotation matrix in (5.24), we find that in the mass eigenbasis the coefficients are given by

$$
\begin{align*}
C_{L} & =\left(a Y_{*}^{3}+b Y_{*}\right) R^{\prime 2} \frac{e}{16 \pi^{2}} \frac{v}{\sqrt{2}} 2 m_{\mu} f_{L_{2}} f_{-E_{1}}  \tag{5.27}\\
C_{R} & =\left(a Y_{*}^{3}+b Y_{*}\right) R^{\prime 2} \frac{e}{16 \pi^{2}} \frac{v}{\sqrt{2}} 2 m_{\mu} f_{L_{1}} f_{-E_{2}} \tag{5.28}
\end{align*}
$$

The $\mu \rightarrow e \gamma$ branching fraction and its experimental bound are given by

$$
\begin{align*}
& \operatorname{Br}(\mu \rightarrow e \gamma)_{\text {thy }}=\frac{12 \pi^{2}}{\left(G_{F} m_{\mu}^{2}\right)^{2}}\left(\left|C_{L}\right|^{2}+\left|C_{R}\right|^{2}\right)  \tag{5.29}\\
& \operatorname{Br}(\mu \rightarrow e \gamma)_{\exp }<1.2 \cdot 10^{-11} \tag{5.30}
\end{align*}
$$

While the generic expression for $\operatorname{Br}(\mu \rightarrow e \gamma)$ depends on the individual wave functions $f_{L,-E}$, the product $C_{L} C_{R}$ is fixed by the physical lepton masses and the relation $C_{L}^{2}+C_{R}^{2} \geq 2 C_{L} C_{R}$ so that one can put a lower bound on the branching ratio

$$
\begin{equation*}
\operatorname{Br}(\mu \rightarrow e \gamma) \geq 6\left|a Y_{*}^{2}+b\right|^{2} \frac{a}{4 \pi}\left(\frac{R^{\prime 2}}{G_{F}}\right)^{2} \frac{m_{e}}{m_{\mu}} \approx 5.1 \cdot 10^{-8}\left|a Y_{*}^{2}+b\right|^{2}\left(\frac{3 \mathrm{TeV}}{M_{\mathrm{KK}}}\right)^{4} \tag{5.31}
\end{equation*}
$$

Thus for a 3 Tev KK gauge boson scale we obtain an upper bound on $Y_{*}$

$$
\begin{equation*}
\left|a Y_{*}^{2}+b\right|\left(\frac{3 \mathrm{TeV}}{M_{\mathrm{KK}}}\right)^{2} \leq 0.015 \tag{5.32}
\end{equation*}
$$

Note that the $b$ coefficient is independent of $Y_{*}$ so that sufficiently large $b$ can rule out the assumption that the ${ }_{5} \mathrm{D}$ Yukawa matrix can be completely anarchic-i.e. with no assumed underlying flavor structure-at a given KK scale no matter how small one picks $Y_{*}$. This is a new type of constraint on anarchic flavor models in a warped extra dimension. Conversely, if $b$ is of the same order as $a$ and has the opposite sign, then the bounds on the anarchic scale $Y_{*}$ are alleviated. We will show below that $b$ is typically suppressed relative to $a$ but can, in principle, take a range of values between $b=-0.5$ and 0.5 . For simplicity we may use the case $b=o$ as a representative and plausible example, in which case the bound on the anarchic Yukawa scale is

$$
\begin{equation*}
Y_{*} \leq 0.12|a|^{-\frac{1}{2}} \tag{5.33}
\end{equation*}
$$

In Section 5.5 .4 we quantify the extent to which the $b$ term may affect this bound. Combined with the lower bounds on $Y_{*}$


Figure 5.5.1: Neutral boson diagrams contributing to the $a$ coefficient defined in (5.22). Fermion arrows denote the zero mode chirality, i.e. the $S U(2)$ representation. External legs whose arrows do not point outward have an implicit external mass insertion. Dotted lines represent the fifth component of a bulk gauge field. Analytic forms for these diagrams are given in Appendix 5.C.
from tree-level processes in Section 5.3, this bound typically introduces a tension in the preferred value of $Y_{*}$ depending on the value of $a$. In other words, it can force one to either increase the kK scale or introduce additional symmetry structure into the ${ }_{5} \mathrm{D}$ Yukawa matrices which can reduce $a$ in (5.22) or force a cancellation in (5.32).

### 5.5 CALCULATION OF $\mu \rightarrow e \gamma$ IN A WARPED EXTRA DIMENSION

In principle, there are a large number of diagrams contributing to the $a$ and $b$ coefficients even when only considering the leading terms in a mass insertion expansion. These are depicted in Figs. 5.5.1-5.5.3. Fortunately, many of these diagrams are naturally suppressed and the dominant contribution to each coefficient is given by the two diagrams shown in Figure 5.5.4. Analytic expressions for the leading and next-to-leading diagrams are given in Appendix 5.C along with an estimate of the size of each contribution.

The flavor structure of the diagrams contributing to the $b$ coefficient is aligned with the fermion zero-mode mass matrix [226,241]. The rotation of the external states to mass eigenstates thus suppresses these diagrams up to the bulk mass (c) dependence of internal propagators which point in a different direction in flavor space and are not aligned. Since KK modes do not carry very strong bulk mass dependence, the diagrams which typically give the largest contribution after alignment are those which permit zero mode fermions in the loop. We provide a precise definition of the term "typically" in Section 5.5.2.

The Ward identity requires that the physical amplitude for a muon of momentum $p$ to decay into a photon of polarization $\varepsilon$ and an electron of momentum $p^{\prime}$ takes the form

$$
\begin{equation*}
\mathcal{M}=\varepsilon_{\mu} \mathcal{M}^{\mu} \sim \varepsilon_{\mu} \bar{u}_{p^{\prime}}\left[\left(p+p^{\prime}\right)^{\mu}-\left(m_{\mu}+m_{e}\right) \gamma^{\mu}\right] u_{p} \tag{5.34}
\end{equation*}
$$

This is the combination of masses and momenta that gives the correct chirality-flipping tensor amplitude in (5.26). This simplifies the calculation of this process since one only has to identify the coefficient of the $\bar{u}_{p^{\prime}}\left(p+p^{\prime}\right)^{\mu} u$ term to determine the entire amplitude; all other terms are redundant by gauge invariance [255]. The general strategy is to use the Clifford algebra and the equations of motion for the external spinors to determine this coefficient. This allows us to directly write the


W

$W, W^{5}$

$W^{5}, W$


$H^{ \pm}$

$H^{ \pm}$


Figure 5.5.2: Charged boson diagrams contributing to the $a$ coefficient following the conventions in Figure 5.5.1. Analytic forms for these diagrams are given in Appendix 5.C.
finite physical contribution to the amplitude without worrying about the regularization of potentially divergent terms which are not gauge invariant. In Section 5.6 .1 we will further use this observation to explain the finiteness of this amplitude in 5 D .

In addition to the diagrams in Figs. 5.5.1-5.5.3, there are higher-order diagrams with an even number of additional mass insertions and brane-to-brane propagators. Following the Feynman rules in Appendix 4.6, each higher-order pair of mass insertions is suppressed by an additional factor of

$$
\begin{equation*}
\left(\frac{k}{k} \frac{R^{\prime 4}}{R^{4}} \cdot(-i) \frac{R^{3}}{R^{\prime 3}} R Y_{*} \frac{v}{\sqrt{2}}\right)^{2} \sim \frac{1}{2}\left(Y_{*} R^{\prime} v\right)^{2} \sim \mathcal{O}\left(10^{-2}\right), \tag{5.35}
\end{equation*}
$$

since we assume anarchic Yukawa matrices, $Y_{*} \sim 2$. We are thus justified in considering only the leading-order terms in the mass insertion approximation.

We now present the leading contributions to the $a$ and $b$ coefficients. Other diagrams give a correction on the order of $10 \%$ of these results. We provide explicit formulas and numerical estimates for the next-to-leading order corrections in Appendix 5.C.

### 5.5.1 Calculation of $a$

We now calculate the leading-order contribution to the amplitude to determine the $a$ coefficient in (5.22). As discussed above, it is sufficient to compute the coefficient of the $\left(p+p^{\prime}\right)^{\mu}$ term in the amplitude. The dominant contribution to $a$ comes from the $W$ boson diagrams in Figure 5.5.4a. This is because diagrams with ${ }_{5} \mathrm{D}$ gauge bosons are enhanced relative to the Higgs diagrams by a factor of $\ln R^{\prime} / R \sim 37$. Further, the $W$ diagrams are enhanced over the $Z$ diagrams due to the size of their respective Standard Model couplings to leptons. Additional suppression factors can arise from the structure of each diagram and are discussed in Appendix 5.B. Explicit calculation confirms that the $W$ loop with two internal mass insertions indeed gives the leading contribution to $a$.

The charged and neutral boson diagrams have independent flavor structures, $\left(Y_{E} Y_{N}^{\dagger} Y_{N}\right)_{\mu e}$ and $\left(Y_{E} Y_{E}^{\dagger} Y_{E}\right)_{\mu e}$ respectively. The anarchic Yukawa assumption implies that both of these terms should be of the same order, $Y_{*}^{3}$. However one must remember that there may be a relative sign between these contributions depending on the specific anarchic $Y_{N}$ and $Y_{E}$ matrices. In other words, $a=a_{\text {charged }} \pm a_{\text {neutral }}$ where the sign cannot be specified generically. However, because


W

$W, W^{5}$


$W^{5}, W$


Z


W


Z


Figure 5.5.3: Diagrams contributing to the $b$ coefficient following the conventions in Figure 5.5.1. Not shown: zero mass-insertion $Z^{5}$ diagram. Analytic forms for these diagrams are given in Appendix 5.C.
$a_{\text {neutral }} \ll a_{\text {charged }}$, we ignore the neutral boson loops, though these neutral boson diagrams may become appreciable if one allows a hierarchy between the overall scales of the $Y_{N}$ and $Y_{E}$ matrices.

The $W$ loop in Figure 5.5.4a contains an implicit mass insertion on the external muon leg. As explained in Appendix 5.B, the ${ }_{5} \mathrm{D}$ fermion propagator between this mass insertion and the loop vertex is dominated by the KK mode which changes fermion chirality. This is because the chirality-preserving piece of the propagator goes like $p$. Invoking the muon equation of motion gives a factor of $f_{\mu}^{(\circ)}\left(v R^{\prime}\right) f_{\mu}^{(\circ)} \sim\left(m_{\mu} R^{\prime}\right)$ for the external leg. This is much smaller than the $f_{\mu}^{(\circ)}\left(v R^{\prime}\right) f_{\mu}^{(\mathrm{KK})}$ factor from the chirality-flipping part of the propagator. Compared to the mass insertion connecting the zero mode external muon to a KK intermediate state, the mass insertion connecting two zero mode fermions is smaller by a factor of the exponentially suppressed zero mode profile ${ }^{1}$.

Using the Feynman rules in Appendix 4.6, the amplitude this diagram is

$$
\left.\mathcal{M}^{\mu}\right|_{\left(p+p^{\prime}\right)}=\frac{i}{16 \pi^{2}}\left(R^{\prime}\right)^{2} f_{c_{L_{\mu}}} Y_{*}^{3} f_{-c_{E_{e}}} \frac{e v}{\sqrt{2}}\left(\frac{g^{2}}{2} \ln \frac{R^{\prime}}{R}\right)\left(\frac{R^{\prime} v}{\sqrt{2}}\right)^{2} I_{2 M I W} \bar{u}_{p^{\prime}}\left(p+p^{\prime}\right)^{\mu} u_{p}
$$

where $I_{2 \mathrm{MIW}}=-0.31$ is a dimensionless loop integral. Taking $R^{\prime} v / \sqrt{2}=.17$ and $g^{2} / 2 \ln \left(R^{\prime} / R\right)=7 \cdot 3$, the $a$ coefficient in (5.22) is

$$
\begin{equation*}
a=-0.065 \tag{5.37}
\end{equation*}
$$

### 5.5.2 CALCULATION OF $b$

As discussed above, the diagrams contributing to $b$ are sensitive to the structure of the anarchic Yukawa matrix relative to that of the non-universal internal bulk fermion masses. For example, if the bulk mass parameters were universal, then the $b$ coefficient operator would be aligned and the off-diagonal element would vanish. The sign of this off-diagonal term is a function of the initial anarchic matrix so that the $b$ term may interfere constructively or destructively with the $a$ term calculated above. We numerically generate anarchic matrices whose elements have random sign and random values between

[^2]

Figure 5.5.4: The leading diagrams contributing to the $a$ and $b$ coefficients following the same conventions as Figure 5.5.1.
0.5 and 2 to determine the distribution of probable Yukawa structures. Such a distribution is peaked about zero so that the choice $b=o$ is a reasonable simplifying assumption. For a more detailed description of the range of bounds accessible by the anarchic RS scenario, one may use the $1 \sigma$ value of $|b|$ as characteristic measure of how large an effect one should expect from generic anarchic Yukawas.

The dominant contributions to the $b$ coefficient are shown in Figure 5.5.4b. These are the diagram with a charged Goldstone and a $W$ in the loop and the diagram with a $Z$ and a single mass insertion in the loop. Following the analysis in in Appendix 5.B.4, these diagrams can have zero mode fermions propagating in the loop and hence are sensitive to the bulk mass parameters of the internal fermions being summed in the loop. This, in turn, implies that the diagrams are more robust against alignment upon rotating to the zero mode mass basis.

The amplitudes associated with this diagram are

$$
\begin{align*}
\left.\mathcal{M}(\mathrm{IMIZ})\right|_{\left(p+p^{\prime}\right)^{\mu}} & =\frac{i}{16 \pi^{2}}\left(R^{\prime}\right)^{2} f_{c_{L}} Y_{E} f_{-c_{\mathrm{E}}} \frac{e v}{\sqrt{2}}\left(g_{Z_{\mathrm{L}}} g_{Z_{R}} \ln \frac{R^{\prime}}{R}\right) \times I_{\mathrm{MIZ}},  \tag{5.38}\\
\left.\mathcal{M}(\mathrm{oMIHW})\right|_{\left(p+p^{\prime}\right)^{\mu}} & =\frac{i}{16 \pi^{2}}\left(R^{\prime}\right)^{2} f_{c_{L}} Y_{E} f_{-c_{\mathrm{E}}} \frac{e v}{\sqrt{2}}\left(\frac{g^{2}}{2} \ln \frac{R^{\prime}}{R}\right) \times I_{\mathrm{oMIH}}, \tag{5.39}
\end{align*}
$$

where $g_{Z_{L, R}}$ is the Standard Model coupling of the $Z$ to left- and right-handed leptons respectively. The values for the dimensionless integrals are given in (5.75) and (5.76).

After scanning over anarchic matrices as defined above, the $1 \sigma$ value for the $b$ coefficient is

$$
\begin{equation*}
\left|b^{1 \sigma}\right|=0.03 \tag{5.40}
\end{equation*}
$$

Here we take the $1 \sigma$ value of the $b$ coefficient assuming the bulk masses of the minimal model $c_{L}=c_{R}$ as a representative benchmark for a plausible general estimate of the generically allowed range of $b$.

### 5.5.3 MODIFICATIONS IN CUSTODIAL MODES

In Section 5.3.2 it was shown that custodial symmetry weakens the bounds from tree-level FCNCs. Since we would like to assess the tension between tree- and loop-level bounds, we should also examine the effect of the additional custodial modes on $\mu \rightarrow e \gamma$. These additional diagrams are described by the same topologies as those in Figs. 5.5.1-5.5.3 but differ by replacing internal lines with custodial bosons and fermions. The expression for the amplitude differs by coupling constants and the use of propagators with different boundary conditions, but not in the overall structure of each amplitude and so are straightforward to extract from the minimal model expressions. The leading topologies are unchanged so that it is sufficient to consider the custodial versions of the diagrams in Figure 5.5.4.

For the two-mass-insertion $W$ diagram, there are two additional diagrams with custodial fermions: one with a $W_{L}$ and the other with a $W_{R}$ in the loop. The $P_{\text {LR }}$ symmetry enforces that the couplings are identical while the different boundary conditions modify the definitions of the internal propagators so that the only difference comes from the value of the
dimensionless integral in (5.36). The each diagram contributes a dimensionless integral $I=-0.2$, so that the $a$ coefficient is modified to

$$
\begin{equation*}
a_{\text {cust. }}=-0.15 \tag{5.41}
\end{equation*}
$$

Custodial diagrams do not contribute to the $b$ coefficient at leading order. For example, one might consider the diagram with a $Z$ loop where the $Z$ is replaced by a $Z_{X}$, the orthogonal mixture of the custodial $X$ and $W_{R}^{3}$ bosons. However, leptons carry no $X$ charge so that the effective coupling is only to right chiral modes. For $\mu_{R} \rightarrow e_{L} \gamma$, such a diagram would not be allowed. The leading custodial $b$ coefficient diagrams are an order of magnitude smaller than the minimal model diagrams and we shall ignore them in this paper.

### 5.5.4 CONSTRAINTS AND TENSION



Figure 5.5.5: Bounds on the anarchic Yukawa and кк scales in the minimal (a) and custodial (b) models from tree- and loop-level constraints, (5.12), (5.19), and (5.32). Each curve rules out the region to its left. The solid hyperbola is the appropriate tree-level bound. The thick solid straight line is the $b=o$ loop-level bound. The red dashed (blue dotted) curve is the loop-level bounds in the case where $b$ has the same (opposite) sign as $a$ and takes its $1 \sigma$ magnitude $|b|=|b|_{1 \sigma}=0.03$.

We can now estimate the upper bound on the anarchic Yukawa scale $Y_{*}$ in (5.32),

$$
\begin{equation*}
\left|a Y_{*}^{2}+b\right|\left(\frac{3 \mathrm{Tev}}{M_{\mathrm{KK}}}\right)^{2} \leq 0.015 \tag{5.32}
\end{equation*}
$$

First let us consider the scenario where the $b$ coefficient takes its statistical mean value, $b=0$, and $M_{\mathrm{KK}}=3 \mathrm{Tev}$. In this case the minimal model suffers a $\mathcal{O}(10)$ tension between the tree-level lower bound on $Y_{*}$ and the loop-level upper bound,

$$
\begin{equation*}
Y_{*}>4 \tag{5.42}
\end{equation*}
$$

$$
Y_{*}<0.5 .
$$

The custodial model slightly alleviates this tension,

$$
\begin{equation*}
Y_{*}>1.25 \quad Y_{*}<0.3 \tag{5.43}
\end{equation*}
$$

These discrepancies should be interpreted as an assessment on the extent to which the ${ }_{5} \mathrm{D}$ Yukawa matrices may be generically anarchic. The tension in the bounds above imply that for $M_{\mathrm{KK}}=3 \mathrm{Tev}$, one must accept some mild tuning in the relative sizes of the ${ }_{5} \mathrm{D}$ Yukawa matrix. This is shown by the hyperbola and solid line in Figure 5.5.5.

Alternately, one may ask that assuming totally anarchic Yukawas, what is the minimum value of $M_{\mathrm{KK}}$ for which the tension is alleviated? In the minimal model the tree- and loop-level bounds allow mutually consistent Yukawa scales for $M_{\mathrm{KK}}>6$ starting at $Y=1$. Similarly, for the custodial model the tree- and loop-level bounds allow consistent values for $M_{\mathrm{KK}}>4.75$ starting at $Y=0.5$.

Next one may consider the effect of the $b$ coefficient which is sensitive to the particular flavor structure of the anarchic ${ }_{5} \mathrm{D}$ Yukawa matrix relative to the choice of fermion bulk mass parameters. The $1 \sigma$ range of $b$ values for randomly generated anarchic matrices is $b \in(-0.03,0.03)$. Because this term is independent of $Y_{*}$, the value of $b$ can directly constraint the KK scale. For the $1 \sigma$ value this sets $M_{\mathrm{KK}} \gtrsim 4 \mathrm{Tev}$, as can be seen from the intersection of the red dashed lines and blue dotted lines with the horizontal axes in Figure 5.5.5.

The most interesting range for $b$, however, is the regime where it can cancel the $a$ term in term in (5.32). In such a regime the loop level bounds can deviate significantly from the prediction with only the $a$ coefficient, allowing one to relax the constraints on $Y_{*}$ and $M_{\mathrm{Kk}}$. However, because the $1 \sigma$ value of $b$ is an order of magnitude smaller than $a$ in the lepton sector, this region is disfavored by tree-level bounds. For broad model-building purposes, the key point is that the effect of the $b$ coefficient lines in Figure 5.5.5 represent the freedom to reduce (or enhance) the loop-level constraints through the misalignment of the anarchic Yukawas relative to the bulk masses. This misalignment comes from the choice of two independent spurions in flavor space and is not a tuning in the hierarchies of the Yukawa matrices.

In Figure 5.5 .5 the red dashed line shows the bound when $b$ takes its $1 \sigma$ magnitude and has an opposite sign from $a$; the cusp at $M_{\mathrm{KK}}=o$ represents the case where the $a$ and $b$ terms cancel. The blue dotted line shows the case where $b$ takes its $1 \sigma$ magnitude and has the same sign as $a$. What is important to note is that as one takes $|b|$ less than $|b|_{1 \sigma}$, these lines continuously converge upon the straight line corresponding to $b=0$ so that any combination of $Y_{*}$ and $M_{\mathrm{KK}}$ between the upper red dashed line and the blue dotted line can be plausibly achieved within the anarchic paradigm. Let us make the caveat that the above values are estimates at $\mathcal{O}(10 \%)$ accuracy. Specific results depend on model-dependent factors such as the extent to which the matrices are anarchic, the relative scale of the charged lepton and neutrino anarchic values, or extreme values for bulk masses. For completeness we provide analytic formulas for the leading and next-to-leading order diagrams in Appendix 5.C.

### 5.6 Power counting and finiteness

We now develop an intuitive understanding of the finiteness of this ${ }_{5} \mathrm{D}$ process, highlight some subtleties associated with the KK versus ${ }_{5} \mathrm{D}$ calculation of the loop diagrams ${ }^{2}$, and estimate the degree of divergence of the two-loop result. Our primary tool is naïve dimensional analysis, from which we may determine the superficial degree of divergence for a given ${ }_{5} \mathrm{D}$ diagram. Special care is given to the treatment of brane-localized fields and the translation between the manifestly 5 D and кк descriptions.

### 5.6.1 4D and 5 D theories of bulk fields

It is instructive to review key properties of $\mu \rightarrow e \gamma$ in the Standard Model. This amplitude was calculated by several authors [255, 257-260]. Two key features are relevant for finiteness:

1. Gauge invariance cancels the leading order divergences. The Ward identity requires $q_{\mu} \mathcal{M}^{\mu}=0$, where $\mathcal{M}^{\mu}$ is the amplitude with the photon polarization peeled off and $q_{\mu}$ is the photon momentum. This imposes a nontrivial $q$-dependence on $\mathcal{M}$ and reduces the superficial degree of divergence by one.

[^3]2. Lorentz invariance prohibits divergences which are odd in the loop momentum, $k$. In other words, $\int d^{4} k k / k^{2 n}=0$. After accounting for the Ward identity, the leading contribution to the dipole operator is odd in $k$ and thus must vanish. Specifically, one of the $k$ terms in a fermion propagator must be replaced by the fermion mass $m$.

Recall that the chiral structure of this magnetic operator requires an explicit internal mass insertion. In the Standard Model this is related to both gauge and Lorentz invariance so that it does not give an additional reduction in the superficial degree of divergence. Before accounting for these two features, naïve power counting in the loop integrals appears to suggest that the Standard Model amplitude is logarithmically divergent from diagrams with two internal fermions and a single internal boson. Instead, one finds that these protection mechanisms force the amplitude to go as $M^{-2}$ where $M$ is the characteristic loop momentum scale.

We can now extrapolate to the case of a 5 D theory. First suppose that the theory is modified to include a noncompact fifth dimension: then we could trivially carry our results from 4 D momentum space to ${ }_{5} \mathrm{D}$ except that there is an additional loop integral. By the previous analysis, this would give us an amplitude that goes as $M^{-1}$ and is thus finite. Such a theory is not phenomenologically feasible but accurately reproduces the UV behavior of a bulk process in a compact extra dimension so long as we consider the Uv limit where the loop momentum is much larger than the compactification and curvature scales. This is because the UV limit of the loop probes very small length scales that are insensitive to the compactification and any warping. This confirms the observation that $\mu \rightarrow e \gamma$ in Randall-Sundrum models with all fields (including the Higgs) in the bulk is UV-finite [241]. In the case where there are brane-localized fields, this heuristic picture is complicated since the $\mu \rightarrow e \gamma$ loop is intrinsically localized near the brane and is sensitive to its physics; we address this issue below.

### 5.6.2 BULK FIELDS IN THE 5 D FORMALISM

We may formalize this power counting in the mixed position/momentum space formalism. This also generalizes the above argument to theories on a compact interval. Each loop carries an integral $d^{4} k$ and so contributes +4 to the superficial degree of divergence. We can now consider how various features of particular diagrams can render this finite.

1. Gauge invariance $\left(p+p^{\prime}\right)$. As argued above and shown explicitly in (5.34), the Ward identity identifies the gauge invariant contribution to this process to be proportional to $\left(p+p^{\prime}\right)^{\mu}$, which reduces the overall degree of divergence by one.
2. Bulk Propagators. The bulk fermion propagators in the mixed position/momentum space formalism have a momentum dependence of the form $k / k \sim 1$ while the bulk boson propagators go like $1 / k$. This matches the power counting from summing a tower of kK modes. Note that this depends on $k=\sqrt{k^{2}}$ so that the Lorentz invariance in Section 5.6.1 for a noncompact extra dimension is no longer valid.
3. Bulk vertices $(d z)$, overall $z$-momentum conservation. Each bulk vertex carries an integral over the vertex position which brings down an inverse power of the momentum flowing through it. This can be seen from the form of the bulk propagators, which depend on $z$ in the dimensionless combination $k z$ up to overall warp factors. In the Wick-rotated UV limit, the integrands reduce to exponentials so that their integrals go like $1 / k$. In momentum space this suppression is manifested as the momentum-conserving $\delta$ function in the far UV limit where the loop momentum is much greater than the curvature scale.
An alternate and practical way to see the $1 / k$ scaling of an individual $d z$ integral comes from the Jacobian as one shifts to dimensionless integration variables,

$$
\begin{equation*}
y=k_{E} R^{\prime} \quad x=k_{E} z \tag{5.44}
\end{equation*}
$$

so that $y \in[0, \infty]$ plays the role of the loop integrand and $x \in\left[y R / R^{\prime}, y\right]$ plays the role of the integral over the interval extra dimension. These are the natural objects that appear as arguments in the Bessel functions contained in the bulk field propagators, as demonstrated in Appendix 4.4.3. In these variables each $d x$ brings down a factor of $1 / y$ from the Jacobian of the integration measure. These variables are natural choices because they relate distance intervals in the extra dimension to the scales that are being probed by the loop process. The physically relevant distance scales are precisely these ratios.
4. Overall $z$-momentum conservation. We must make one correction to the bulk vertex suppression due to overall $z$-momentum conservation. This is most easily seen in momentum space where one $\delta$-function from the bulk vertices conserves overall external momentum in the extra dimension and hence does not affect the loop momentum. In mixed position/momentum space this is manifested as one $d z$ integral bringing down an inverse power of only external momenta without any dependence on the loop momentum. We review this in Appendix 5.D, where we discuss the passage between position and momentum space. The overall $z$-momentum conserving $\delta$-function thus adds one unit to the superficial degree of divergence to account for the previous overcounting of $d z \sim 1 / k$ suppressions.
5. Derivative coupling. The photon couples to charged bosons through a derivative coupling which is proportional to the momentum flowing through the vertex. This gives a contribution that is linear in the loop momentum, $k^{\mu}$.
6. Chirality: mass insertion, equation of motion. To obtain the correct chiral structure for a dipole operator, each diagram must either have an explicit fermion mass insertion or must make use of the external fermion equation of motion (EOM). For a bulk Higgs field, each fermion mass insertion carries a $d z$ integral which goes like $1 / k$. As described in Section 5.5, the use of the EOM corresponds to an explicit external mass insertion. Thus fermion chirality reduces the degree of divergence by one unit.

We may now straightforwardly count the powers of the loop momentum to determine the superficial degree of divergence for the case where the photon is emitted from a fermion (one boson and two fermions in the loop) or a boson (two bosons and one fermion in the loop). The latter case differs from the former in the number of boson propagators and the factor of $k^{\mu}$ in the photon Feynman rule.

|  | Neutral | Charged |
| ---: | :--- | :--- |
| Boson | Boson |  |
| Loop integral $\left(d^{4} k\right)$ | +4 | +4 |
| Gauge invariance $\left(p+p^{\prime}\right)$ | -1 | -1 |
| Bulk fermion propagators | 0 | 0 |
| Bulk boson propagator | -1 | -2 |
| Bulk vertices $(d z)$ | -3 | -3 |
| Overall $z$-momentum | +1 | +1 |
| Derivative coupling | 0 | +1 |
| Mass insertion/EOM | -1 | -1 |
| Total degree of divergence | -1 | -1 |

The $W H^{ \pm}$diagram in Figure 5.5 .3 is a special case since it has neither a derivative coupling nor an additional chirality flip, but these combine to make no net change to the superficial degree of divergence. We confirm our counting in Section 5.6.1 that the superficial degree of divergence for universal extra dimension where all fields propagate in the bulk is -1 so that the flavor-changing penguin is manifestly finite.

Before moving on to the case of a brane-localized boson, let us remark that this bulk counting may straightforwardly be generalized to the case of a bulk boson with brane-localized mass insertions. To do this, we note that the brane-localized mass insertion breaks momentum conservation in the $z$ direction and this no longer contributes +1 to the degree of divergence. On the other hand, each mass insertion no longer contributes -1 from the $d z$ integral so that the changes in the "overall $z$-momentum" and "mass insertion/EOM" counting cancel out. We find that diagrams with a bulk gauge boson and brane-localized mass insertions have the same superficial degree of divergence as the lowest order diagrams in a bulk mass insertion expansion.

### 5.6.3 BULK FIELDS IN THE KK FORMALISM

All of the power counting from the 5 D position/momentum space formalism carries over directly to the Kk formalism with powers of $m_{\mathrm{KK}}$ treated as powers of $k$. The position/momentum space propagators already carry the information about the

Figure 5.6.1: One-mass-insertion neutral scalar diagrams. The leading order $k$-dependence of each diagram cancels when the two are summed together.
entire KK tower as well as the profiles of each KK mode. Explicitly converting from a ${ }_{5} \mathrm{D}$ propagator to a KK reduction,

$$
\begin{equation*}
\Delta_{5 D}\left(k, z, z^{\prime}\right)=\sum_{n} f^{(n)}(z) \Delta_{\mathrm{KK}}^{(n)}(k) f^{(n)}\left(z^{\prime}\right) \tag{5.45}
\end{equation*}
$$

where $f^{(n)}$ is the profile of the $n^{\text {th }}$ KK mode. The sum over KK modes is already accounted for in the 5 D propagator; for example, for a boson $\Delta_{\mathrm{KK}}^{(n)} \sim 1 / k^{2}$ while $\Delta_{5 D} \sim 1 / k$. The vertices between KK modes are given by the $d z$ integral over each profile, which reproduces the same counting since each profile depends on $z$ as a function of $m_{\mathrm{KK}}^{(n)} z$. Conservation of $z$-momentum is replaced by conservation of KK number in the UV limit of large KK number.

Indeed, it is almost tautological that the KK and position/momentum space formalisms should match for bulk fields since the process of KK reducing a ${ }_{5} \mathrm{D}$ theory implicitly passes through the position/momentum space construction. This will become slightly more nontrivial in the case of brane-localized fields. We shall postpone a discussion of mixing between KK states until Section 5.6.5.

### 5.6.4 BRANE FIELDS IN THE 5 D FORMALISM

The power counting above appears to fail for loops containing a brane-localized Higgs field. The brane-localized Higgs propagator goes like $1 / k^{2}$ rather than $1 / k$ for the bulk propagator, but this comes at the cost of two vertices that must also be brane-localized, thus negating the suppression from the $d z$ integrals. The charged Higgs has two brane-localized Higgs propagators, but loses a third $d z$ integral from the brane-localized photon emission. Finally, there are no additional contributions from the brane-localized fermion mass insertions nor are there any corrections from the conservation of overall $z$-momentum since it is manifestly violated by the brane-localized vertices (see Appendix 5.D for a detailed discussion). In the absence of any additional brane effects, both types of loops would be logarithmically divergent, as discussed in [241].

Fortunately, two such brane effects appear. First consider the two neutral Higgs diagrams in Figure 5.5.1. The diagram with no mass insertion requires the use of an external fermion equation of motion which still reduces the superficial degree of divergence by one so that it is finite. The diagram with a single mass insertion is finite in the Standard Model due to a cancellation between the Higgs and neutral Goldstone diagrams, as discussed in Section 5.5. More generally, even for a single type of brane-localized field, there is a cancellation between diagrams in Figure 5.6.1 where the photon is emitted before and after the mass insertion. This can be seen by writing down the Dirac structure coming from the fermion propagators to leading order in the loop momentum,

$$
\begin{align*}
& \mathcal{M}_{a} \sim k \gamma^{\mu} k k-k \gamma^{\mu} k k=k^{2}\left(k \gamma^{\mu}-\gamma^{\mu} k\right)  \tag{5.46}\\
& \mathcal{M}_{b} \sim k k \gamma^{\mu} k-k k \gamma^{\mu} k=k^{2}\left(\gamma^{\mu} k-k \gamma^{\mu}\right) \tag{5.47}
\end{align*}
$$

The terms with three factors of $k$ are contributions where "correct-chirality" fermions propagate into the bulk, while the terms with only one $k$ are contributions where "wrong-chirality" fermions propagate into the bulk. The structure of the latter terms comes from the $\gamma^{5} \partial_{z}$ term in the Dirac operator. The structures above multiply scalar functions which, to leading order in $k$, are identical for each term. From the Clifford algebra it is clear that (5.46) and (5.47) cancel so that the contribution that is nonvanishing in the UV must be next-to-leading order in the loop momentum. In Appendix 5.E this cancellation is connected to the chiral boundary conditions on the brane and is demonstrated with explicit flat-space fermion propagators. We thus find that the brane-localized neutral Higgs diagrams have an additional - 1 contribution to the superficial degree of divergence.

Next we consider the charged Goldstone diagrams. These diagrams have an additional momentum suppression coming from a positive power of the charged Goldstone mass $M_{W}^{2}$ appearing in the numerator due to a cancellation within each diagram. In fact, we have already seen in Section 5.5 .1 how such a cancellation appears. For the single-mass-insertion charged

Goldstone diagram in Figure 5.5.2, we saw in (5.69) that the form of the 4 D scalar propagators and the photon-scalar vertex cancels the leading-order loop momentum term multiplying the required $\left(p+p^{\prime}\right)^{\mu}$. The cancellation introduces an additional factor of $M_{W}^{2} /\left(k^{2}-M_{W}^{2}\right)$ so that the superficial degree of divergence is reduced by two. Note that the position/momentum space propagators for a bulk Higgs have a different form than that of the 4 D brane-localized Higgs and do not display the same cancellation. In the KK picture this is the observation that the cancellation in (5.69) takes the form $M_{\mathrm{KK}}^{2} /\left(k^{2}-M_{\mathrm{KK}}^{2}\right)$, which does not provide any suppression for heavy kK Higgs modes.

Finally, the diagrams where the photon emission vertex mixes the $W$ and brane-localized charged Goldstone are special cases. The photon vertex carries neither a $d z$ integral nor a $k^{\mu}$ Feynman rule and hence makes no net contribution to the degree of divergence. A straightforward counting including the brane-localized Goldstone, bulk $W$, and the single bulk vertex thus gives a degree of divergence of $-\mathbf{1}$.

We summarize the power counting for a brane-localized Higgs as follows:

|  | Neutral <br> boson | Charged <br> boson | $W-H^{ \pm}$ <br> mixing |
| ---: | :--- | :--- | :--- |
| Loop integral $\left(d^{4} k\right)$ | +4 | +4 | +4 |
| Gauge invariance $\left(p+p^{\prime}\right)$ | -1 | -1 | -1 |
| Brane boson propagators | -2 | -4 | -2 |
| Bulk boson propagator | $\circ$ | 0 | -1 |
| Bulk vertices $(d z)$ | -1 | $\circ$ | -1 |
| Photon Feynman rule | $\circ$ | +1 | $\circ$ |
| Brane chiral cancellation | -1 | 0 | $\circ$ |
| Brane $M_{W}^{2}$ cancellation | $\circ$ | -2 | $\circ$ |
| Total degree of divergence | -1 | -2 | -1 |

It may seem odd that the brane-localized charged Higgs loop has a different superficial degree of divergence than the other ${ }_{5} \mathrm{D}$ cases, which heretofore have all been -1 . This, however, should not be surprising since the case of a brane-localized Higgs is manifestly different from the universal extra dimension scenario. It is useful to think of the brane-localized Higgs as a limiting form of a кк reduction where the zero mode profile is sharply peaked on the IR brane. The difference between the bulk and brane-localized scenarios corresponds to whether or not one includes the rest of the kr tower.

### 5.6.5 BRANE FIELDS IN THE KK FORMALISM

Let us now see how the above power counting for the brane-localized Higgs manifests itself in the Kaluza-Klein picture [241]. Observe that this power counting for both the $W-H^{ \pm}$and the charged boson loops are trivially identical to the ${ }_{5} \mathrm{D}$ case due to the arguments in Section 5.6.3. For example, the $M_{W}^{2}$ cancellation is independent of how one treats the bulk fields. The neutral Higgs loop, however, is somewhat subtle since the "chiral cancellation" is not immediately obvious in the кк picture.

We work in the mass basis where the fermion line only carries a single кK sum (not independent sums for each mass insertion) and the zero mode photon coupling preserves KK number due to the flat $A^{(\circ)}$ profile. In this basis the internal fermion line carries one KK sum and it is sufficient to show that for a single arbitrarily large KK mode the process scales like $1 / M_{\mathrm{KK}}^{2}$. The four-dimensional power counting in Section 5.6 .1 appears to give precisely this, except that Lorentz invariance no longer removes a degree of divergence. This is because this suppression came from the replacement of a loop momentum $k$ by the fermion mass $m$. For an arbitrarily large кк mode, the fermion mass itself is the loop momentum scale and so does not reduce the degree of divergence. In the absence of any additional suppression coming from the mixing of кK modes, it would appear that the KK power counting only goes like $1 / M_{\mathrm{KK}}$ so that the sum over KK modes should be logarithmically divergent, in contradiction with the power counting for the same process in the ${ }_{5} \mathrm{D}$ formalism.

We shall now show that the pair of Yukawa couplings for the neutral Higgs also carries the expected $1 / k$ factor that renders these diagrams finite and allows the superficial degrees of divergence to match between the KK and ${ }_{5} \mathrm{D}$ counting. It is instructive to begin by defining a basis for the zero and first кк modes in the weak (chiral) basis. We denote left (right) chiral fields of кK number $a$ by $\chi_{L, R}^{(a)}\left(\psi_{L, R}^{(a)}\right)$ where the $L, R$ refers to $\operatorname{SU}(2)_{L}$ doublets and singlets respectively. We can arrange these


Figure 5.6.2: The fermion line in the mass basis for diagrams with an internal KK mode $(J>3)$. For simplicity we do not show the internal photon insertion.
into vectors

$$
\begin{equation*}
\chi=\left(\chi_{L_{i}}^{(\circ)}, \chi_{R_{i}}^{(1)}, \chi_{L_{i}}^{(1)}\right) \quad \psi=\left(\psi_{R_{i}}^{(\circ)}, \psi_{R_{i}}^{(1)}, \psi_{L_{i}}^{(1)}\right) \tag{5.48}
\end{equation*}
$$

where $i$ runs over flavors. It is helpful to introduce a single index $J=3 a+i$ where $i=1,2,3$ according to flavor and $a=0,1,2$ according to KK mode (writing $a=2$ to mean the first KK mode with opposite chirality as the zero mode). Thus the external muon and electron are $\chi_{2}$ and $\psi_{1}$ respectively, while an internal KK mode takes the form $\chi_{J}$ or $\psi_{J}$ with $J>3$. This convention in (5.48) differs from that typically used in the literature (e.g. [241]) in the order of the last two elements of $\psi$. This basis is useful because the KK terms are already diagonal in the mass matrix ( $\psi M \chi+$ h.c. $)$,

$$
M=\left(\begin{array}{ccc}
m^{11} & 0 & m^{13}  \tag{5.49}\\
m^{21} & M_{\mathrm{KK}, 1} & m^{23} \\
\circ & \circ & M_{\mathrm{KK}, 2}
\end{array}\right)
$$

where each element is a $3 \times 3$ block in flavor space and we have written

$$
\begin{equation*}
m=\frac{v}{\sqrt{2}} f_{R_{i}}^{(a)} Y_{*} f_{L_{j}}^{(b)} \ll M_{\mathrm{KK}} \tag{5.50}
\end{equation*}
$$

with indices as appropriate and $M_{\mathrm{KK}}$ diagonal. Let us define $\varepsilon=v / M_{\mathrm{KK}}$ to parameterize the hierarchies in the mass matrix. For a bulk Higgs, these terms are replaced by overlap integrals and the $M_{32}$ block is nonzero, though this does not affect our argument. Note that $M_{\mathrm{KK}, 1}$ and $M_{\mathrm{KK}, 2}$ are typically not degenerate due to $\mathcal{O}(m)$ differences in the doublet and singlet bulk masses. In the gauge eigenbasis the Yukawa matrix is given by

$$
y=\left.\frac{\sqrt{2}}{v} M\right|_{M_{\mathrm{KK}}=0} \sim\left(\begin{array}{lll}
1 & 0 & 1  \tag{5.51}\\
1 & 0 & 1 \\
0 & 0 & 0
\end{array}\right)
$$

where we have assumed $f_{L}, f_{R}, Y_{*} \sim \mathcal{O}(1)$ for simplicity since the hierarchies in the $f^{(\circ)} s$ do not affect our argument. The 1 elements thus refer to blocks of the same order of magnitude that are not generically diagonal. The o blocks must vanish by gauge invariance and chirality.

We now rotate the fields in (5.48) to diagonalize the mass matrix (5.49); we indicate this by a caret, e.g. $\hat{\chi}$. In this basis the Yukawa matrix is also rotated $y \rightarrow \hat{y}$. The fermion line for this process is shown in Figure 5.6.2; the Yukawa dependence of the amplitude is

$$
\begin{equation*}
\mathcal{M} \sim \hat{y}_{1} \hat{y}_{J_{2}} \tag{5.52}
\end{equation*}
$$

First let us note that in the unrealistic case where $\hat{y}=y$, one of the Yukawa factors in (5.52) is identically zero for all internal KK modes, $J>3$. One might then expect that the mass rotation would induce a mixing of the zero modes with the KK modes
that induces $\mathcal{O}(\varepsilon)$ blocks into the Yukawa matrix,

$$
\hat{y} \stackrel{?}{\sim}\left(\begin{array}{ccc}
1 & \varepsilon & 1  \tag{5.53}\\
1 & \cdots & \ldots \\
\varepsilon & \cdots & \ldots
\end{array}\right) .
$$

If this were the case then the product $\hat{y}_{1} \hat{y}_{J_{2}}$ would not vanish, but would be proportional to $\varepsilon \sim 1 / M_{\mathrm{KK}}$, which is precisely the KK dependence that we wanted to show. While this intuition is correct and captures the correct physics, the actual Yukawa matrix in the mass basis has the structure (c.f. (67) in [241])

$$
\hat{y} \sim\left(\begin{array}{ccc}
1 & 1+\varepsilon & -1+\varepsilon  \tag{5.54}\\
1+\varepsilon & \cdots & \cdots \\
1-\varepsilon & \cdots & \cdots
\end{array}\right)
$$

The new $\mathcal{O}(1)$ elements come from the large rotations induced by the $m^{21}$ and $m^{13}$ blocks. These factors cancel out so that we still have the desired $\hat{y}_{1} \hat{y}_{j_{2}} \sim \varepsilon$ relation. Physically this is because these $\mathcal{O}(1)$ factors come from the "large" rotation from chiral zero modes to light Dirac SM fermions. Thus they represent the "wrong-chirality" coupling of the external states induced by the usual mixing of Weyl states from a Dirac mass. This does not include the mixing with the heavy kK modes, which indeed carries the above $\varepsilon$ factors so that the final result is

$$
\begin{equation*}
\hat{y}_{1} \hat{y}_{J_{2}} \sim \varepsilon \sim \frac{1}{M_{\mathrm{KK}}}, \tag{5.55}
\end{equation*}
$$

giving the correct -1 contribution to the superficial degree of divergence for the neutral Higgs diagrams to render them manifestly finite.

A few remarks are in order. First let us emphasize again that promoting the Higgs to a bulk field makes the 3-2 block of the $y$ matrix nonzero. This does not affect the above argument so that the кк decomposition confirms the observation that the amplitude with a bulk Higgs is also finite [241]. Of course, for a bulk Higgs the power counting in Section 5.6.2 gives a more direct check of finiteness. Next, note that without arguing the nature of the zeros in the gauge basis Yukawa matrix or the physical nature of the $\varepsilon$ mixing with KK modes, it may appear that the $1 / M_{\mathrm{KK}}$ dependence of $\hat{y}_{\mathrm{J}} \hat{y}_{J_{2}}$ requires a "miraculous" fine tuning between the matrix elements of ( 5.54 ). Our discussion highlights the physical nature of this cancellation as the mixing with heavy states that is unaffected by the $\mathcal{O}(1)$ mixing of light chiral states.

Finally, let us point out that the above arguments are valid for the neutral Higgs diagram where $y=y_{E}$, the charged lepton Yukawa matrix. The analogous charged Higgs diagram contains neutrino Yukawa matrices $y_{N}$ so that there is no additional $1 / k$ from mixing.

### 5.6.6 MATCHING KK AND LOOP CUTOFFS

There is one particularly delicate point in the single-mass-insertion neutral Higgs loop in the KK reduction that is worth pointing out because it highlights the relation between the KK scales $M_{\mathrm{KK}}^{(n)}$ and the ${ }_{5} \mathrm{D}$ loop momentum. To go from the ${ }_{5} \mathrm{D}$ to the 4 D formalism we replace our position/momentum space propagators with a sum of Kaluza-Klein propagators,

$$
\begin{equation*}
\Delta_{\mathrm{sD}}\left(k, z, z^{\prime}\right)=\sum_{n=\mathrm{o}}^{N} f^{(n)}(z) \frac{k+M_{n}}{k^{2}-M_{n}^{2}} f^{(n)}\left(z^{\prime}\right) . \tag{5.56}
\end{equation*}
$$

The full ${ }_{5} \mathrm{D}$ propagator is exactly reproduced by summing the infinite tower of states, $N \rightarrow \infty$. More practically, the ${ }_{5} \mathrm{D}$ propagator with characteristic momentum scale $k$ is well-approximated by at least summing up to modes with mass $M_{n} \approx k$. Modes that are much heavier than this decouple and do not give an appreciable contribution. Thus, when calculating low-energy, tree-level observables in 5 D theories, it is sufficient to consider only the effect of the first few kr modes. On the other hand, this means that one must be careful in loop diagrams where internal lines probe the uv structure of the theory. In particular, significant contributions from internal propagators near the threshold $M_{n} \approx k$ would be missed if one sums only to a finite KK number while taking the loop integral to infinity. This is again a concrete manifestation of the remarks below (5.44)
that the length scales probed by a process depend on the characteristic momentum scale of the process.
Indeed, a Kaluza-Klein decomposition for a single neutral Higgs yields

$$
\begin{equation*}
|\mathcal{M}|_{\left(p+p^{\prime}\right)^{\mu}}=\frac{g v}{16 \pi^{2}} f_{\mu} f_{-e} \bar{u}_{e}\left(p+p^{\prime}\right)^{\mu} u_{\mu} \times \frac{1}{M^{2}}\left[c_{0}+c_{1}\left(\frac{v}{M}\right)^{2}+\mathcal{O}\left(\frac{v}{M}\right)^{3}\right] \tag{5.57}
\end{equation*}
$$

for some characteristic кK scale $M \approx M_{\mathrm{KK}}$ and dimensionless coefficients $c_{i}$ that include a loop integral and KK sums. In order to match the ${ }_{5} \mathrm{D}$ calculation detailed above, we shall work in the mass insertion approximation so that there are now two kK sums in each coefficient. The leading $c_{0}$ term is especially sensitive to the internal loop momentum cutoff $\Lambda$ relative to the internal kк masses,

$$
\begin{equation*}
c_{\circ}=-\lambda^{2} \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{\lambda^{2}\left(n^{2}+m^{2}\right)+2 n^{2} m^{2}}{4\left(n^{2}+\lambda^{2}\right)^{2}\left(m^{2}+\lambda^{2}\right)^{2}} \equiv-\frac{1}{\lambda^{2}} \sum_{n=1}^{N} \sum_{m=1}^{N} \hat{c}_{0}(n, m), \tag{5.58}
\end{equation*}
$$

where we have written mass scales in terms of dimensionless numbers with respect to the mass of the first kK mode: $M_{n} \sim n M_{\mathrm{KK}}$ and $\Lambda \sim \lambda M_{\mathrm{KK}}$. It is instructive to consider the limiting behavior of each term $\hat{c}(n, m)$ for different ratios of the кк scale (assume $n=m$ ) to the cutoff scale $\lambda$ :

$$
\begin{array}{ll}
\hat{c}_{0}(n, n) \longrightarrow\left(\frac{n}{\lambda}\right)^{2} & \text { for } n \ll \lambda \\
\hat{c}_{0}(n, n) \longrightarrow\left(\frac{n}{\lambda}\right)^{\circ} & \text { for } n \approx \lambda \\
\hat{c}_{0}(n, n) \longrightarrow\left(\frac{\lambda}{n}\right)^{4} & \text { for } n \gg \lambda \tag{5.61}
\end{array}
$$

We see that the dominant contribution comes from modes whose KK scale is near the loop momentum cutoff while the other modes are suppressed by powers of the ratio of scales. In particular, if one calculates the loop for any internal mode of finite кK number while taking the loop cutoff to infinity, then the $c_{0}$ contribution vanishes because the $n \approx \lambda$ contributions are dropped. From this one would incorrectly conclude that the leading order term is $c_{1}$ and that the amplitude is orders of magnitude smaller than our 5 D calculation. Thus one cannot consistently take the 4 D momentum to infinity without simultaneously taking the ${ }_{5} \mathrm{D}$ momentum (i.e. KK number) to infinity. Or, in other words, one must always be careful to include the nonzero contribution from modes with $n \approx \lambda$. One can see from power counting on the right-hand side of ( 5.58 ) that so long as the highest kK number $N$ and the dimensionless loop cutoff $\lambda$ are matched, $c_{\circ}$ gives a nonzero contribution even in the $\lambda \rightarrow \infty$ limit.

This might seem to suggest UV sensitivity or a nondecoupling effect ${ }^{3}$. However, we have already shown that $\mu \rightarrow e \gamma$ is uv-finite in ${ }_{5}$ D. Indeed, our previous arguments about uv finiteness tell us that the overall contribution to the amplitude from large loop momenta (and hence high KK numbers) must become negligible; we see this explicitly in the UV limit of ( 5.58 ). The key statement is that the KK scale and the Uv cutoff of the loop integral must be matched, $N \gtrsim \lambda$. This can be understood as maintaining momentum-space rotational invariance in the microscopic limit of the effective theory (much smaller than the curvature scale). Further, the prescription that one must match our KK and loop cutoffs $N \gtrsim \lambda$ is simply the statement that we must include all the available modes of our effective theory. It does not mean that one must sum a large number of modes in an effective KK theory. In particular, one is free to perform the loop integrals with a low cutoff $\Lambda \sim M_{\mathrm{KK}}$ so that only a single KK mode runs in the loop. This result gives a nonzero value for $c_{0}$ which matches the order of magnitude of the full ${ }_{5} \mathrm{D}$ calculation and hence confirms the decoupling of heavy modes.

### 5.6.7 Two-LOOP STRUCTURE

As with any ${ }_{5}$ D effective theory, the Rs framework is not UV complete. This nonrenormalizability means that it is possible for processes to be cutoff-sensitive. Since an effective $\mu \rightarrow e \gamma$ operator (in the sense of Appendix 5.A) cannot be written at tree level, there can be no tree-level counter term and so we expect the process to be finite at one-loop order, as we have indeed

[^4]

Figure 5.6.3: Yin-Yang and double rainbow topologies of two-loop diagrams. The dotted line represents either a gauge or Higgs boson. We have omitted the photon emission and an odd number of mass insertions.
confirmed above. In principle, however, higher loops need not be finite.
The one-loop analysis presented thus far assumes that we may work in a regime where the relevant couplings are perturbative. In other words, we have assumed that higher-loop diagrams are negligible due to an additional $g^{2} / 16 \pi^{2}$ suppression, where $g$ is a generic internal coupling. This naturally depends on the divergence structure of the higher-loop diagrams. If such diagrams are power-law divergent then it is possible to lose this window of perturbativity even for relatively low UV cutoff $\Lambda \sim M_{\mathrm{KK}}$. We have shown that even though naïve dimensional analysis suggests that the $\mu \rightarrow e \gamma$ amplitude should be linearly divergent in 5 D , the one-loop amplitudes are manifestly finite.

Here we argue that the two-loop diagrams should be no more than logarithmically divergent for bulk bosons so that there is an appreciable region of parameter space where the process is indeed perturbative and the one-loop analysis can be trusted. This case is also addressed in [241]. The relevant topologies are shown in Figure 5.6.3. In this case, the power counting arguments that we have developed in this section carry over directly to the two-loop diagrams:

| Loop integrals $\left(d^{4} k\right)$ | +8 |
| ---: | ---: |
| Gauge invariance $\left(p+p^{\prime}\right)$ | -1 |
| Bulk boson propagators | -2 |
| Bulk vertices $(d z)$ | -5 |
| Total degree of divergence | $\circ$ |

We find that the superficial degree of divergence is zero so that the process is, at worst, logarithmically divergent.
The power counting for the brane-localized fields is more subtle, as we saw above. Naïve power counting suggests that the two-loop, brane-localized diagrams are no more than quadratically divergent. However, just as additional cancellations manifested themselves in the one-loop, brane-localized case, it may not be unreasonable to expect that those cancellations might carry over to the two-loop diagrams. Checking the existence of such cancellations requires much more work we leave this to a full two-loop calculation.

### 5.7 Outlook and Conclusion

We have presented a detailed calculation of the $\mu \rightarrow e \gamma$ amplitude in a warped rs model using the mixed position/momentum representation of 5 D propagators and the mass insertion approximation, where we have assumed that the localized Higgs vev is much smaller than the KK masses in the theory. Our calculation reveals potential sensitivity to the specific flavor structure of the anarchic Yukawa matrices since this affects the relative signs of coefficients that may interfere constructively or destructively. We thus find that while generic flavor bounds can be placed on the lepton sector of rs models, one can systematically adjust the structure of the $Y_{E}$ and $Y_{N}$ matrices to alleviate the bounds while simultaneously maintaining anarchy. In other words, there are regions of parameter space which can improve agreement with experimental constraints without fine tuning. Conversely, one may generate anarchic flavor structures which—for a given KK scale—cannot satisfy the $\mu \rightarrow e \gamma$ constraints for any value of the anarchic scale $Y_{*}$. Over a range of randomly generated anarchic matrices, the parameter controlling this $Y_{*}$-independent structure has a mean value of zero and a $1 \sigma$ value which can push the KK scale to 4 Tev.

It is interesting to consider the case where $M_{\mathrm{KK}}=3 \mathrm{Tev}$ where KK excitations are accessible to the LHC. When the $b$ coefficient takes its statistical mean value, $b=0$, the minimal model suffers a $\mathcal{O}(10)$ tension between the tree-level lower bound on $Y_{*}$ and the loop-level upper bound,

$$
\begin{equation*}
Y_{*}>4 \tag{5.62}
\end{equation*}
$$

$$
Y_{*}<0.5
$$

This tension is slightly alleviated in the custodial model,

$$
\begin{equation*}
Y_{*}>1.25 \quad Y_{*}<0.3 \tag{5.63}
\end{equation*}
$$

Thus for $M_{K K}=3 \mathrm{TeV}$ one must one must accept some mild tuning in the relative sizes of the ${ }_{5} \mathrm{D}$ Yukawa matrix. Figure 5.5.4 summarizes the bounds including the effect of the $b$ coefficient.

On the other hand, we know that anarchic models generically lead to small mixing angles (see however [245]). These fit the observed quark mixing angles well but are in stark contrast with the lepton sector where neutrino mixing angles are large, $\mathcal{O}(1)$, and point to additional flavor structure in the lepton sector. For example in [242] a bulk $A_{4}$ non-Abelian discrete symmetry is imposed on the lepton sector. This leads to a successful explanation of both the lepton mass hierarchy and the neutrino mixing angles (see also [261]) while all tree-level lepton number-violating couplings are absent, so the only bound comes from the $\mu \rightarrow e \gamma$ amplitude.

We have also provided different arguments for the one-loop finiteness of this amplitude which we verified explicitly through calculations. We have illuminated how to correctly perform the power counting to determine the degree of divergence from both the 5 D and 4 D formalisms. The transition between these two pictures is instructive and we have demonstrated the importance of matching the number of KK modes in a 4 D EFT to any 4 D momentum cutoff in loop diagrams. The power-counting analysis can be particularly subtle for the case of brane-localized fields and we have shown how one-loop finiteness can be made manifest. Finally, we have addressed the existence of a perturbative regime in which these one-loop results give the leading result by arguing that the bulk field two-loop diagrams should be at most logarithmically divergent and that it is at least feasible that the brane-localized two-loop diagrams may follow this power counting.

In addition to $\mu \rightarrow e \gamma$, there is an analogous flavor-changing dipole-mediated process in the quark sector, $b \rightarrow s \gamma$ with additional gluon diagrams with the same topology as the $Z$ diagrams described here. Because of operator mixing, connecting the $b \rightarrow s \gamma$ amplitude to QCD observables requires the Wilson coefficients for both the photon penguin $C_{7 \gamma}$ and the gluon penguin $C_{8 g}$. A discussion can be found in [226], though there it was expected that these penguins would be logarithmically divergent. Further, it would be interesting to note whether the experimental bounds on this process admits the small- $Y_{*}$ region of parameter space where the $b$ term may be of the same order as the $a$ term. We leave the explicit evaluation of the $b \rightarrow s \gamma$ amplitude in warped space to future work [3].

## 5.A Matching 5D amplitudes to 4D EFTs

The standard procedure for comparing the loop-level effects of new physics on low-energy observables is to work with a low-energy effective field theory in which the UV physics contributes to the Wilson coefficient of an appropriate local effective operator by matching the amplitudes of full and effective theories. In this appendix we briefly remark on the matching of 5 D mixed position/momentum space amplitudes to 4 D effective field theories, where some subtleties arise from notions of locality in the extra dimension.

The only requirement on the 5 D amplitudes that must match to the 4 D effective operator is that they are local in the four Minkowski directions. There is no requirement that the operators should be local in the fifth dimension since this dimension is integrated over to obtain the 4 D operator. Thus the 5 D amplitude should be calculated with independent external field positions in the extra dimension. Heuristically, one can write this amplitude as a nonlocal 5 D operator

$$
\begin{equation*}
\mathcal{O}_{5}\left(x, z_{H}, z_{L}, z_{E}, z_{A}\right)=H_{5}\left(x, z_{H}\right) \cdot \bar{L}_{5}\left(x, z_{L}\right) \sigma^{M N} E_{5}\left(x, z_{E}\right) F_{M N}\left(x, z_{A}\right) \tag{5.64}
\end{equation*}
$$

Note that this object has mass dimension 8. In the ${ }_{5} \mathrm{D}$ amplitude the fields are replaced by external state wavefunctions and this is multiplied by a "nonlocal coefficient" $c_{5}\left(z_{H}, z_{L}, z_{E}, z_{A}\right)$ which includes integrals over internal vertices and loop momenta as well as the mixed position/momentum space propagators to the external legs. To match with the low-energy 4 D
operator we impose that the external states are zero modes and decompose them into 4 D zero-mode fields multiplied by a ${ }_{5} \mathrm{D}$ profile $f(z)$ of mass dimension $1 / 2$,

$$
\begin{equation*}
\Phi_{5}(x, z) \rightarrow \Phi^{(\circ)}(x) f^{(\circ)}(z) \tag{5.65}
\end{equation*}
$$

Further, we must integrate over each external field's $z$-position. Thus the 4 D Wilson coefficient and operator are given by

$$
\begin{equation*}
c_{4} \mathcal{O}_{4}(x)=\int\left[\prod_{i} d z_{i}\right] c_{5}\left(z_{H}, z_{E}, z_{L}, z_{A}\right) f_{H}^{(\circ)}\left(z_{H}\right) f_{E}^{(\rho)}\left(z_{E}\right) f_{L}^{(\circ)}\left(z_{L}\right) f_{A}^{(\rho)}\left(z_{A}\right) H \cdot \bar{L} \sigma^{\mu \nu} E F_{\mu \nu} \tag{5.66}
\end{equation*}
$$

where the fields on the right-hand side are all zero modes evaluated at the local 4 D point $x$. Note that these indeed have the correct 4 D mass dimensions, $\left[\mathcal{O}_{4}\right]=6$ and $[c]=-2$.

Finally, let us remark that we have treated the ${ }_{5} \mathrm{D}$ profiles completely generally. In particular, there are no ambiguities associated with whether the Higgs field propagates in the bulk or is confined to the brane. One can take the Higgs profile to be brane-localized,

$$
\begin{equation*}
f_{H}\left(z_{H}\right) \sim \sqrt{R^{\prime}} \delta\left(z-R^{\prime}\right) \tag{5.67}
\end{equation*}
$$

where the prefactor is required by the dimension of the profiles. With such a profile (or any limiting form thereof) the passage from 5 D to 4 D according to the procedure above gives the correct matching for brane-localized fields.

## 5.B Estimating the size of each diagram

As depicted in Figs. 5.5.1-5.5.3, there are a large number of diagrams contributing to the $a$ and $b$ coefficients even when only considering the leading terms in a mass-insertion expansion. Fortunately, many of these diagrams are naturally suppressed and the dominant contribution to each coefficient is given by the two diagrams shown in Figure 5.5.4. This can be verified explicitly by using the analytic expressions for the leading and next-to-leading diagrams are given in Appendix 5.C. In this appendix we provide some heuristic guidelines for estimating the relative sizes of these diagrams.

## 5.B.1 Relative sizes of couplings

First note that after factoring out terms in the effective operator in ( 5.21 ), Yukawa couplings give order one contributions while gauge couplings give an enhancement of $g_{S M}^{2} \ln R^{\prime} / R$, where $g_{S M}$ is the appropriate Standard Model coupling. This gives a factor of $\sim 5(7)$ enhancement in diagrams with a $W$ over those with a $Z(H)$.

## 5.B. 2 SUPPRESSION MECHANISMS IN DIAGRAMS

Next one can count estimate suppressions to each diagram coming from the following factors
A. Mass insertion, $\sim 10^{-1} /$ insertion. Each fermion mass insertion on an internal line introduces a factor of $\mathcal{O}\left(\nu R^{\prime}\right)$. This comes from the combination of dimensionful factors in the Yukawa interaction and the additional fermion propagator.
$B_{1}$. Equation of motion, $\sim 10^{-4}$. Higgs diagrams without an explicit chirality-flipping internal mass insertion must swap chirality using the muon equation of motion $\bar{u}(p) \not p=m_{\mu} u(p)$. This gives a factor of $\mathcal{O}\left(m_{\mu} R^{\prime}\right)$ and is equivalent to external mass insertion that picks up the zero-mode mass.
$B_{2}$. External mass insertion, $\sim 10^{-1}$. Alternately, when a loop vertex is in the bulk, an external mass insertion can pick up the diagonal piece of the propagator-see (5.127) -representing the propagation of a zero mode into a 'wrong-chirality' кк mode. Unlike the off-diagonal piece which imposes the equation of motion, this is only suppressed by the $\mathcal{O}\left(\nu R^{\prime}\right)$ mentioned above ${ }^{4}$. One can equivalently think of this as an insertion of the KK mass which

[^5]mixes the physical zero and kK modes.
C. Higgs/Goldstone cancellation, $\sim 10^{-3}$. The $H^{\circ}$ and $G^{\circ}$ one-mass-insertion loops cancel up to $\mathcal{O}\left(\left(m_{H}^{2}-m_{Z}^{2}\right) / m_{\mathrm{KK}}^{2}\right)$ because the two Goldstone couplings appear with factors of $i$ relative to the neutral Higgs couplings ${ }^{5}$.
D. Proportional to charged scalar mass, $\sim 10^{-2}$. The leading loop-momentum term in the one-mass-insertion brane-localized $H^{ \pm}$loop cancels due to the form of the photon coupling relative to the propagators. The gauge-invariant contribution from such a diagram is proportional to $\left(M_{W} R^{\prime}\right)^{2}$. This is shown explicitly in (5.69) below.

To demonstrate the charged scalar mass proportionality, we note that the amplitude for the one mass insertion charged Higgs diagram in Figure 5.5.2 is

$$
\begin{equation*}
\mathcal{M}^{\mu}=-R^{2}\left(\frac{R}{R^{\prime}}\right)^{6} \frac{e v}{\sqrt{2}} f_{c_{L_{\mu}}} Y_{*}^{3} f_{-c_{E_{e}}} \int \frac{d^{4} k}{(2 \pi)^{4}} \bar{u}_{p^{\prime}} \Delta_{k}^{R} \Delta_{k}^{L} u_{p} \frac{\left(2 k-p-p^{\prime}\right)^{\mu}}{\left[\left(k-p^{\prime}\right)^{2}-M_{W}^{2}\right]\left[(k-p)^{2}-M_{W}^{2}\right]} \tag{5.68}
\end{equation*}
$$

Remembering that the ${ }_{5} \mathrm{D}$ fermion propagators go like $\Delta \sim \not k / k$, this amplitude naïvely appears to be logarithmically divergent. However, the Ward identity forces the form of the photon coupling to the charged Higgs to be such that the leading order term in $k^{2}$ cancels. This can be made manifest by expanding the charged Higgs terms in $p$ and $p^{\prime}$,

$$
\begin{equation*}
\frac{\left(2 k-p-p^{\prime}\right)^{\mu}}{\left[\left(k-p^{\prime}\right)^{2}-M_{W}^{2}\right]\left[(k-p)^{2}-M_{W}^{2}\right]}=\frac{\left(p+p^{\prime}\right)^{\mu}}{\left(k^{2}-M_{W}^{2}\right)^{2}}\left[\frac{k^{2}}{k^{2}-M_{W}^{2}}-1\right]=\frac{M_{W}^{2}\left(p+p^{\prime}\right)^{\mu}}{\left(k^{2}-M_{W}^{2}\right)^{3}} \tag{5.69}
\end{equation*}
$$

where we have dropped terms of order $\mathcal{O}\left(m_{\mu}^{2} / M_{W}^{2}\right)$. Thus see that the coefficient of the gauge-invariant contribution is finite by power counting. After Wick rotation, this amplitude takes the form

$$
\begin{equation*}
\left.\mathcal{M}^{\mu}\left(1 \mathrm{MI} H^{ \pm}\right)\right|_{\left(p+p^{\prime}\right)}=\frac{2 i}{16 \pi^{2}}\left(R^{\prime}\right)^{2} f_{c_{L_{\mu}}} Y_{*}^{3} f_{-c_{E_{e}}} \frac{e v}{\sqrt{2}}\left(R^{\prime} M_{W}\right)^{2} I_{1 \mathrm{MI} H^{ \pm}} \bar{u}_{p^{\prime}}\left(p+p^{\prime}\right) u_{p} \tag{5.70}
\end{equation*}
$$

where $I_{1 \mathrm{MI} H^{ \pm}}$is a dimensionless integral given in (5.C). We see that the amplitude indeed carries a factor of $\left(M_{W} R^{\prime}\right)^{2}$.

## 5.B. 3 DIMENSIONLESS INTEGRALS

Estimating the size of dimensionless integrals over the loop momentum and bulk field propagators (such as $I_{1 \mathrm{MI} H^{ \pm}}$) is more subtle and is best checked through explicit calculation. However, one may develop an intuition for the relative size of these integrals.

Note that the fifth component of a bulk gauge field naturally has boundary conditions opposite that of the four-vector [177] so that the fifth components of Standard Model gauge fields have Dirichlet boundary conditions. This means that diagrams with a $W^{s} H^{ \pm} A$ vertex vanish since the brane-localized Higgs and bulk $W^{s}$ do not have overlapping profiles. Further, loops with fifth components of Standard Model gauge fields and internal mass insertions tend to be suppressed since the mass insertions attach the loop to the IR brane. In the UV limit the loop shrinks towards the brane and has reduced overlap with the fifth component gauge field.

Otherwise the loop integrals are typically $\mathcal{O}(0.1)$. The particular value depends on the propagators and couplings in the integrand.

## 5.B. 4 Robustness Against ALIGnment

As discussed in Section 5.5.2, the flavor structure of the diagrams contributing to the $b$ coefficient is aligned with the fermion zero-mode mass matrix [226,241]. Contributions to this coefficient vanish in the zero mode mass basis in the absence of additional flavor structure from the bulk mass (c) dependence of the internal fermion propagators. The diagrams which
${ }^{5}$ We thank Yuko Hori and Takemichi Okui for pointing this out.

$$
\xrightarrow{\psi^{(0)}} \boldsymbol{x}^{x^{(n)}} \bar{\psi}^{(n)}\left\{\bar{\psi}^{(m)} \xrightarrow{\left.\chi^{(m)}\right\}}\right.
$$

Figure 5.B.1: Alignment of the external mass insertion diagrams with Standard Model gauge bosons. $\chi$ and $\psi$ are left- and right-chiral Weyl spinors respectively. The gauge boson vertices don't change fermion chirality so that the internal fermion must be a chirality-flipping KK mode. We have neglected the contribution where the external mass insertion connects two zero mode fermions since this is suppressed by $m_{\mu} R^{\prime}$.
generally give the largest contribution after passing to the zero mode mass basis are those with with the strongest dependence on the fermion bulk masses. Since zero mode fermion profiles are exponentially dependent on the bulk mass parameter, a simple way to identify potential leading diagrams is to identify those which may have zero mode fermions propagating in the loop.

This allows us to neglect diagrams with an external mass insertion and a 4 D vector boson in the loop. As shown in Figure 5.B.1, such diagrams do not permit intermediate zero modes to leading order. Note, however, that diagrams with an external mass insertion and the fifth component of gauge boson are allowed to have zero mode fermions in the loop. Indeed, a diagram with a $W^{s}$ and $W^{\mu}$ in the loop would permit zero mode fermions but is numerically small due to the size of the $W^{\top} A W^{\mu}$ coupling. The dominant diagrams for the $b$ coefficient are the $H^{ \pm} W^{ \pm}$loop and the $Z$ loop with an internal mass insertion. In the KK reduction, the misalignment comes from diagrams with zero mode fermions and KK gauge bosons.

## 5.C Analytic expressions

We present analytic expressions for the leading and next-to-leading diagrams contributing to $\mu \rightarrow e \gamma$. We label the diagrams in Figs. 5.5.1-5.5.3 according to the number of Higgs-induced mass insertions and the internal boson(s). For example, the two-mass-insertion $W$ diagram in Figure 5.5 .4 a is referred to as ${ }_{2}$ MIW. Estimates for the size of each contribution are given in Appendix 5 .B. We shall only write the coefficient of the $\bar{u}_{p^{\prime}}\left(p+p^{\prime}\right)^{\mu} u_{p}$ term since this completely determines the gauge-invariant contribution.

## 5.C. 1 Dominant diagrams

As discussed in Section 5.5 , the leading diagrams contributing to the $a$ and $b$ coefficients are

$$
\begin{align*}
\mathcal{M}(2 \mathrm{MIW}) & =\frac{i}{16 \pi^{2}}\left(R^{\prime}\right)^{2} f_{c_{L_{\mu}}} Y_{E} Y_{N}^{\dagger} Y_{N} f_{-c_{E_{e}}} \frac{e v}{\sqrt{2}}\left(\frac{g^{2}}{2} \ln \frac{R^{\prime}}{R}\right)\left(\frac{R^{\prime} v}{\sqrt{2}}\right)^{2} I_{2 \mathrm{MIW}}  \tag{5.71}\\
\mathcal{M}(\mathrm{oMIHW}) & =\frac{i}{16 \pi^{2}}\left(R^{\prime}\right)^{2} f_{c_{L}} Y_{E} f_{-c_{E}} \frac{e v}{\sqrt{2}}\left(\frac{g^{2}}{2} \ln \frac{R^{\prime}}{R}\right) I_{\mathrm{oMIHW}},  \tag{5.72}\\
\mathcal{M}(\mathrm{IMIZ}) & =\frac{i}{16 \pi^{2}}\left(R^{\prime}\right)^{2} f_{c_{L}} Y_{E} f_{-c_{E}} \frac{e v}{\sqrt{2}}\left(g_{Z_{L}} g_{Z_{R}} \ln \frac{R^{\prime}}{R}\right) I_{\mathrm{IMIZ}}, \tag{5.73}
\end{align*}
$$

We have explicitly labeled the 4 D (dimensionless) anarchic Yukawa matrices whose elements assumed to take values of order $\left(Y_{E}\right)_{i j} \sim\left(Y_{N}\right)_{i j} \sim Y_{*}$, but have independent flavor structure. Note that we have suppressed the flavor indices of the Yukawas and the dimensionless integrals. Diagrams with a neutral boson and a Yukawa structure $Y_{E} Y_{E}^{\dagger} Y_{E}$ also contribute to the $a$ coefficient, but these contributions are suppressed relative to the dominant charged boson diagrams above. These diagrams may become appreciable if one permits a hierarchy in the relative $Y_{E}$ and $Y_{N}$ anarchic scales, in which case one should also
consider the $Z$ boson diagrams whose analytic forms are given below. The dimensionless integrals are

$$
\begin{align*}
& I_{2 \mathrm{MI} W}=-\frac{3}{2} \int d y d x_{1} d x_{2} d x_{3} y^{3}\left(\frac{y}{x_{1}}\right)^{c_{L}+2}\left(\frac{y}{x_{2}}\right)^{4}\left(\frac{y}{x_{3}}\right) \\
& \tilde{F}_{+, y}^{L 1 y} \tilde{F}_{-, y}^{R y y} \tilde{D}_{-} \tilde{F}_{-, y}^{L y_{2}} \tilde{F}_{+, y_{\mu}}^{L y_{\mu}} \frac{\partial}{\partial k_{E}}\left(G_{y}^{13} G_{y}^{32}\right)  \tag{5.74}\\
& I_{\mathrm{OMIHW}}=\int d y d x\left(\frac{y}{x}\right)^{2+c_{L}}\left(\frac{1}{2 \sqrt{2}} \frac{y^{2}}{y^{2}+m_{H}^{2} R^{\prime 2}} \tilde{F}_{+, y}^{L_{1 y}} y \partial_{k_{E}} G_{y}^{x y}\right)  \tag{5.75}\\
& I_{1 \mathrm{MIZ}}=-\int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{2+c_{L}}\left(\frac{y}{x_{2}}\right)^{2-c_{E}}\left(\frac{y}{x_{3}}\right)^{4}\left(y \partial_{k_{E}} G^{12}\right) y^{2} \times \\
& \left(-\tilde{D}_{+} \tilde{F}_{+, y}^{R 23} \tilde{D}_{-} \tilde{F}_{-, y}^{R_{3 y}} \tilde{F}_{+, y}^{L_{1}}+\tilde{F}_{-, y}^{R 2 x_{3}} \tilde{F}_{-, y}^{R_{3 y}} \tilde{F}_{+, y}^{L_{y 1}}\right. \\
& \left.-\tilde{F}_{-, y}^{R 2 y} \tilde{D}_{-} \tilde{F}_{-, y}^{L_{y}} \tilde{D}_{+} \tilde{F}_{+, y}^{L_{31}}+\tilde{F}_{-, y}^{R 2 y} \tilde{F}_{+, y}^{L_{3}} \tilde{F}_{+, y}^{L_{31}}\right) . \tag{5.76}
\end{align*}
$$

where $x=k_{E} z, y=k_{E} R^{\prime}$, and $y_{\mu}=m_{\mu} R^{\prime}$. The significance of these dimensionless variables is discussed below (5.44). The dimensionless Euclidean-space propagator functions $\tilde{F}$ are defined in ( $4.65-4.66$ ), where the upper indices of the $F$ functions define the propagation positions. For example, $F^{R 3 y}$ represents a propagator from $z=R^{\prime}$ to $z=z_{3}$. Similarly, $G_{y}$ and $\bar{G}_{y}$ are defined in (4.115) and (4.116).

## 5.C. 2 SUbDOMINANT $a$ COEFFICIENT DIAGRAMS

The diagrams containing a brane-localized Higgs loop are

$$
\begin{align*}
\mathcal{M}\left(n \mathrm{MI} H^{ \pm}\right) & =\frac{i}{16 \pi^{2}}\left(R^{\prime}\right)^{2} f_{c_{L}} Y_{E} Y_{N}^{\dagger} Y_{N} f_{-c_{E}} \frac{e v}{\sqrt{2}} I_{n \mathrm{MI} H^{ \pm}},  \tag{5.77}\\
\mathcal{M}\left(n \mathrm{MI} H^{\circ}\right) & =\frac{i}{16 \pi^{2}}\left(R^{\prime}\right)^{2} f_{c_{\mathrm{L}}} Y_{E} Y_{E} Y_{E}^{\dagger} f_{-c_{E}} \frac{e v}{\sqrt{2}} I_{o \mathrm{MIH}} . \tag{5.78}
\end{align*}
$$

Here $n=0,1$ counts the number of internal mass insertions in the diagram. The gauge boson loops are

$$
\left.\begin{array}{rl}
\mathcal{M}(n \mathrm{MIZ}
\end{array}{ }^{(s)}\right)=\frac{i}{16 \pi^{2}}\left(R^{\prime}\right)^{2} f_{c_{L}} Y_{E} Y_{E}^{\dagger} Y_{E} f_{-c_{E}} \frac{e v}{\sqrt{2}}\left(g_{Z_{L}} g_{Z_{\mathrm{R}}} \ln \frac{R^{\prime}}{R}\right)\left(\frac{v}{\sqrt{2}} R^{\prime}\right)^{2} I_{n \mathrm{MIZ}(\mathrm{~s})}, ~\left(\frac{i}{16 \pi^{2}}\left(R^{\prime}\right)^{2} f_{c_{L}} Y_{E} Y_{N}^{\dagger} Y_{N} f_{-c_{\mathrm{E}}} \frac{e v}{\sqrt{2}}\left(\frac{g^{2}}{2} \ln \frac{R^{\prime}}{R}\right)\left(\frac{v}{\sqrt{2}} R^{\prime}\right)^{2} \times I_{2 \mathrm{MI} w w} .\right.
$$

Where $n=2,(1+2), 3$ with $(1+2)$ referring to a single internal mass insertion and two external mass insertions. $2 \mathrm{MI} w w$ represents $2 \mathrm{MI} W^{s} W^{s}, 2 \mathrm{MI} W W^{s}$ and ${ }_{2} \mathrm{MI}^{\boldsymbol{s}} W$. The dimensionless integrals are

$$
\begin{align*}
& I_{1 \text { МIH }}=\int d y d x y^{2}\left(\frac{y}{x}\right)^{4}\left[-2 \tilde{F}_{+, y}^{L y x} \tilde{F}_{+, y}^{L x y} \tilde{F}_{-, y}^{R y y} \frac{y^{2}}{y^{2}+\left(M_{H} R^{\prime}\right)^{2}}\right. \\
& +\tilde{F}_{+, y}^{L y x} \tilde{F}_{+, y}^{L x y} \tilde{F}_{-, y}^{R y y} \frac{y^{4}}{\left(y^{2}+\left(m_{H} R^{\prime}\right)^{2}\right)^{2}}-\frac{1}{2}\left(y \partial_{k_{B}} \tilde{F}_{+, y}^{L y x}\right) \tilde{F}_{+, y}^{L x y} \tilde{F}_{-, y}^{R y y} \frac{y^{2}}{y^{2}+\left(M_{H} R^{\prime}\right)^{2}} \\
& -\frac{1}{2}\left(y \partial_{k_{E}} \tilde{D}_{-} \tilde{F}_{-, y}^{L y x}\right) \tilde{D}_{+} \tilde{F}_{+, y}^{L x y} \tilde{F}_{-, y}^{R y y} \frac{1}{y^{2}+\left(M_{H} R^{\prime}\right)^{2}}+2 \tilde{F}_{+, y}^{L y y} \tilde{D}_{+} \tilde{F}_{+, y}^{R y x} \tilde{D}_{-} \tilde{F}_{-, y}^{R x y} \frac{1}{y^{2}+\left(M_{H} R^{\prime}\right)^{2}} \\
& -\tilde{F}_{+, y}^{L y y} \tilde{D}_{+} \tilde{F}_{+, y}^{R y x} \tilde{D}_{-} \tilde{F}_{-, y}^{R x y} \frac{y^{2}}{\left(y^{2}+\left(M_{H} R^{\prime}\right)^{2}\right)^{2}}+\frac{1}{2}\left(y \partial_{k_{k}} \tilde{F}_{+, y}^{L y y}\right) \tilde{D}_{+} \tilde{F}_{+, y}^{R y x} \tilde{D}_{-} \tilde{F}_{-, y}^{R x y} \frac{1}{y^{2}+\left(M_{H} R^{\prime}\right)^{2}} \\
& +\tilde{F}_{+, y}^{L y y} \tilde{F}_{-, y}^{R y x} \tilde{F}_{-, y}^{R x y} \frac{y^{2}}{y^{2}+\left(M_{H} R^{\prime}\right)^{2}}+\frac{1}{2}\left(y \partial_{k_{k}} \tilde{F}_{+, y}^{L y y}\right) \tilde{F}_{-, y}^{R y x} \tilde{F}_{-, y}^{R x y} \frac{y^{2}}{y^{2}+\left(M_{H} R^{\prime}\right)^{2}} \\
& \left.+\frac{1}{2} \tilde{F}_{+, y}^{L y y}\left(y \partial_{k_{E}} \tilde{R}_{-, y}^{R y y}\right) \tilde{F}_{-, y}^{R x y} \frac{y^{2}}{y^{2}+\left(M_{H} R^{\prime}\right)^{2}}+\frac{1}{2} \tilde{F}_{+, y}^{L y y}\left(y \partial_{k_{E}} \tilde{D}_{+} \tilde{F}_{+, y}^{R y x}\right) \tilde{D}_{-} \tilde{F}_{-, y}^{R x y} \frac{1}{y^{2}+\left(M_{H} R^{\prime}\right)^{2}},\right] .  \tag{1}\\
& I_{1 \mathrm{MI} H^{ \pm}}=\int d y \tilde{F}_{+, y}^{L y y} \tilde{F}_{+, y}^{R y y} \frac{2 y^{5}}{\left(y^{2}+\left(M_{W} R^{\prime}\right)^{2}\right)^{3}}  \tag{5.82}\\
& I_{\mathrm{oMIH}}{ }^{ \pm}=\int d y \tilde{F}_{-, y}^{R y} \frac{y^{5}}{\left(y^{2}+\left(M_{H} R^{\prime}\right)^{2}\right)^{3}}  \tag{5.83}\\
& I_{\text {oMIH }}=\int d y d x y^{2}\left(\frac{y}{x}\right)^{4} \tilde{F}_{+, y}^{L y x} \tilde{F}_{+, y}^{L x y} \frac{y^{2}}{\left(y^{2}+\left(M_{H} R^{\prime}\right)^{2}\right)^{2}}  \tag{5.84}\\
& I_{2 \mathrm{MIZ}}=\int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{2+c_{L}}\left(\frac{y}{x_{2}}\right)^{4}\left(\frac{y}{x_{3}}\right)^{4} \times \\
& \left\{y \partial _ { k _ { E } } G _ { y } ^ { 1 3 } \tilde { D } _ { + } \tilde { F } _ { + , y _ { \mu } } ^ { L _ { 3 y } y _ { \mu } } \left[y ^ { 2 } \left(\tilde{F}_{+, y}^{L_{12}} \tilde{y}_{+, y}^{L_{+y}} \tilde{F}_{-, y}^{R y} \tilde{D}_{-} \tilde{F}_{-, y}^{L y_{3}}+\tilde{F}_{+, y}^{L_{11}} \tilde{F}_{-, y}^{R y 2} \tilde{F}_{-, y}^{R 2 y} \tilde{D}_{-} \tilde{F}_{-, y}^{L y_{3}}\right.\right.\right. \\
& \left.+\tilde{F}_{+, y}^{L_{1 y}} \tilde{F}_{-, y}^{R y y} \tilde{D}_{-} \tilde{F}_{-, y}^{R 2 y} \tilde{F}_{-, y}^{L_{23}}+\tilde{F}_{+, y}^{L_{11}} \tilde{F}_{-, y}^{R y y} \tilde{F}_{+, y}^{L y 2} \tilde{D}_{-} \tilde{F}_{-, y}^{L_{23}}\right) \\
& \left.\left.-\left(\tilde{D}_{-} \tilde{F}_{-, y}^{L_{12}} \tilde{D}_{+} \tilde{F}_{+, y}^{L 2 y} \tilde{F}_{-, y}^{R y y} \tilde{D}_{-} \tilde{F}_{-, y}^{L y_{3}}+\tilde{F}_{+, y}^{L_{1 y}} \tilde{D}_{+} \tilde{F}_{+, y}^{R y 2} \tilde{D}_{-} \tilde{F}_{-, y}^{R 2 y} \tilde{D}_{+} \tilde{F}_{+, y}^{L_{y}}\right)\right]\right\},  \tag{5.85}\\
& I_{2 \mathrm{MIZ}^{5}}=-\int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{2+c_{L}}\left(\frac{y}{x_{2}}\right)^{4}\left(\frac{y}{x_{3}}\right)^{4} \times \\
& \frac{1}{2}\left[y \partial_{k_{E}} \bar{G}_{y}^{13} \tilde{D}_{+} \tilde{F}_{+, y_{\mu}}^{L_{3} y_{\mu}}\left(y^{2} \tilde{F}_{-, y}^{L_{12}} \tilde{D}_{+} \tilde{F}_{+, y}^{L_{2}} \tilde{F}_{+, y}^{R y y}\right)\right],  \tag{5.86}\\
& I_{(1+2) \text { MIZ }}=-\int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{4}\left(\frac{y}{x_{2}}\right)^{4}\left(\frac{y}{x_{3}}\right)^{4} \times \\
& {\left[\tilde{D}_{+} \tilde{F}_{+, y_{e}}^{R y_{c_{c}}} \tilde{D}_{+} \tilde{F}_{+, y_{\mu}}^{L_{2} y_{\mu}} G_{y}^{21}-\left(4+y \partial_{k_{E}}\right)\left(\tilde{D}_{-} \tilde{F}_{-, y}^{R 1 y} \tilde{F}_{-, y}^{R_{3 y}} \tilde{D}_{-} \tilde{F}_{-, y}^{L_{32}}+\tilde{F}_{+, y}^{R_{13}} \tilde{D}_{-} \tilde{F}_{-, y}^{R_{3 y}} \tilde{D}_{-} \tilde{F}_{-, y}^{L_{32}}\right.\right.} \\
& \left.\left.+\tilde{D}_{-} \tilde{F}_{-, y}^{R 1 y} \tilde{D}_{-} \tilde{F}_{-, y}^{L_{y}} \tilde{F}_{-, y}^{L_{32}}+\tilde{D}_{-} \tilde{F}_{-, y}^{R 1 y} \tilde{L}_{+, y}^{L_{3}} \tilde{D}_{-} \tilde{F}_{-, y}^{L_{32}}\right)\right],  \tag{5.87}\\
& I_{(1+2) \mathrm{MIZ}^{s}}=-\int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{4}\left(\frac{y}{x_{2}}\right)^{4}\left(\frac{y}{x_{3}}\right)^{4} \tilde{D}_{+} \tilde{F}_{+, y_{e}}^{R y_{y_{e}}} \tilde{D}_{+} \tilde{F}_{+, y_{\mu}}^{L_{3 y_{\mu}}} \bar{G}_{y}^{13} \\
& \frac{1}{2}\left[\tilde{F}_{-, y}^{R 12} y \partial_{k_{E}}\left(\tilde{F}_{-, y}^{R 2 y} \tilde{F}_{+, y}^{L y_{3}}\right)+y \partial_{k_{E}}\left(\tilde{D}_{+} \tilde{F}_{+, y}^{R 12}\right) \tilde{D}_{-} \tilde{F}_{-, y}^{R 2 y} \tilde{F}_{+, y}^{L y_{3}}\right. \\
& \left.\tilde{F}_{-, y}^{R 1 y} \tilde{D}_{-} \tilde{F}_{-, y}^{L y 2} y \partial_{k_{E}}\left(\tilde{D}_{+} \tilde{F}_{+, y}^{L 23}\right)+y \partial_{k_{E}}\left(y^{2} \tilde{F}_{-, y}^{R 1 y} \tilde{F}_{+, y}^{L y 2}\right) \tilde{F}_{+, y}^{L 23}\right], \tag{5.88}
\end{align*}
$$

The integral for ${ }_{3} \mathrm{MIZ}$ and ${ }_{3} \mathrm{MIZ}^{5}$ can be written as

$$
\begin{equation*}
I_{3 \mathrm{MIZ} / \mathrm{Z}^{S}}=\frac{1}{2} \int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{2+c_{L}}\left(\frac{y}{x_{2}}\right)^{2-c_{E}}\left(\frac{y}{x_{3}}\right)^{4} G_{y}^{13} \sum_{i=1}^{8} M_{i} y \partial_{k_{E}} N_{i} . \tag{5.89}
\end{equation*}
$$

For ${ }_{3}$ MIZ, the $(M, N)$ pairs are

$$
\begin{aligned}
& \left(M_{1}, N_{1}\right)=\left(\tilde{F}_{+, y}^{L 2}, y^{4} \tilde{F}_{+, y}^{L 2 y} \tilde{y}_{-, y}^{R y y} \tilde{F}_{+, y}^{L y y} \tilde{F}_{-, y}^{R y_{3}}\right), \\
& \left(M_{2}, N_{2}\right)=\left(-y^{2} \tilde{D}_{+} \tilde{F}_{+, y}^{L 2 y} \tilde{F}_{-, y}^{R y y} \tilde{F}_{+, y}^{L y y} \tilde{F}_{-, y}^{R y_{3}}, \tilde{D}_{-} \tilde{F}_{-, y}^{L_{12}}\right), \\
& \left(M_{3}, N_{3}\right)=\left(-y^{2} \tilde{F}_{-, y}^{R 2 y} \tilde{F}_{+, y}^{L y y} \tilde{F}_{-, y}^{R y_{3}},-y^{2} \tilde{F}_{+, y}^{L 1 y} \tilde{F}_{-, y}^{R y 2}\right) \text {, } \\
& \left(M_{4}, N_{4}\right)=\left(\tilde{F}_{+, y}^{L y} \tilde{D}_{+} \tilde{F}_{+, y}^{R y 2},-y^{2} \tilde{D}_{-} \tilde{F}_{-, y}^{R 2 y} \tilde{F}_{+, y}^{L y y} \tilde{F}_{-, y}^{R y /}\right), \\
& \left(M_{5}, N_{5}\right)=\left(-y^{2} \tilde{F}_{+, y}^{L 1 y} \tilde{F}_{-, y}^{R y y} \tilde{F}_{+, y}^{L y_{2}},-y^{2} \tilde{F}_{+, y}^{L 2 y} \tilde{F}_{-, y}^{R y{ }_{3}}\right) \text {, } \\
& \left(M_{6}, N_{6}\right)=\left(\tilde{D}_{+} \tilde{F}_{+, y}^{L 2 y} \tilde{F}_{-, y}^{R y /},-y^{2} \tilde{F}_{+, y}^{L 1 y} \tilde{F}_{-, y}^{R y y} \tilde{D}_{-} \tilde{F}_{-, y}^{L y 2}\right) \text {, } \\
& \left(M_{7}, N_{7}\right)=\left(\tilde{F}_{-, y}^{R_{23}}, y^{4} \tilde{F}_{+, y}^{L_{11}} \tilde{F}_{-, y}^{R y y} \tilde{F}_{+, y}^{L_{y}} \tilde{F}_{-, y}^{R y 2}\right), \\
& \left(M_{8}, N_{8}\right)=\left(-y^{2} \tilde{F}_{+, y}^{L y} \tilde{F}_{-, y}^{R y y} \tilde{F}_{+, y}^{L y y} \tilde{D}_{+} \tilde{F}_{+, y}^{R y 2}, \tilde{D}_{-} \tilde{F}_{-, y}^{R 23}\right) .
\end{aligned}
$$

For ${ }_{3} \mathrm{MIZ}^{5}$, the $(M, N)$ pairs are

$$
\begin{align*}
& \left(M_{1}, N_{1}\right)=\left(-y^{2} \tilde{D}_{+} \tilde{F}_{+, y}^{L 1 y} \tilde{F}_{-, y}^{R y y} \tilde{F}_{+, y}^{L y y} \tilde{F}_{-, y}^{R y 2}, \tilde{D}_{+}+\tilde{F}_{+, y}^{R 23}\right),  \tag{5.98}\\
& \left(M_{2}, N_{2}\right)=\left(\tilde{F}_{+, y}^{R 23},-y^{2} \tilde{D}_{+} \tilde{F}_{+, y}^{L 1 y} \tilde{F}_{-, y}^{R y y} \tilde{F}_{+, y}^{L y y} \tilde{D}_{+} \tilde{F}_{+, y}^{R y 2}\right) \text {, }  \tag{5.99}\\
& \left(M_{3}, N_{3}\right)=\left(\tilde{D}_{+} \tilde{F}_{+, y}^{L y_{1}} \tilde{F}_{-, y}^{R y y} \tilde{D}_{-} \tilde{F}_{-, y}^{L y_{2}}, \tilde{D}_{+} \tilde{F}_{+, y}^{L 2 y} \tilde{D}_{+} \tilde{F}_{+, y}^{R y /}\right), \\
& \left(M_{4}, N_{4}\right)=\left(\tilde{F}_{+, y}^{L_{2 y}} \tilde{D}_{+} \tilde{F}_{+, y}^{R y_{3}},-y^{2} \tilde{D}_{+} \tilde{F}_{+, y}^{L y} \tilde{F}_{+, y}^{R y y} \tilde{F}_{+, y}^{L_{2}}\right), \\
& \left(M_{5}, N_{5}\right)=\left(\tilde{D}_{+} \tilde{F}_{+, y}^{L 1 y} \tilde{F}_{-, y}^{R y 2},-y^{2} \tilde{F}_{-, y}^{R 2 y} \tilde{F}_{+, y}^{L y} \tilde{D}_{+} \tilde{F}_{+, y}^{R y 3}\right), \\
& \left(M_{6}, N_{6}\right)=\left(\tilde{D}_{-} \tilde{F}_{-, y}^{R 2 y} \tilde{F}_{+, y}^{L y y} \tilde{D}_{+} \tilde{F}_{+, y}^{R y_{3}}, \tilde{D}_{+} \tilde{F}_{+, y}^{L_{1 y}} \tilde{D}_{+} \tilde{F}_{+, y}^{R y 2}\right) \text {, } \\
& \left(M_{7}, N_{7}\right)=\left(\tilde{F}_{-, y}^{L 2},-y^{2} \tilde{D}_{+} \tilde{F}_{+, y}^{L 2 y} \tilde{F}_{-, y}^{R y y} \tilde{F}_{+, y}^{L y y} \tilde{D}_{+} \tilde{F}_{+, y}^{R y /}\right) \text {, } \\
& \left(M_{8}, N_{8}\right)=\left(-y^{2} \tilde{F}_{+, y}^{L 2 y} \tilde{F}_{-, y}^{R y y} \tilde{F}_{+, y}^{L y y} \tilde{D}_{+} \tilde{F}_{+, y}^{R y /}, \tilde{D}_{+} \tilde{F}_{+, y}^{L L 2}\right) .
\end{align*}
$$

The integrals for the $W^{s}$ loops are

$$
\begin{align*}
& I_{2 \text { MIWsws }^{s}}=-\int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{2+c_{L}}\left(\frac{y}{x_{2}}\right)^{4}\left(\frac{y}{x_{3}}\right) \times \\
& \left\{\frac{1}{2} y^{2} \tilde{D}_{+} \tilde{F}_{+, y}^{L 1 y} \tilde{F}_{-, y}^{R y y} \tilde{F}_{+, y}^{L y_{2}} \tilde{D}_{+} \tilde{F}_{+, y_{y}}^{L 2 y_{\mu}}\left[4 \bar{G}_{y}^{13} \bar{G}_{y}^{23}+y \partial_{k_{E}}\left(\bar{G}_{y}^{13} \bar{G}_{y}^{23}\right)\right]\right\},  \tag{5.106}\\
& I_{2 \text { MIWsW }}=-\int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{2+c_{L}}\left(\frac{y}{x_{2}}\right)^{4}\left(\frac{y}{x_{3}}\right) \times \\
& {\left[\frac{1}{2} y^{2} \tilde{F}_{+, y}^{L 1} \tilde{F}_{-, y}^{R y y} \tilde{F}_{+, y}^{L y_{2}} \tilde{D}_{+} \tilde{F}_{+, y_{y}}^{L 2 y_{\mu}}\left(y \partial_{k_{E}} G_{y}^{13} \partial_{z} \bar{G}_{y}^{23}-y \partial_{k_{E}} \partial_{z} G_{y}^{13} \bar{G}_{y}^{23}\right)\right],} \tag{5.107}
\end{align*}
$$

$$
\begin{align*}
I_{2 \mathrm{MIWW}^{s}}= & -\int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{2+c_{L}}\left(\frac{y}{x_{2}}\right)^{4}\left(\frac{y}{x_{3}}\right) \times \\
& {\left[\frac{1}{2} \tilde{D}_{+} \tilde{F}_{+, y}^{L^{1 y}} \tilde{F}_{-, y}^{R y y} \tilde{D}_{-} \tilde{F}_{-, y}^{L_{2} /} \tilde{D}_{+}+\tilde{F}_{+, y_{\mu}}^{2 y_{\mu}}\left(y \partial_{k_{E}} G_{y}^{23} \partial_{z} \bar{G}_{y}^{13}-y \partial_{k_{E}} \partial_{z} G_{y}^{23} \bar{G}_{y}^{13}\right)\right] . } \tag{5.108}
\end{align*}
$$

## 5.C. 3 Subdominant $b$ coefficient diagrams

$$
\begin{align*}
\mathcal{M}\left(n \mathrm{MIZ} / Z^{5}\right) & =\frac{i}{16 \pi^{2}}\left(R^{\prime}\right)^{2} f_{c_{L}} Y_{E} f_{-c_{\mathrm{E}}} \frac{e v}{\sqrt{2}}\left(g_{Z_{L}} g_{Z_{\mathrm{R}}} \ln \frac{R^{\prime}}{R}\right) I_{n \mathrm{MIZ} / Z^{5}},  \tag{5.109}\\
\mathcal{M}(\mathrm{oMI} W) & =\frac{i}{16 \pi^{2}}\left(R^{\prime}\right)^{2} f_{c_{L}} Y_{E} f_{-c_{\mathrm{E}}} \frac{e v}{\sqrt{2}}\left(\frac{g^{2}}{2} \ln \frac{R^{\prime}}{R}\right) I_{\mathrm{oMIW}},  \tag{5.110}\\
\mathcal{M}\left(\mathrm{oMI} W^{\delta}\right) & =\frac{i}{16 \pi^{2}}\left(R^{\prime}\right)^{2} f_{c_{L_{\mu}}} Y_{E} f_{-c_{E_{e}}} \frac{e v}{\sqrt{2}}\left(\frac{g^{2}}{2} \ln \frac{R^{\prime}}{R}\right) I_{\mathrm{oMIW}}{ }^{5} \tag{5.111}
\end{align*}
$$

where $n=0,1$ counts the number of internal mass insertions.

$$
\begin{align*}
& I_{\mathrm{IMIZ}}{ }^{S}=\int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{2+c_{\mathrm{L}}}\left(\frac{y}{x_{2}}\right)^{2-c_{\mathrm{E}}}\left(\frac{y}{x_{3}}\right)^{4} \times \\
& \frac{1}{2}\left[\tilde{F}_{-, y}^{L_{13}} y \partial_{k E}\left(\tilde{D}_{+} \tilde{F}_{+, y}^{L_{3 y}} \tilde{D}_{+} \tilde{F}_{+, y}^{R y 2}\right) \bar{G}_{y}^{12}-\tilde{D}_{+} \tilde{F}_{+, y}^{L_{13}} y \partial_{k E}\left(\tilde{F}_{+, y}^{L_{3 y}} \tilde{D}_{+} \tilde{F}_{+, y}^{R y 2} \bar{G}_{y}^{12}\right)\right. \\
& -4 \tilde{D}_{+} \tilde{F}_{+, y}^{L_{13}} \tilde{F}_{+, y}^{L_{3 y}} \tilde{D}_{+} \tilde{F}_{+, y}^{R y 2} \bar{G}_{y}^{12}+\tilde{D}_{+} \tilde{F}_{+, y}^{L_{1 y}} \tilde{F}_{-, y}^{R y y_{3}}\left(y \partial_{k_{E}} \tilde{D}_{+} \tilde{F}_{+, y}^{R_{32}}\right) \bar{G}_{y}^{12} \\
& \left.-\tilde{D}_{+} \tilde{F}_{+, y}^{L 1 y} \tilde{D}_{+} \tilde{F}_{+, y}^{R y} y \partial_{k_{E}}\left(\tilde{F}_{+, y}^{R_{32}} \bar{G}_{y}^{12}\right)-4 \tilde{D}_{+} \tilde{F}_{+, y}^{L 1} \tilde{D}_{+} \tilde{F}_{+, y}^{R y 3} \tilde{F}_{+, y}^{R_{32}} \bar{G}_{y}^{12}\right] .  \tag{5.112}\\
& I_{\text {oMIZ }}=\int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{2+c_{\mathrm{L}}}\left(\frac{y}{x_{2}}\right)^{4}\left(\frac{y}{x_{3}}\right)^{4} \times \\
& y \partial_{k_{E}} G_{y}^{13} \tilde{D}_{+} \tilde{F}_{+, y_{\mu}}^{L_{3} y_{\mu}}\left(\tilde{D}_{-} \tilde{F}_{-, y}^{L 12} \tilde{F}_{-, y}^{L 23}+\tilde{F}_{+, y}^{L 21} \tilde{D}_{-} \tilde{F}_{-, y}^{L 23}\right) \text {, }  \tag{5.113}\\
& I_{\mathrm{oMIZs}}=-\int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{2+c_{L}}\left(\frac{y}{x_{2}}\right)^{4}\left(\frac{y}{x_{3}}\right)^{4} \times \\
& \left\{\frac{1}{4} \tilde{D}_{+} \tilde{F}_{+, y}^{L_{2}} \tilde{D}_{+} \tilde{F}_{+, y_{\mu}}^{L_{3} y_{\mu}}\left[\tilde{F}_{-, y}^{L_{12}}\left(4 \bar{G}_{y}^{13}+y \partial_{k_{E}} \bar{G}_{y}^{13}\right)+y \partial_{k_{E}} \tilde{F}_{-, y}^{L_{12}} \bar{G}_{y}^{13}\right]\right\}, \\
& I_{\text {OMIW }}=-\int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{2+c_{L}}\left(\frac{y}{x_{2}}\right)\left(\frac{y}{x_{3}}\right)^{4} \times \\
& \frac{3}{2} y \partial_{k_{E}}\left(G_{y}^{13} G_{y}^{32}\right) \tilde{D}_{-} \tilde{F}_{-, y}^{L L 2} \tilde{D}_{L} \tilde{F}_{L, y_{\mu}}^{+3 y_{\mu}} \\
& I_{\mathrm{oMI} W^{s}}=\int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{c_{L}+2}\left(\frac{y}{x_{2}}\right)^{4}\left(\frac{y}{x_{3}}\right) \\
& \left\{\frac{y}{2} \tilde{F}_{+, y}^{L 1 y} \tilde{D}_{+} \tilde{F}_{+, y_{\mu}}^{L y_{\mu}}\left(\frac{\partial}{\partial k_{E}} \frac{\partial}{\partial x_{3}} G_{y}^{13}\right) \bar{G}_{y}^{32}+\frac{y}{2} \tilde{F}_{+, y}^{L 12} \tilde{D}_{+} \tilde{F}_{+, y_{\mu}}^{L 2 y_{\mu}}\left(\frac{\partial}{\partial k_{E}} \frac{\partial}{\partial x_{3}} G_{y}^{32}\right) \bar{G}_{y}^{13}\right. \\
& \left.-\tilde{D}_{+} \tilde{F}_{+, y}^{L 12} \tilde{D}_{+} \tilde{F}_{+, y_{\mu}}^{L 2 y_{\mu}}\left[2 \bar{G}_{y}^{13} \bar{G}_{y}^{23}+\frac{y}{2} \frac{\partial}{\partial k_{E}}\left(\bar{G}_{y}^{13} \bar{G}_{y}^{32}\right)\right]\right\} . \tag{5.116}
\end{align*}
$$

## 5.C. 4 Custodial Models

For custodially protected models, one must include loops with the custodial partners of fermions and gauge bosons. See, e.g., [209] for details of the additional field content of such models. The new particles have mixed boundary conditions, $(-+)$
or (+-). For the chirality flipping process $\mu \rightarrow e \gamma$, Yukawa insertions on the IR brane only allow fermions carrying either $(++)$ or $(-+)$ boundary conditions running in the loop. This limits the number of the new diagrams to be considered. The new fermion propagators can be obtained by making the replacement $\tilde{F} \rightarrow \tilde{E}$. Writing the boundary condition in terms of the Weyl components of the Dirac spinor, $\tilde{E}^{L}$ corresponds to the boundary condition $\left(\psi_{(+-)}, \bar{\chi}_{(-+)}\right)$, while $\tilde{E}^{R}$ corresponds to $\left(\psi_{(-+)}, \bar{\chi}_{(+-)}\right)$. For $x>x^{\prime}$, the $\tilde{E}$-functions can be written as follows:

$$
\begin{array}{ll}
\tilde{E}_{-}^{L}=\frac{\left(x x^{\prime}\right)^{5 / 2}}{y^{5}} \frac{S_{c}\left(x_{-}, y_{-}\right) T_{c}\left(x_{-}^{\prime}, w y_{+}\right)}{T_{c}\left(y_{-}, w y_{+}\right)} & \tilde{E}_{+}^{L}=-\frac{\left(x x^{\prime}\right)^{5 / 2}}{y^{5}} \frac{T_{c}\left(x_{+}, y_{-}\right) S_{c}\left(x_{+}^{\prime}, w y_{+}\right)}{T_{c}\left(y_{-}, w y_{+}\right)} \\
\tilde{E}_{-}^{R}=-\frac{\left(x x^{\prime}\right)^{5 / 2}}{y^{5}} \frac{T_{c}\left(x_{-}, y_{+}\right) S_{c}\left(x_{-}^{\prime}, w y_{-}\right)}{T_{c}\left(y_{+}, w y_{-}\right)} & \tilde{E}_{+}^{R}=\frac{\left(x x^{\prime}\right)^{5 / 2}}{y^{5}} \frac{S_{c}\left(x_{+}, y_{+}\right) T_{c}\left(x_{+}^{\prime}, w y_{-}\right)}{T_{c}\left(y_{+}, w y_{-}\right)} .
\end{array}
$$

The $x<x^{\prime}$ expressions are obtained by replacing $x \leftrightarrow x^{\prime}$. Gauge bosons with ( -+ ) boundary conditions can also appear in custodial loops. The corresponding propagator for $x>x^{\prime}$ is $G \rightarrow H$ with

$$
\begin{equation*}
H_{k}\left(x, x^{\prime}\right)=\frac{\left(R^{\prime}\right)^{2}}{R} \frac{x x^{\prime}}{y} \frac{T_{10}(x, y) S_{11}\left(x^{\prime}, w y\right)}{T_{10}(w y, y)} \tag{5.119}
\end{equation*}
$$

The $T$ and $S$ are defined in Appendix. (4.6), and the $x<x^{\prime}$ case can be obtained by $x \leftrightarrow x^{\prime}$.

## 5.D Position, momentum, and position/momentum space

In order to elucidate the power counting in Section 5.6 and to provide some motivation for the structure of the propagators in Appendix 4.4.1, we review the passage between Feynman rules in position, momentum, and mixed position/momentum space. For simplicity we shall work with massless scalar fields on a flat (Minkowski) $d$-dimensional background, but the generalization of the salient features to higher spins is straightforward. In position space, the two-point Green's function for a particle propagating from $x^{\prime}$ to $x$ is

$$
\begin{equation*}
D\left(x, x^{\prime}\right)=\int d^{d} k \frac{i}{k^{2}} e^{-i k \cdot\left(x-x^{\prime}\right)}, \tag{5.120}
\end{equation*}
$$

a momentum-space integral over a power-law in $k$ times a product of exponentials in $k \cdot x$ and $k \cdot x^{\prime}$. Each vertex carries a $d^{d} x$ integral representing each spacetime point at which the interaction may occur. When some dimensions are compact, the associated integrals are reverted to discrete sums and the particular linear combination of exponentials is shifted to maintain boundary conditions. Further, when dimensions are warped the exponentials become Bessel functions. In this Appendix we will neglect these differences and focus on general features since the $U v$ behavior of each of the aforementioned scenarios (i.e. for momenta much larger than any mass, compactification, or warping scales) reduces to the flat noncompact case presented here.

In 4 D it is conventional to work in full momentum space where the Feynman rules are derived by performing the $d^{d} x$ integrals at each vertex over the exponential functions from each propagator attached to the vertex and amputating the external propagators. This generates a momentum-conserving $\delta$-function at each vertex which can be used to simplify the $\hbar^{d} k$ integrals in each propagator. For each diagram one such $\delta$-function imposes overall conservation of the external momenta and hence has no dependence on any internal momenta. For a loop diagram this means that there is a leftover $\rrbracket^{d} k$ which corresponds to the integration over the loop momentum. Thus the momentum space formalism involves separating the exponentials in $k \cdot x$ from the rest of the Green's function and performing the $d^{d} x$ integral to obtain $\delta$-functions.

To go to the mixed position/momentum space formalism we pick one direction, $z$, and leave the dependence on that position in the propagator while integrating over the $z$-component of the momentum, $k^{z}$ in ( 5.120 ). We shall write the Minkowski scalar product of the $(d-1)$ momentum-space directions as $k^{2}$ so that the full $d$-dimensional scalar product is $k^{2}-k_{z}^{2}$. The Feynman rule for each vertex now includes an explicit $d z$ integral which must be performed after including each


Figure 5.D.1: A simple loop diagram to demonstrate the power counting principles presented. The lines labeled $p_{i}$ represent the net external momentum flowing into each vertex so that $p_{i}^{z}$ corresponds to the KK mass of the $i^{\text {th }}$ external particle.
of the position/momentum space propagators, which take the form

$$
\begin{equation*}
\Delta\left(k, z, z^{\prime}\right)=\int d k_{z} \frac{i}{k^{2}-k_{z}^{2}} e^{i k_{z}\left(z-z^{\prime}\right)} \tag{5.121}
\end{equation*}
$$

The $(d-1)$ other exponentials and momentum integrals are accounted in the usual momentum-space formalism. This object goes like $\Delta \sim 1 / k$, which indeed has the correct dimensionality for the sum over a KK tower of scalar propagators. Similarly, the massless bulk fermion propagator is

$$
\begin{equation*}
\Delta\left(k, z, z^{\prime}\right)=\int \pi k_{z} \frac{i\left(k-k_{z} \gamma^{5}\right)}{k^{2}-k_{z}^{2}} e^{i k_{z}\left(z-z^{\prime}\right)}, \tag{5.122}
\end{equation*}
$$

where we may now identify the scalar functions $F \sim d k_{z} e^{i k_{z}\left(z-z^{\prime}\right)} /\left(k^{2}-k_{z}^{2}\right)$ in (4.33) and (4.49).
It is thus apparent that the mixed formalism contains all of the same integrals and factors as the momentum-space formalism, but that these are packaged differently between vertex and propagator Feynman rules. By identifying features between the two pictures one may glean physical intuition in one picture that is not manifest in the other. For example, the observation in the mixed formalism that each bulk vertex on a loop brings down a power of $1 / k$ is straightforwardly understood to be a manifestation of momentum conservation in the momentum space picture.

On the other hand, the mixed formalism is much more intuitive for brane-localized effects. Interactions with fields on the brane at $z=L$ carry $\delta(z-L)$ factors in the vertex Feynman rules. Such interactions violate momentum conservation in the $z$-direction. In the KK formalism this manifests itself as the question of when it is appropriate to sum over an independent tower of кк modes. This is easily quantified in the mixed formalism since the $d z$ integrals are not yet performed in the Feynman rules and we may directly insert $\delta(z-L)$ terms in the expression for the amplitude.

As a concrete example, consider the loop diagram with three vertices shown in Figure 5.D.1. It is instructive to explicitly work out loop $z$-momentum structure of this diagram in the case where all vertices are in the bulk and observe how this changes as vertices are localized on the brane. To simplify the structure, let us define the product of momentum-space propagators

$$
\begin{equation*}
f\left(k_{1}, k_{2}, k_{3}\right) \equiv \prod_{i=1}^{3} \frac{i}{k_{i}^{2}-\left(k_{i}^{2}\right)^{2}} \tag{5.123}
\end{equation*}
$$

Using $\int d z \exp (i z k)=\delta(k)$, the bulk amplitude is proportional to

$$
\begin{align*}
\mathcal{M} & \sim \int d z_{1} d z_{2} d z_{3} d k_{1}^{z} d k_{2}^{z} d k_{3}^{z} f\left(k_{1}, k_{2}, k_{3}\right) e^{i z_{1}\left(k_{1}+p_{1}-k_{2}\right)^{z}} e^{i z_{2}\left(k_{2}+p_{3}-k_{3}\right)^{z}} e^{i z_{3}\left(k_{3}+p_{3}-k_{1}\right)^{z}} \\
& \sim \int d z_{2} d z_{3} d k_{2}^{z} d k_{3}^{z} f\left(k_{2}-p_{1}, k_{2}, k_{3}\right) e^{i z_{2}\left(k_{2}+p_{3}-k_{3}\right)^{z}} e^{i z_{3}\left(k_{3}+p_{3}-k_{2}+p_{1}\right)^{z}} \\
& \sim \int d z_{3} d k_{3}^{z} f\left(k_{3}-p_{2}-p_{1}, k_{3}-p_{2}\right) e^{i z_{3}\left(p_{1}+p_{2}+p_{3}\right)^{z}} \tag{5.126}
\end{align*}
$$

We have implicitly performed the associated $d^{(d-1)} x$ integrals at each step. The final $d z_{3}$ integral gives the required $\delta$-function of external momenta while leaving an unconstrained $d k_{3}^{z}$ loop integral. Each $d k^{z} /\left(k^{2}-k_{z}^{2}\right) \sim 1 / k$ represents the entire кк tower associated with an internal line. The removal of two $d k^{z}$ integrals by $\delta$-functions is a manifestation of the $1 / k$ suppression coming from each $d z$ integral with the caveat that the "last" $d z$ integral only brings down powers of external momenta and hence does not change the power of loop momenta. This explains the "overall $z$-momentum" contribution to the superficial degree of divergence in Section 5.6.2.

Next consider the case when the $z_{3}$ vertex is brane localized so that its Feynman rule is proportional to $\delta\left(z_{3}-L\right)$. This only affects the last line of the simplification by removing the $d z_{3}$ integral. Physically this means that $z$-momentum (KK number) needn not be conserved for this process. Since the $z_{3}$ exponential is independent of any loop momenta, this does not affect the superficial degree of divergence.

On the other hand, if $z_{2}$ is also brane localized, then the $\delta\left(z_{2}-L\right)$ from the vertex prevents the $d z_{2}$ integral in the second line from giving the $\delta\left(k_{2}+p_{2}-k_{3}\right)$ that cancels the $d k_{2}^{z}$ integral. Thus the process has an additional $d k_{2}^{z}$ integral which now increases the degree of divergence. In the 4 D formalism this is manifested as an additional independent sum over KK states. It is now also clear that setting $z_{1}$ to be brane localized prevents the $d k_{1}^{z}$ from being cancelled and hence adds another unit to the degree of divergence. This counting is trivially generalized to an arbitrary number of vertices and different types of internal propagators. For a loop with $V$ vertices, $V_{B}$ of which are in the bulk, the key points are:

1. If $V=V_{B}$, then the $d z$ integrals reduce the superficial degree of divergence by $\left(V_{B}-1\right)$.
2. If, on the other hand, $V>V_{B}$ so that there is at least one brane-localized vertex, then the $d z$ integrals reduce the superficial degree of divergence by $V_{B}$.
Intuitively the $z$-momentum nonconservation coming from brane-localized interactions can be understood as the particle picking up an arbitrary amount of momentum as it bounces off the brane (a similar picture can be drawn for the orbifold [262]). Alternately, it reflects the uniform spread in momentum associated with complete localization in $z$-position. While this may seem to imply sensitivity to arbitrarily high scale physics on the brane, a negative degree of divergence will prevent the loop from being sensitive to uv physics. In other words, we are free to treat brane-localized fields as having $\delta$-function profiles independent of the physics that generates the brane.

Finally, note that we have assumed that each fermion mass insertion is brane localized. In ${ }_{5} \mathrm{D}$ this means that higher-order diagrams in the fermion mass-insertion approximation are not suppressed by momentum since each additional brane-to-brane propagator goes like $\sim k / k$ after accounting for the $d k^{z}$ integrals. Instead, these mass insertions are suppressed only by the relative sizes of the Higgs vEv and compactification scale, $\left(v R^{\prime}\right)^{2} \sim$.o1. It is perhaps interesting to note that our analysis further suggests that in 6 D with a Higgs localized on a 4 D subspace, there are two additional momentum integrals coming from a mass insertion so that each vev-to-vev propagator goes like a positive power of the momentum $\sim k k$ causing the mass-insertion approximation to break down.

## 5.E Finiteness of the brane-localized neutral Higgs diagram

As explained in Section 5.6.4, the finiteness of the one-loop result and logarithmic divergence at two-loop order becomes opaque to naïve ${ }_{5} \mathrm{D}$ power counting arguments when the Higgs is brane-localized. Additional cancellations of leading-order terms in loop momentum are required to sensibly interpolate between the superficial degree of divergence of the bulk and brane-localized scenarios. For the charged Higgs this cancellation mechanism came from an $M_{W}^{2}$ insertion, which led to an
additional $1 / k^{2}$ factor relative to the bulk field. Here we shall elucidate the finiteness of the single-mass-insertion brane-localized neutral scalar loop.

At one-loop order this finiteness can be seen explicitly by the cancellation between the neutral Higgs and the neutral Goldstone. However, there is an additional chiral cancellation that occurs between the two diagrams associated a single intermediate neutral boson. Indeed, because the Higgs and neutral Goldstone do not appear to completely cancel at two-loop order, this additional cancellation is necessary for the power-counting arguments given in Section 5.6.7.

We highlight this cancellation in two ways. The pure momentum space calculation highlights the role of the chiral boundary conditions, while the mixed position/momentum space calculation shows an explicit cancellation while including the full scalar structure the amplitude.

## 5.E. 1 Momentum space

Here we shall see that 4D Lorentz invariance combined with the chiral boundary conditions forces the UV divergence of the two diagrams in Figure 5.6.1 to cancel.

We first note that the propagators to the photon vertex each have an endpoint in the bulk. This implies that the leading-order contributions to these propagators in the UV limit are proportional to the uncompactified flat-space ${ }_{5} \mathrm{D}$ propagators,

$$
\Delta=\left(\begin{array}{cc}
\Delta_{\psi \chi} & \Delta_{\psi \psi}  \tag{5.127}\\
\Delta_{\chi \chi} & \Delta_{\chi \psi}
\end{array}\right) \sim \frac{1}{k^{2}-k_{s}^{2}}\left(\begin{array}{cc}
i k_{5} & k_{\mu} \sigma^{\mu} \\
k_{\mu} \bar{\sigma}^{\mu} & -i k_{5}
\end{array}\right)=\frac{k_{\mu} \gamma^{\mu}+k_{5} \gamma^{5}}{k^{2}-k_{5}^{2}}
$$

where we have written $\Delta_{\psi \chi}$ to mean the propagation of a left-handed Weyl spinor $\chi$ into a right-handed spinor $\psi$. The terms along the diagonal come from $k_{5} \gamma^{5}$ and represent the chirality-flipping part of the propagator. The boundary conditions require the wrong-chirality modes, the $\operatorname{SU}(2)$ doublet $\psi_{L}$ and $\operatorname{SU}(2)$ singlet $\chi_{R}$, to vanish on the IR brane. Thus, the fermion may propagate to the wrong-chirality spinor in the bulk only if it propagates back to the correct-chirality spinor when it returns to the brane. For an internal left-handed Weyl fermion $\chi_{L}$, the portion of the amplitude coming from the photon emission takes the form

$$
\begin{equation*}
\Delta_{\chi \chi} \sigma^{\mu} \Delta_{\chi \chi}+\Delta_{\chi \psi} \bar{\sigma}^{\mu} \Delta_{\psi \chi} \sim\left(k_{\alpha} \bar{\sigma}^{\alpha}\right) \sigma^{\mu}\left(k_{\beta} \bar{\sigma}^{\beta}\right)+\left(k_{5}\right)^{2} \bar{\sigma}^{\mu} . \tag{5.128}
\end{equation*}
$$

Combining with the analogous expression for a right-handed Weyl fermion in the loop, the relevant part of the photon emission amplitude can be written as

$$
\begin{equation*}
\frac{k \gamma^{\mu} k+\left(k_{5}\right)^{2} \gamma^{\mu}}{\left(k^{2}-k_{s}^{2}\right)^{2}} \tag{5.129}
\end{equation*}
$$

where these terms correspond to a fermion of the correct and incorrect chirality propagating into the brane. The second term can be simplified using

$$
\begin{equation*}
\int d k_{5} \frac{\left(k_{5}\right)^{2}}{\left(k^{2}-k_{5}^{2}\right)^{2}}=\int d k_{5} \frac{-k^{2}}{\left(k^{2}-k_{5}^{2}\right)^{2}} \tag{5.130}
\end{equation*}
$$

which can be confirmed by Wick rotating both sides, $k^{2} \rightarrow-k_{E}^{2}$, and performing the $d k_{5}$ integral explicitly. Now it is easy to see that the divergent contributions from the diagrams in Figure 5.6.1 cancel. The boundary conditions force brane-to-brane propagators to go like $k$ with no $\gamma^{5}$ part. Thus we may write the internal fermion structure of the amplitudes as

$$
\begin{equation*}
\mathcal{M}_{(a)}+\mathcal{M}_{(b)} \sim k\left(k \gamma^{\mu} k-k^{2} \gamma^{\mu}\right)+\left(k \gamma^{\mu} k-k^{2} \gamma^{\mu}\right) k=0 . \tag{5.131}
\end{equation*}
$$

The key minus sign between the two terms in the photon emission comes from the chiral boundary conditions that force the second term to pick up the relative sign between the two diagonal blocks of $\gamma^{5}$.

Let us remark that it is crucial that the denominator in ( 5.130 ) contains exactly two propagators or else the equality would not hold. One might be concerned that the brane-to-brane propagator should also contribute an additional factor of $\left(k^{2}-k_{5}^{2}\right)$ to the denominator (the $k_{5} \gamma^{5}$ term vanishes in the numerator from boundary conditions). Such a factor is indeed present in
the full calculation, but because ${ }_{5} \mathrm{D}$ Lorentz invariance is broken on the brane, $k_{5}$ is not conserved there and this factor actually includes a different, uncorrelated fifth momentum component, $\tilde{k}_{5}$, which can be taken the be independent of the $d k_{5}$ integral. This is a manifestation of the principles in Appendix 5.A. As a check, one can perform the $d \tilde{k}_{5}$ integral for this brane-to-brane propagator and obtain the same $k /|k|$ uv behavior found in the careful derivation performed in Appendix 4.4.1.

## 5.E. 2 Position/momentum space

In Appendix 4.4.1 we derived the flat-space bulk fermion propagator,

$$
\Delta\left(p, x_{5}, x_{5}^{\prime}\right)=\left(\not p-i \gamma^{5} \partial_{5}+m\right) \frac{-i \cos \chi_{p}\left(L-\left|x_{5}-x_{5}^{\prime}\right|\right)+\gamma^{5} \wp(X) \cos \chi_{p}\left(L-\left(x_{5}+x_{5}^{\prime}\right)\right)}{2 \chi_{p} \sin \chi_{p} L}
$$

where the zero mode chirality is given by $X=\{L, R\}$ with $\wp(L)=+1$ and $\wp(R)=-1$. We then argued at the end of Appendix 4.4.2 that the propagators in a warped extra dimension reduce to this case up to overall phases. Thus we expect the amplitudes to have the same uv behavior up to finite factors. The relevant flat-space one-loop diagrams contributing to the operator (5.20) are shown in Figure 5.6.1. We start with Figure ?? and assume that the decay is from $\mu_{L}$ to $e_{R}$. The loop propagators with $\left(x_{5}, x_{5}^{\prime}\right)=(L, z),(z, L)$ and $(L, L)$ can be written as

$$
\begin{align*}
\Delta\left(k^{\prime}, L, z\right) & =-i \frac{l k^{\prime \prime} \cos \chi_{k^{\prime}} z-i \gamma^{5} \chi_{k^{\prime}} \sin \chi_{k^{\prime}} z}{\chi_{k^{\prime}} \sin \chi_{k^{\prime}} L} P_{\mathrm{R}}  \tag{5.133}\\
\Delta(k, z, L) & =-i \frac{k \cos \chi_{k} z+i \gamma^{5} \chi_{k} \sin \chi_{k} z}{\chi_{k} \sin \chi_{k} L} P_{R}  \tag{5.134}\\
\Delta(k, L, L) & =-i \frac{k \cos \chi_{k} L}{\chi_{k} \sin \chi_{k} L} P_{R} \tag{5.135}
\end{align*}
$$

where $k^{\prime}=k+q$. We have used the chiral boundary conditions to simplify $\Delta(k, L, L)$. Since we are interested in the UV behavior we have dropped the terms proportional to the bulk mass $m$ from the internal propagators because these are finite. Combining the propagators together and doing the same calculation for Figure ??, the amplitudes become

$$
\begin{align*}
& \mathcal{M}_{(a)}^{\mu}=\int \frac{d^{4} k}{(2 \pi)^{4}} d z \bar{u}\left(p^{\prime}\right)\left\{\frac{k^{\prime \prime} \gamma^{\mu} k f(k, z)+\chi_{k} \chi_{k^{\prime}} \gamma^{\mu} g(k, z)}{\chi_{k} \chi_{k^{\prime}}\left[(p+k)^{2}-m_{H}^{2}\right]}\right\} \frac{k \cot \chi_{k} L}{\chi_{k}} u(p)  \tag{5.136}\\
& \mathcal{M}_{(b)}^{\mu}=\int \frac{d^{4} k}{(2 \pi)^{4}} d z \bar{u}\left(p^{\prime}\right) \frac{k^{\prime \prime} \cot \chi_{k^{\prime}} L}{\chi_{k^{\prime}}}\left\{\frac{k^{\prime \prime} \gamma^{\mu} k f(k, z)+\chi_{k} \chi_{k^{\prime}} \gamma^{\mu} g(k, z)}{\chi_{k} \chi_{k^{\prime}}\left[(p+k)^{2}-m_{H}^{2}\right]}\right\} u(p) \tag{5.137}
\end{align*}
$$

where we have written

$$
\begin{align*}
& f(k, z)=-\frac{\cos \left(\chi_{k+q} z\right) \cos \left(\chi_{k} z\right)}{\sin \chi_{k+q} L \sin \chi_{k} L}  \tag{5.138}\\
& g(k, z)=-\frac{\sin \left(\chi_{k+q} z\right) \sin \left(\chi_{k} z\right)}{\sin \chi_{k+q} L \sin \chi_{k} L} \tag{5.139}
\end{align*}
$$

Note that all of the $z$ dependence is manifestly contained in sines and cosines. Further we have neglected the flavor-dependence of the $\chi_{k}$ factors since these also come from the bulk masses via (4.36) and are negligible in the UV.

Upon Wick rotation the trigonometric functions become hyperbolic functions which are exponentials in the Euclidean
momentum,

$$
\begin{align*}
& \cos \chi_{k} z \rightarrow \cosh \left(\chi_{k_{E}} z\right)=\frac{1}{2}\left(e^{\chi_{k_{E}} z}+e^{-\chi_{k_{E}} z}\right)  \tag{5.140}\\
& \sin \chi_{k} z \rightarrow i \sinh \left(\chi_{k_{E}} z\right)=\frac{i}{2}\left(e^{\chi_{k_{E}} z}-e^{-\chi_{k_{B}} z}\right) . \tag{5.141}
\end{align*}
$$

We may now replace the trigonometric functions with the appropriate Euclidean exponentials. Since we are concerned with the UV behavior, we may drop terms which are exponentially suppressed for large $k$ over the entire range of $z$. The remaining terms are simple exponentials and can be integrated over the interval. One finds that the trigonometric terms in (5.136) and ( 5.137 ) yield the expression

$$
\begin{equation*}
\frac{i}{\chi_{k_{E}+q}+\chi_{k_{E}}} \rightarrow \frac{-1}{\chi_{k+q}+\chi_{k}} \tag{5.142}
\end{equation*}
$$

where on the right we have reversed our Wick rotation to obtain a Minkowski space expression for the terms which are not exponentially suppressed in Euclidean momentum. After doing this, the leading order term in cot $\chi \mathrm{L}$ in (5.136) and (5.137) equals $i^{-1}$ and the terms in the braces become

$$
\begin{equation*}
\left\{\frac{(k+q) \gamma^{\mu} k-\chi_{k+q} \chi_{k} \gamma^{\mu}}{\chi_{k} \chi_{k+q}\left(\chi_{k}+\chi_{k+q}\right)\left[(p+k)^{2}-m_{H}^{2}\right]}\right\} \tag{5.143}
\end{equation*}
$$

which gives the numerator of (5.131).
In terms of these quantities the potentially divergent amplitudes can be written as

$$
\begin{align*}
& \mathcal{M}_{(a)}^{\mu}=\int \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{\left(\chi_{k+q}+\chi_{k}\right)\left[(p+k)^{2}-m_{H}^{2}\right]} \bar{u}(p)\left\{\frac{(k+q)}{\chi_{k+q}} \gamma^{\mu}-\gamma^{\mu} \frac{\not k}{\chi_{k}}\right\} u(p+q) \\
& \mathcal{M}_{(b)}^{\mu}=\int \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{\left(\chi_{k+q}+\chi_{k}\right)\left[(p+k)^{2}-m_{H}^{2}\right]} \bar{u}(p)\left\{\gamma^{\mu} \frac{k}{\chi_{k}}-\frac{(k+q)}{\chi_{k+q}} \gamma^{\mu}\right\} u(p+q) \tag{5.145}
\end{align*}
$$

therefore these two terms cancel each other in the UV and the operator (5.20) is finite.
Higher mass insertions do not spoil this cancellation since these are associated with internal brane-to-brane propagators whose UV limit goes like $\Delta(k) \sim \nless / \chi_{k}$. The chiral structure of the effective operator $(5.20)$ requires that only diagrams with an odd number of mass insertions contribute. Using the UV limit $\Delta(k)^{2} \rightarrow 1$ one notes that the divergence structure reduces to the case above.

Is it this, + ? Or this, - ? Maybe this =? No, the most impor-
tant symbol is this $\sim$. Tell me the order of magnitude, the [scaling].
That is the physics.
Yuval Grossman, 21 August 2008


## Warped hadronic penguins

In the previous chapter we explored the effects of a 'warped' extra dimension on the simplest penguin processes involving leptons. In this chapter we extend that analysis to the phenomenology of quarks. Unlike the previous chapter, quarks are stuck in bound states so that their penguin transitions must be unraveled from the interactions of the hadrons that contain them.

### 6.1 Overview:

We calculate contributions to the photon and gluon magnetic dipole operators that mediate $b \rightarrow s \gamma$ and $b \rightarrow d \gamma$ transitions in the Randall-Sundrum model of a warped extra dimension with anarchic bulk fermions and a brane localized Higgs. Unlike the Standard Model, there are large contributions to the left-handed $b$ quark decays, parameterized by the Wilson coefficient $C_{7}^{\prime}$, due to the pattern of bulk fermion localization, and sizable contributions from the gluonic penguins, $\mathrm{C}_{8}^{(\prime)}$, through renormalization group mixing. Further, unlike the Randall-Sundrum result for $\mu \rightarrow e \gamma$, the unprimed Wilson coefficients receive non-negligible contributions from the misalignment of the bulk fermion spectrum with the Standard Model flavor sector. We compare the size of effects and the constraints imposed by the branching ratios $\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)$ and $\left\langle\operatorname{Br}\left(B \rightarrow X_{d} \gamma\right)\right\rangle$ within the minimal and the custodial model. Within the custodial framework, we study the effect on a number of benchmark observables and find that $\operatorname{Br}\left(B \rightarrow X_{s} \mu^{+} \mu^{-}\right)$and the forward-backward asymmetry in $B \rightarrow K^{*} \mu^{+} \mu^{-}$remain close to their Standard Model predictions. On the other hand, there can be large enhancements of the time-dependent CP asymmetry in $B \rightarrow K^{*} \gamma$ and the transverse asymmetry $A_{T}^{(2)}$.

### 6.2 INTRODUCTION

The Randall-Sundrum (RS) scenario of a warped extra dimension provides an elegant solution to the hierarchy problem [175, 188-190, 219,220] and a way to understand strongly coupled dynamics through the AdS/CFT correspondence [197, 200, 203, 219]. For reviews see [176, 177, 180, 201]. One of the promising phenomenological features to come out of this framework is an explanation of the Standard Model (SM) flavor structure through the split-fermion scenario [189, 190, 224, 263]. In these models the Yukawa matrices are anarchic and the spectrum of fermion masses is generated by the exponential suppression of zero mode wavefunctions with a brane-localized Higgs [226]. This also
automatically generates hierarchical mixing angles [226-228] and suppresses many tree-level flavor-changing neutral currents (FCNCS) through the RS-GIM mechanism [226]. In order to protect against large contributions to the $T$ parameter coming from bulk gauge fields, one may introduce a gauged custodial symmetry [191] that is broken on the boundaries; a straightforward discrete extension of such a symmetry also protects against corrections to the $Z b \bar{b}$ vertex [211,212] and flavor changing couplings of the $Z$ boson to left-handed down-type quarks [209, 234].

These flavor protection mechanisms are not always sufficient to completely protect rs models from stringent experimental flavor constraints. In the quark sector, the tree-level exchange of Kaluza-Klein (KK) gluons and neutral electroweak gauge bosons contributes to meson-antimeson mixing and induces left-right operators. These operators are not present in the SM and receive a significant enhancement through QCD effects due to their large anomalous dimension. In the kaon system they are also chirally enhanced by a factor of $m_{K}^{2} / m_{s}^{2}$. These contributions lead to new CP violating effects in the kaon system, namely the well-measured observable $\varepsilon_{\mathrm{K}}$, and result in generic bounds of $\mathcal{O}(10-20 \mathrm{Tev})$ for the KK gluon mass [208, 209, 230, 231,233,264, 265]. To reduce this bound, one must introduce additional structure such as horizontal symmetries [235,236], flavor alignment [237,238], or an extended strong sector [266]. Alternately, one may promote the Higgs to a bulk field [239] to localize the fermion zero modes closer to the UV brane.

Additional constraints on the rS flavor sector come from loop-induced dipole operators through penguin diagrams. The first estimates for these operators were performed in [225-227,241] assuming UV sensitivity at all loops within the ${ }_{5} \mathrm{D}$ effective theory and a calculation within the two-site approach was performed in [239]. In [226] the bound $M_{\mathrm{KK}}>\mathcal{O}(10 \mathrm{Tev})$ was derived from the constraint on the neutron electric dipole moment. The rS dipole contributions lead to dangerously large effects in direct CP violation in the $K \rightarrow \pi \pi$ decays measured by the ratio $\varepsilon^{\prime} / \varepsilon$ [240]. Combining the bound from the latter ratio with the $\varepsilon_{K}$ constraint leads to a lower bound on the KK scale independent of the strength of the ${ }_{5 D}$ Yukawa. More recently it was shown that even for the brane Higgs scenario the one-loop induced magnetic penguin diagrams are finite in RS and can be calculated effectively in a manifestly ${ }_{5} \mathrm{D}$ formalism [2]. The lepton flavor violating penguin $\mu \rightarrow e \gamma$ sets bounds on the кк and anarchic Yukawa scales that are complementary to tree-level processes, so the tension between these bounds quantifies the degree of tuning required in the ${ }_{5}$ D Yukawa matrix [241].

In this chapter we examine the calculation and phenomenological observables of the quark sector processes $b \rightarrow q \gamma$ ( $q=s, d$ ) in the Rs framework with a brane-localized Higgs field using the mixed position-momentum space formalism. These processes differ from their leptonic analogs for various reasons beyond the spectrum and diagrams involved. Firstly, while the branching ratio of $\mu \rightarrow e \gamma$ is only bounded from above, the branching ratios for $B \rightarrow X_{s} \gamma$ and, to a lesser extent, $B \rightarrow X_{d} \gamma$ are well-measured and in good agreement with the SM. Secondly, theoretical predictions are more involved due to the renormalization group ( RG ) evolution from the кк scale to the $B$ meson scale and hadronic effects at the latter scale. The RG running over this large range of energy scales introduces a sizable mixing between the various effective operators, so that one must also include the effects of the magnetic gluon penguin $C_{8}^{(\prime)}$ in addition to the magnetic photon penguin $C_{7}^{(\prime)}$.

We calculate the $C_{7}^{(\prime)}$ and $C_{8}^{(\prime)}$ Wilson coefficients of the quark dipole operators in Section 6.3. We provide explicit formulae for the dominant RS contributions to the Wilson coefficients at the KK scale in both the minimal and custodial models and analyze the size of these contributions. In Sections 6.4 and 6.5 , we subsequently perform the RG evolution down to the $B$ meson scale and obtain predictions for the branching ratios $\operatorname{Br}\left(B \rightarrow X_{s, d} \gamma\right)$.

Finally, in Section 6.6, we investigate the phenomenological implications on a number of benchmark observables related to the photon and gluon penguin operators. We first show that these operators give non-negligible constraints for both minimal and custodial models. We then restrict our attention to realistic models with a bulk custodial symmetry $S U(2)_{L} \times S U(2)_{R} \times U(1)_{X} \times P_{L R}$ and consider the effect of benchmark observables on points in parameter space that pass tree-level constraints as evaluated in [231]. Rather than performing a detailed analysis of all observables provided by the $B \rightarrow X_{s} \gamma, B \rightarrow K^{*} \gamma, B \rightarrow X_{s} \mu^{+} \mu^{-}$and $B \rightarrow K^{*} \mu^{+} \mu^{-}$decay modes, we focus on a number of benchmark observables in order to illustrate the pattern of effects and leave a more detailed analysis for future work. Specifically we study:

- The branching ratio $\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)$ and the CP averaged branching ratio $\left\langle\operatorname{Br}\left(B \rightarrow X_{d} \gamma\right)\right\rangle$ which we impose as constraints on our parameter scan.
- The branching ratio $\operatorname{Br}\left(B \rightarrow X_{s} \mu^{+} \mu^{-}\right)$and the forward backward asymmetry $A_{\mathrm{FB}}$ in $B \rightarrow K^{*} \mu^{+} \mu^{-}$. Stringent data that are in good agreement with the SM exist for both observables, placing strong bounds on various new physics (NP) scenarios. The custodial rs model naturally predicts small effects in these observables since they are rather insensitive to NP contributions to the primed magnetic Wilson coefficients.
- The time-dependent CP asymmetry $S_{K^{*} \gamma}$ in $B \rightarrow K^{*} \gamma$ and the transverse asymmetry $A_{T}^{(2)}$ in $B \rightarrow K^{*} \mu^{+} \mu^{-}$, evaluated in the region of low dimuon invariant mass $1 \mathrm{GeV}^{2}<q^{2}<6 \mathrm{GeV}^{2}$.

Since the RS contributions generally exhibit the hierarchy $\Delta C_{7}^{\prime} \gg \Delta C_{7}[225,226]$ the latter observables are particularly suited to look for RS contributions. CP asymmetries in radiative $B$ decays were already suggested in [225,226] as good probes to look for RS effects. We quantify the possible size of effects and study the possible rS contributions to the various observables in a correlated manner. We also included the transverse asymmetry $A_{T}^{(2)}$, which has not been considered in the context of RS models before.

### 6.3 Calculation of the $b \rightarrow q \gamma$ Penguin in rs

We now calculate the RS contributions to the $b \rightarrow q \gamma$ and $b \rightarrow q g(q=d, s)$ decays. These contributions are calculated at the KK scale $M_{\mathrm{KK}} \sim{ }_{1} / R^{\prime}$; in subsequent sections we will relate these to renormalization group (RG) evolved coefficients and observables at the low scale $\sim m_{b}$.

We only evaluate the dominant diagrams, working in Feynman gauge and the mass insertion approximation, where the expansion parameter is $v R^{\prime} / \sqrt{2} \sim \mathcal{O}(0.1)$. We have checked explicitly that the diagrams presented here dominate those that were neglected by at least an order of magnitude; a more detailed calculation is beyond the scope of this work and, in our opinion, premature before the discovery of rS KK modes. We refer to [ 2 ] for details of the ${ }_{5}$ D calculation, Feynman rules, and guidelines for estimating the dominant diagrams. For additional notation and conventions, especially with respect to the custodially protected model, see [209]. See Appendix 6.D for comments on theory uncertainties.

### 6.3.1 Effective Hamiltonian for $b \rightarrow q \gamma$ transitions

The $b \rightarrow q \gamma(q=d, s)$ transitions are most conveniently described by an effective Hamiltonian in the operator product expansion, see e.g. [134] for a review. The dipole terms most sensitive to new physics are

$$
\begin{equation*}
\mathcal{H}_{\mathrm{eff}}=-\frac{G_{F}}{\sqrt{2}} V_{t q}^{*} V_{t b}\left[C_{7}(\mu) Q_{7}(\mu)+C_{7}^{\prime}(\mu) Q_{7}^{\prime}(\mu)+C_{8}(\mu) Q_{8}(\mu)+C_{8}^{\prime}(\mu) Q_{8}^{\prime}(\mu)\right]+\text { h.c. } \tag{6.1}
\end{equation*}
$$

where we neglect terms proportional to $V_{u q}^{*} V_{u b}$. The effective operators are

$$
\begin{array}{ll}
Q_{J}=\frac{e}{4 \pi^{2}} m_{b}\left(\bar{q} \sigma_{\mu \nu} P_{R} b\right) F^{\mu \nu} & Q_{\overline{7}}^{\prime}=\frac{e}{4 \pi^{2}} m_{b}\left(\bar{q} \sigma_{\mu \nu} P_{L} b\right) F^{\mu \nu} \\
Q_{s}=\frac{g_{s}}{4 \pi^{2}} m_{b}\left(\bar{q} \sigma_{\mu \nu} T^{a} P_{R} b\right) G^{\mu v, a} & Q_{s}^{\prime}=\frac{g_{s}}{4 \pi^{2}} m_{b}\left(\bar{q} \sigma_{\mu \nu} T^{a} P_{L} b\right) G^{\mu \nu, a}
\end{array}
$$

where $P_{L, R}=\left(1 \mp \gamma^{5}\right) / 2$. In this document we will focus on new contributions from the RS model to these operators. There are also contributions from non-dipole operators $Q_{, \ldots, 6}$ and their chirality-flipped (primed) counterparts, but these are far less sensitive to NP and can be assumed to be equal to their SM contributions ${ }^{1}$.

At leading order in the $S M$, the primed Wilson coefficients $\mathrm{C}_{7,8}^{\prime}$ are suppressed by $m_{s} / m_{b}$ and therefore negligible, so the relevant Wilson coefficients at the scale $M_{W}$ are

$$
\begin{equation*}
C_{7}^{S M}\left(M_{W}\right)=-\frac{1}{2} D_{0}^{\prime}\left(x_{t}\right), \quad C_{8}^{S M}\left(M_{W}\right)=-\frac{1}{2} E_{0}^{\prime}\left(x_{t}\right), \tag{6.4}
\end{equation*}
$$

where $x_{t}=m_{t}^{2} / M_{W}^{2}$, and $D_{\circ}^{\prime}\left(x_{t}\right) \approx 0.37$ and $E_{o}^{\prime}\left(x_{t}\right) \approx 0.19$ are loop functions given explicitly in (3.15-3.16) of [135]. In what follows we refer to the rS contributions to these operators as $\Delta C_{7,8}^{(\prime)}$.

[^6]
### 6.3.2 Structure of the amplitude

In order to calculate the $b \rightarrow(s, d) \gamma$ and $b \rightarrow(s, d) g$ penguins, we work in a manifestly ${ }_{5} \mathrm{D}$ framework. Unlike the 4D KK reduction, this procedure automatically incorporates the entire KK tower $^{2}$ at the cost of an expansion with respect to the Higgs-induced mass term $\left(\sim v R^{\prime}\right)$.

Using the on-shell condition for the photon, the general form of the left-to-right chirality $f_{i}^{L}(p) \rightarrow f_{j}^{R}\left(p^{\prime}\right) \gamma$ amplitude, $\mathrm{C}_{7}$, in a ${ }_{5}$ D theory can be written as $[2,255$ ]

$$
\begin{equation*}
\mathcal{M}_{L^{i} \rightarrow R^{j}}=\frac{i e}{16 \pi^{2}} \frac{\nu R^{\prime 2}}{\sqrt{2}} \sum_{k, \ell}\left(a_{k \ell} Y_{i k}^{\dagger} Y_{k \ell} Y_{\ell j}^{\dagger}+b_{i j} Y_{i j}^{\dagger}\right) f_{Q_{i}:} f_{D_{j}} \bar{u}_{p^{\prime}}^{R}\left[\left(p+p^{\prime}\right)^{\mu}-\left(m_{b}+m_{q}\right) \gamma^{\mu}\right] u_{p}^{L} \varepsilon_{\mu} \tag{6.5}
\end{equation*}
$$

where $\varepsilon$ is the photon polarization. The chirality flipped amplitude is given by the conjugate of this result, $\mathcal{M}_{R^{i} \rightarrow L^{j}}=\left(\mathcal{M}_{L^{j} \rightarrow R^{i}}\right)^{\dagger}$. The expression for the gluon penguin is analogous with the appropriate substitutions. Using the fermion equations of motion, the term in the square brackets gives the required dipole structure $\sigma^{\mu \nu} F_{\mu v}$, so a simple way to identify the gauge-invariant contribution to the amplitude is to determine the coefficient of the $\left(p+p^{\prime}\right)^{\mu}$ term [255]. In [2] this observation was used to show the manifest one-loop finiteness of these dipole transitions in 5 D theories. Matching (6.5) to the effective Hamiltonian (6.1) yields expressions for the RS contributions to the Wilson coefficients, $\Delta C$.

We refer to the coefficients $a_{k \ell}$ and $b_{i j}$ in (6.5) as the anarchic and the misalignment contributions, respectively. They are products of couplings and dimensionless integrals whose flavor indices reflect the bulk mass dependence of internal propagators. Upon diagonalizing the SM fermion mass matrix, the anarchic term $a$ is not diagonalized and generally remains anarchic. On the other hand, in the limit where the bulk masses are degenerate, the flavor structure of the $b$ term is aligned with the SM Yukawa matrices and thus contains no flavor-changing transitions in the mass basis [226,239,241]. This alignment is pronounced for the first and second generation fermions because their bulk masses are nearly degenerate, but special care is required for the third generation quarks since these are localized towards the IR brane. The physical contribution of the $b$ coefficient comes from the robustness of off-diagonal elements of $b_{i j} Y_{i j} f_{Q_{j}} f_{D_{j}}$ after passing to the basis in which $Y_{i j} f_{Q_{i}} f_{D_{j}}$ is diagonalized. Contrary to the usual assumption of Yukawa anarchy, the overall size of the $b$ term depends on the misalignment of the specific anarchic Yukawa matrix relative to the set of bulk masses as flavor spurions. One expects diagrams with internal zero modes to give the dominant contributions, since these are the most sensitive to the bulk mass spectrum and hence robust against diagonalization; this intuition is confirmed by our numerical scans. One measure of this effect is the $1 \sigma$ standard deviation from $b=o$ in a scan over random anarchic matrices [2]; we use this to identify the dominant contributions to this misalignment term.

By assumption, the anarchic contribution is independent of the SM flavor sector, so there is no analogous alignment suppression to the $a$ coefficient. However, depending on the internal modes in the loop, each diagram contributing to this term carries one of two possible independent flavor spurions that can be built out of the Yukawa matrices that may enter this product: $Y_{u}^{\dagger} Y_{u} Y_{d}^{\dagger}$ and $Y_{d}^{\dagger} Y_{d} Y_{d}^{\dagger}$. These matrices may have arbitrary relative phase, so the two terms may add either constructively or destructively. The misalignment contribution is a third independent flavor spurion, which also carries a relative phase dependent on the particular choice of parameters.

We express the anarchic ( $a$ ) and misalignment (b) coefficients in terms of dimensionless integrals, which are defined in Appendix 6.A. To explicitly demonstrate the calculation of diagrams in the 5 D mixed position/momentum space formalism, we present a sample calculation of the anarchic contribution to $C_{7}$ in Appendix 6.B. The $C_{8}$ diagrams where a gluon is emitted from an internal gluon have integral results that are typically $\mathcal{O}(1)$ while the integrals for the other diagrams are typically $\mathcal{O}\left(10^{-1}\right)$ in magnitude. Note that the contribution to $a$ from each diagram matches what is expected from a naive dimensional analysis. This is in contrast to the analogous calculation for $\mu \rightarrow e \gamma$, where the leading diagrams are smaller than the naive estimated size. There are thus no problems with the two-loop contribution yielding a larger contribution than expected from the perturbative expansion.

Below we present the calculation for the right-to-left chirality (unprimed) Wilson coefficients $\Delta C_{7,8}$ for $b \rightarrow q$; the left-to-right chirality (primed) Wilson coefficients are obtained by Hermitian conjugation of the $q \rightarrow b$ amplitude. The anarchic contribution to the left-to-right chirality coefficients are enhanced over the right-to-left coefficients by a factor of

[^7]$f_{b_{L}} / f_{b_{R}}$, while the misalignment contribution is of the same order of magnitude. This behavior is explained qualitatively in Appendix 6.C and demonstrated numerically in Section 6.6.

### 6.3.3 Calculation of $\Delta C_{7}^{(\prime)}$

Figure 6.3.1 shows the dominant contributions to the $C_{7}$ photon penguin operator. The RS contribution to the $b \rightarrow q \gamma$

$H^{-}$
(a) Charged Goldstone loop


G


G
(b) Gluon $\left(G_{\mu}\right.$ or $\left.G_{5}\right)$ loops with a single mass insertion

Figure 6.3.1: Leading contributions to the anarchic (a) and misalignment (b) terms of the $C_{7}$ Wilson coefficient. Arrows indicate $\mathrm{SU}(2)_{\mathrm{L}}$ representation; this is equivalent to labeling the chirality of the zero mode for SM fields. Here $Q, U$ and $D$ denote the 5 D chiral fermion fields containing the $s m$ left-handed doublets and righthanded up and down singlets, respectively. $H^{-}$is the charged component of the Higgs doublet that serves as the Goldstone boson of $W^{-}$after electroweak symmetry breaking, and $G$ is the ${ }_{5} \mathrm{D}$ gluon field. Additional diagrams related by exchanging the order of the mass insertion and photon emission are left implicit.

Wilson coefficient is

$$
\begin{equation*}
\Delta C_{7}=\frac{-v R^{\prime 2}}{8 m_{b} G_{F}}\left(V_{t q}^{*} V_{t b}\right)^{-1} \sum_{i j k \ell}\left(U_{q i}^{D_{L}}\right)^{\dagger} f_{Q_{i}^{d}} f_{D_{j}}\left[\sum_{k, \ell} a_{k k} Y_{i k}^{u \dagger} Y_{k \ell}^{u} Y_{\ell j}^{d \dagger}+b_{i j} Y_{i j}^{d \dagger}\right] U_{j b}^{D_{\mathrm{R}}} . \tag{6.6}
\end{equation*}
$$

$U^{D_{L, R}}$ are the rotation matrices between the ${ }_{5} \mathrm{D}$ gauge and the light down quark mass bases.
Note that throughout our analysis we use the tree level matching condition for the ${ }_{5} \mathrm{D}$ gauge couplings and neglect possible brane kinetic terms that may alter this matching. While this affects the misalignment contribution to $C_{7}^{(\prime)}$ and the calculation for $C_{8}^{(\prime)}$, the anarchic contribution to $C_{7}^{(\prime)}$, containing only one gauge coupling vertex instead of three, remains relatively unaffected. Since the latter gives the dominant contribution to the observables discussed in section 6.6, we do not expect this assumption to have a significant impact on our predictions.

## $\Delta C_{7}$ : ANARCHIC CONTRIBUTION

The dominant anarchic contribution is the diagram with one mass insertion and a charged Higgs (Goldstone) in the loop, Figure 6.3.1a. Note that this diagram is not present in the analogous leptonic penguin, which has a neutrino in the loop. The analogous diagram with the photon emitted off the charged Higgs propagator is found to be suppressed by a factor of $\left(m_{W} R^{\prime}\right)^{2} \sim 10^{-2}$ due to an algebraic cancellation [2], while the one with the mass insertion on an external fermion leg is suppressed by $m_{q} / v$ since the external brane-to-brane fermion propagator must be a zero-mode. All other diagrams contain two additional mass insertions-necessary to obtain the required structure of a product of three Yukawas-and are therefore
also suppressed by a factor of $\left(\nu R^{\prime} / \sqrt{2}\right)^{2} \sim 10^{-2}$. Of these neglected diagrams, the next-to-leading diagrams contributing to this coefficient are gluon loops with three mass insertions, which carry a gauge coupling enhancement of $g_{s}^{2} \ln R^{\prime} / R \approx 36$ but are suppressed due to the two additional mass insertions, the quark charge ( $Q_{d}=-1 / 3$ ), and different topologies; they are $\sim 5 \%$ corrections to the leading contribution. Note that these diagrams carry an independent flavor structure ( $Y_{d}^{\dagger} Y_{d} Y_{d}^{\dagger}$ ) and can interfere either constructively or destructively with Figure 6.3.1a.

The value for the $a$ coefficient in (6.6) coming from the penguin in Figure 6.3.1a is a dimensionless integral whose explicit form is given in (6.63),

$$
\begin{equation*}
a=Q_{u} I_{C_{7 a}}, \tag{6.7}
\end{equation*}
$$

where $Q_{u}=2 / 3$ is the charge of the internal up-type quark.

## $\Delta C_{7}$ : MISALIGNMENT CONTRIBUTION

The dominant misalignment contributions come from gluon diagrams with a single mass insertion. As shown in Figure 6.3.1b, this insertion can either be on an internal or external fermion line. All other diagrams contain electroweak couplings and hence are subdominant. The final misalignment contribution in (6.6) is

$$
\begin{equation*}
b=Q_{d} \frac{4}{3}\left(g_{s}^{2} \ln \frac{R^{\prime}}{R}\right) I_{C_{7 b}} . \tag{6.8}
\end{equation*}
$$

Here $Q_{d}$ is the charge of the internal down-type quark, $4 / 3$ is a color factor, $\ln R^{\prime} / R$ is a warp factor associated with bulk gauge couplings, and $I_{C_{7 b}}$ is a dimensionless integral defined in (6.64).

### 6.3.4 Calculation of $\Delta C_{8}^{(\prime)}$

The gluon penguin operators $C_{8}$ and $C_{8}^{\prime}$ differ from their photon counterparts due to additional QCD vertices available and the magnitude of the QCD coupling, $g_{s D}^{2} / R=g_{s}^{2} \ln R^{\prime} / R \approx 36$. Because of this, the dominant diagrams contributing to $b \rightarrow q g$ cannot be obtained from $b \rightarrow q \gamma$ by simply replacing the photon with a gluon in the leading diagrams for $C_{7}^{(\prime)}$. The general expression for $\Delta C_{8}$ is the same as that for $\Delta C_{7}$ in (6.6), with coefficients $a$ and $b$ coming from the diagrams shown in Figure 6.3.2.


Figure 6.3.2: Leading contributions to the $a$ and $b$ terms of the $C_{8}$ Wilson coefficient following the notation of Figure 6.3.1. $G_{\mu}$ refers to only the gluon four-vector.

## $\Delta C_{8}$ : ANARCHIC CONTRIBUTION

There are two classes of dominant contributions to the anarchic (a) coefficient in $\mathrm{C}_{8}^{(\prime)}$. In addition to the charged Higgs diagrams analogous to Figure 6.3.1a, there are gluon diagrams with three mass insertions on the fermion lines, which are now sizable due to the size of the strong coupling constant and the three-point gauge boson vertex (as mentioned earlier, the dimensionless integral associated with this digram is $\mathcal{O}(1)$, while all other diagrams have $\mathcal{O}(0.1)$ integrals $)$. Of the latter class, one only needs to consider diagrams with at most one mass insertion on each external leg since sequential insertions on an external leg are suppressed by factors of $m_{q} R^{\prime}$. Note that these two sets of diagrams contribute with different products of Yukawa matrices; while the Higgs diagrams are proportional to $Y_{u}^{\dagger} Y_{u} Y_{d}^{\dagger}$, the gluon diagrams are proportional to $Y_{d}^{\dagger} Y_{d} Y_{d}^{\dagger}$. Thus these two terms may add either constructively or destructively and may even add with different relative sizes if there is a hierarchy between the overall scale of the up- and down-type ${ }_{5} \mathrm{D}$ anarchic Yukawas. The $a$ coefficient is

$$
\begin{equation*}
a=I_{C_{7 a}} \oplus \frac{3}{2}\left(g_{s}^{2} \ln \frac{R^{\prime}}{R}\right)^{2}\left(\frac{R^{\prime} v}{\sqrt{2}}\right)^{2} I_{C_{8 a}}^{G}, \tag{6.9}
\end{equation*}
$$

where we have written $\oplus$ to indicate that the two terms carry independent flavor spurions. Here $I_{\mathrm{C}_{7 a}}$ is the same dimensionless integral appearing in (6.7). The second term includes color factors, warped bulk gauge couplings, and explicit mass insertions in addition to the dimensionless integral $I_{C_{8 \alpha}}$ defined in (6.67).

## $\Delta C_{8}$ : MISALIGNMENT CONTRIBUTION

The single mass insertion gluon emission diagram in Figure 6.3.2b gives the dominant misalignment term. Additional diagrams with the gluon emission from the quark line are suppressed by a relative color factor of $1 / 6$ versus $3 / 2$ and can be neglected. Diagrams with a scalar $\left(G_{5}\right)$ gluon or the mass insertion on an external leg do not carry an internal fermion zero mode and become negligible after rotation to the mass basis, as discussed earlier. Diagrams with electroweak gauge bosons in the loop are suppressed due to the smaller size of the gauge coupling. The expression for the dominant diagram is

$$
\begin{equation*}
b=\frac{3}{2}\left(g_{s}^{2} \ln \frac{R^{\prime}}{R}\right) I_{C_{s b}} \tag{6.10}
\end{equation*}
$$

with $I_{C_{s b}}$ defined in (6.71). We have again pulled out an explicit color factor and the warped bulk gauge coupling.

### 6.3.5 MODIFICATIONS FROM CUSTODIAL SYMMETRY

In models with a gauged bulk custodial symmetry, the additional matter content may also contribute to the $b \rightarrow q \gamma(g)$ transitions. By construction, boundary conditions for custodial fermions are chosen such that they have no zero modes. The misalignment (b) coefficients do not receive any significant corrections from custodial diagrams: diagrams with custodial gauge bosons are suppressed due to electroweak couplings, while those with custodial fermions do not carry internal fermion zero modes and become negligible after rotation to the mass basis.

The leading custodial contributions to the anarchic (a) coefficients are shown in Figure 6.3.3; these are the same diagrams that contribute to the anarchic $(a)$ terms of the $C_{7}$ and $C_{8}$ Wilson coefficients and now appear with additional custodial fermions, denoted by $U^{\prime}, U^{\prime \prime}$, and $D^{\prime}$. These are the only custodial diagrams that give contributions comparable to those in Figure 6.3.1 and Figure 6.3.2. The remaining diagrams consist of $W$ and $Z$ loops, which, as mentioned earlier, are suppressed by a factor of $\sim 10^{-2}$ relative to the Higgs loops due to the two additional mass insertions, and remain negligible despite the larger multiplicity due to the extended electroweak sector.

Since the custodial fermions $U^{\prime}, U^{\prime \prime}$, and $D^{\prime}$ have the same IR boundary condition as their SM counterparts but the opposite UV boundary condition, and since the localization of the Higgs pulls the loop towards the IR brane, the contribution of these custodial diagrams is well-approximated by the contributions of their SM counterparts. Since the minimal model diagrams are dominated by the кк fermion contribution, it is reasonable that the custodial modes should contribute approximately equally to the process.


Figure 6.3.3: Additional custodial diagrams contributing to the $C_{7}$ and $C_{8}$ coefficients.

Observe that each of these custodial contributions is proportional to $Y_{d}^{\dagger} Y_{d} Y_{d}^{\dagger}$. In particular, the custodial Higgs diagrams carry a flavor structure that is independent of that of their minimal model counterparts. Also, note that the $U^{\prime}$ and $U^{\prime \prime}$ couplings to the charged Higgs come with a factor of $1 / \sqrt{2}$ while the $D^{\prime}$ coupling to the Higgs does not [231]. Thus the additional custodial diagrams contribute an analytic structure that is nearly identical to the minimal model diagrams except for the Yukawa matrices, which now come with the product $Y_{d}^{\dagger} Y_{d} Y_{d}^{\dagger}$. Since this is independent of the $Y_{d}^{\dagger} Y_{u} Y_{u}^{\dagger}$ flavor spurion in the minimal model diagrams, the addition of the custodial diagrams generically enhances the penguin amplitude by less than the factor of two that one would obtain in the limit $Y_{d}=Y_{u}$. This shows that while custodial symmetry can be used to suppress tree-level flavor changing effects in RS models, this comes at the cost of generically enhancing loop-level flavor processes.

### 6.4 Radiative $B$ DECAYS

We now examine the physical observables most directly related to the parton-level $b \rightarrow q(\gamma, g)$ operators derived above: $B$ meson decays with an on-shell photon.

### 6.4.1 The $B \rightarrow X_{s, d} \gamma$ Decay

The sm predictions for the inclusive decays $B \rightarrow X_{s, d} \gamma$ are [269, 270]

$$
\begin{equation*}
\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)_{\mathrm{SM}}=(3.15 \pm 0.23) \cdot 10^{-4}, \quad\left\langle\operatorname{Br}\left(B \rightarrow X_{d} \gamma\right)\right\rangle_{\mathrm{SM}}=\left(15 \cdot 4_{-3.1}^{+2.6}\right) \cdot 10^{-6} . \tag{6.11}
\end{equation*}
$$

These can be compared to the measured values [271]

$$
\begin{equation*}
\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)_{\exp }=(3.55 \pm 0.27) \cdot 10^{-4}, \quad\left\langle\operatorname{Br}\left(B \rightarrow X_{d} \gamma\right)\right\rangle_{\exp }=(14 \pm 5) \cdot 10^{-6} \tag{6.12}
\end{equation*}
$$

Here $\left\langle\operatorname{Br}\left(B \rightarrow X_{d} \gamma\right)\right\rangle$ refers to the CP averaged branching ratio in which the hadronic uncertainties cancel to a large extent [272]. We have extrapolated the experimental value for $\left\langle\operatorname{Br}\left(B \rightarrow X_{d} \gamma\right)\right\rangle$ to the photon energy cut $E_{\gamma}>1.6 \mathrm{GeV}$ used for the theory prediction.

Rather than performing an extensive error analysis, we simply require the new RS contributions to fulfill the constraints

$$
\begin{align*}
\Delta \operatorname{Br}\left(B \rightarrow X_{s} \gamma\right) & =\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)_{\exp }-\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)_{\text {SM }}=(0.4 \pm 0.7) \cdot 10^{-4},  \tag{6.13}\\
\Delta \operatorname{Br}\left(B \rightarrow X_{d} \gamma\right) & =\left\langle\operatorname{Br}\left(B \rightarrow X_{d} \gamma\right)\right\rangle_{\exp }-\left\langle\operatorname{Br}\left(B \rightarrow X_{d} \gamma\right)\right\rangle_{\text {SM }}=-(1 \pm 11) \cdot 10^{-6} . \tag{6.14}
\end{align*}
$$

Neglecting all uncertainties associated with NP contributions, these constraints represent the $2 \sigma$ ranges when combining experimental and theoretical uncertainties in quadrature. Although the data and prediction for $B \rightarrow X_{d} \gamma$ are currently less precise than those for $B \rightarrow X_{s} \gamma$, an important and partly complementary constraint can be obtained from the former decay, as recently pointed out in [270]. Since the data for $B \rightarrow X_{d} \gamma$ lie slightly below the sm prediction, $\Delta \operatorname{Br}\left(B \rightarrow X_{d} \gamma\right)<0$ is
somewhat favored, leaving little room for NP contributing to $C_{7}^{\prime}$. In contrast, a positive nP contribution to $\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)$ is welcome to bring the theory prediction closer to the data. We note that if the tree level values for the CKM parameters are used instead of the SM best fit values, the predicted central value for $\left\langle\operatorname{Br}\left(B \rightarrow X_{d} \gamma\right)\right\rangle_{\text {SM }}$ rises to about $19 \cdot 10^{-6}$, increasing the tension with the data.

### 6.4.2 MASTER FORMULA FOR $\operatorname{BR}\left(B \rightarrow X_{s} \gamma\right)$

Following the strategy of $[267,273,274]$, which use the results of [275], the "master formula" for the inclusive $B \rightarrow X_{s} \gamma$ branching ratio in terms of the SM branching ratio, $\mathrm{Br}_{S M}$, and NP contributions to the Wilson coefficients is

$$
\begin{equation*}
\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)=\operatorname{Br}_{\mathrm{SM}}+0.00247\left[\left|\Delta C_{7}\left(\mu_{b}\right)\right|^{2}+\left|\Delta C_{7}^{\prime}\left(\mu_{b}\right)\right|^{2}-0.706 \operatorname{Re}\left(\Delta C_{7}\left(\mu_{b}\right)\right)\right] \tag{6.15}
\end{equation*}
$$

The rs contributions to $\Delta C_{7}^{(\prime)}\left(\mu_{b}\right)$ are obtained from the RG evolution of $\Delta C_{7}^{(\prime)}$ and $\Delta C_{8}^{(\prime)}$, calculated in Section 6.3 at the high scale $M_{\mathrm{KK}}=2.5 \mathrm{TeV}$, down to the $B$ scale, $\mu_{b}=2.5 \mathrm{GeV}$,

$$
\begin{equation*}
\Delta C_{7}^{(\prime)}\left(\mu_{b}\right)=0.429 \Delta C_{7}^{(\prime)}\left(M_{\mathrm{KK}}\right)+0.128 \Delta C_{8}^{(\prime)}\left(M_{\mathrm{KK}}\right) \tag{6.16}
\end{equation*}
$$

All known SM non-perturbative contributions have been taken into account while the RS contribution is included at leading order neglecting uncertainties. This approach is an approximation to studying the effects of re physics on the decay in question; however, in view of the other uncertainties involved-such as the the mass insertion approximation and taking into account only the leading diagrams - this approach gives sufficiently accurate results to estimate the size of rs contributions. A more accurate and detailed analysis is beyond the scope of our analysis and, in our view, premature before the discovery of RS KK modes.

### 6.4.3 MASTER FORMULA FOR $\left\langle\operatorname{Br}\left(B \rightarrow X_{d} \gamma\right)\right\rangle$

A master formula can be obtained in a similar manner for the CP-averaged $B \rightarrow X_{d} \gamma$ branching ratio. Using the expressions collected in [270, 274, 276] we find

$$
\begin{align*}
\left\langle\operatorname{Br}\left(B \rightarrow X_{d} \gamma\right)\right\rangle=\left\langle\operatorname{Br}_{\mathrm{SM}}\right\rangle+10^{-5} & {\left[1.69\left(\left|\Delta C_{7}\right|^{2}+\left|\Delta C_{7}^{\prime}\right|^{2}\right)+0.24\left(\left|\Delta C_{8}\right|^{2}+\left|\Delta C_{8}^{\prime}\right|^{2}\right)\right.} \\
& +1.06 \operatorname{Re}\left[\Delta C_{7} \Delta C_{8}^{*}+\Delta C_{7}^{\prime} \Delta C_{8}^{\prime *}\right]-3.24 \operatorname{Re}\left(\Delta C_{7}\right) \\
& \left.-0.16 \operatorname{Im}\left(\Delta C_{7}\right)-1.03 \operatorname{Re}\left(\Delta C_{8}\right)-0.04 \operatorname{Im}\left(\Delta C_{8}\right)\right] \tag{6.17}
\end{align*}
$$

where all of the RS contributions to the $b \rightarrow d$ Wilson coefficients $\Delta C_{7,8}^{(\prime)}$ are evaluated at $M_{\mathrm{KK}}$.

### 6.4.4 ANALYTIC ESTIMATE OF CONSTRAINTS

Assuming anarchic Yukawa couplings, one may estimate the size of the RS contributions to the Wilson coefficients in terms of the anarchic coefficients in Section 6.3.3,

$$
\begin{align*}
\left|\Delta C_{7}\left(M_{\mathrm{KK}}\right)^{b \rightarrow s, d \gamma}\right| & \sim \frac{1}{4 \sqrt{2} G_{F}} a Y_{*}^{2} R^{\prime 2} \tag{6.18}
\end{align*} \sim 0.015 a Y_{*}^{2}\left(\frac{R^{\prime}}{1 \mathrm{TeV}^{-1}}\right)^{2}, ~=\frac{1}{4 \sqrt{2} G_{F}} a Y_{*}^{2} R^{\prime 2} \frac{m_{s}}{m_{b}\left|V_{t s}\right|^{2}} \sim 0.18 a Y_{*}^{2}\left(\frac{R^{\prime}}{1 \mathrm{TeV}^{-1}}\right)^{2}, ~=\frac{1}{4 \sqrt{2} G_{F}} a Y_{*}^{2} R^{\prime 2} \frac{m_{d}}{m_{b}\left|V_{t d}\right|^{2}} \sim 0.20 a Y_{*}^{2}\left(\frac{R^{\prime}}{1 \mathrm{TeV}^{-1}}\right)^{2},
$$

where we neglect the misalignment contributions. Here $Y_{*}$ is the average size of the anarchic Yukawa couplings $Y_{i j}$ which we assume to be equal for $Y_{u}$ and $Y_{d}$.

Generically the contribution to the chirality-flipped operator $C_{7}^{\prime}$ is larger than the one to $C_{7}$ by more than an order of magnitude. This is a direct consequence of the hierarchical pattern of quark masses and CKM angles: in order to fit the observed spectrum, the left-handed $b_{L}$ quark has to be localized close to the IR brane, and consequently its flavor violating interactions are far more pronounced than those of the right-handed $b_{R}$.

Neglecting the subdominant contributions from $\Delta C_{7}$ and $\Delta C_{8}^{(1)}$, we can constrain the size of $\Delta C_{7}^{\prime}$ by making use of the data on $\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)$ and $\left\langle\operatorname{Br}\left(B \rightarrow X_{d} \gamma\right)\right\rangle$. We obtain the following constraints from the master formulas and the experimental constraints quoted above:

$$
\begin{equation*}
\left|\Delta C_{7}^{\prime}\left(M_{\mathrm{KK}}\right)^{b \rightarrow s \gamma}\right|<0.47, \quad\left|\Delta \mathrm{C}_{7}^{\prime}\left(M_{\mathrm{KK}}\right)^{b \rightarrow d \gamma}\right|<0.77 . \tag{6.21}
\end{equation*}
$$

Using (6.19-6.20) and $a \sim 0.33$ we can derive an upper bound on the size of the Yukawa couplings, $Y_{*}$,

$$
\begin{array}{ll}
\frac{Y_{*} R^{\prime}}{\mathrm{TeV}^{-1}}<2.8 & \text { from } B \rightarrow X_{s} \gamma \\
\frac{Y_{*} R^{\prime}}{\mathrm{TeV}^{-1}}<3.4 & \text { from } B \rightarrow X_{d} \gamma \tag{6.23}
\end{array}
$$

For $R^{\prime}=1 \mathrm{Tev}^{-1}$ these are of the same order as the perturbativity bound on the Yukawa coupling [230]. We see that the generic constraint from $B \rightarrow X_{s} \gamma$ is slightly stronger than that from $B \rightarrow X_{d} \gamma$ due to the larger uncertainties in the latter case. However, since they only differ by an $\mathcal{O}(1)$ factor, in specific cases the latter constraint may be more restrictive, so one must take both processes into account when constraining the rs parameter space.

### 6.4.5 CP ASYMMETRY IN $B \rightarrow K^{*} \gamma$

Like many extensions of the SM , RS generally induces large CP violating phases. It is thus of great interest to also study CP violation in $b \rightarrow s \gamma$ transitions. While the direct CP asymmetry in the inclusive $B \rightarrow X_{s} \gamma$ decay is in principle highly sensitive to NP contributions, in practice the SM contribution is dominated by long-distance physics and therefore plagued by large non-perturbative uncertainties [277]. Consequently, a reliable prediction in the presence of NP is difficult.

Fortunately, a theoretically much cleaner observable is provided by the $B \rightarrow K^{*} \gamma$ decay. While its branching ratio is plagued by the theoretical uncertainty of the $B \rightarrow K^{*}$ form factors, this form factor dependence largely drops out of the time-dependent CP asymmetry [278-280]

$$
\begin{equation*}
\frac{\Gamma\left(\bar{B}^{\circ}(t) \rightarrow \bar{K}^{* o} \gamma\right)-\Gamma\left(B^{\circ}(t) \rightarrow K^{* o} \gamma\right)}{\Gamma\left(\bar{B}^{\circ}(t) \rightarrow \bar{K}^{* o} \gamma\right)+\Gamma\left(B^{\circ}(t) \rightarrow K^{* o} \gamma\right)}=S_{K^{*} \gamma} \sin \left(\Delta M_{d} t\right)-C_{K^{*} \gamma} \cos \left(\Delta M_{d} t\right) \tag{6.24}
\end{equation*}
$$

The coefficient $S_{K^{*} \gamma}$ is highly sensitive to new RS contributions. At leading order it is given by [279, 281]

$$
\begin{equation*}
S_{K^{*} \gamma} \simeq \frac{2}{\left|C_{7}\right|^{2}+\left|C_{7}^{\prime}\right|^{2}} \operatorname{Im}\left(e^{-i \varphi_{d}} C_{7} C_{7}^{\prime}\right) \tag{6.25}
\end{equation*}
$$

where the Wilson coefficients are to be taken at the scale $\mu_{b} \cdot \varphi_{d}$ is the phase of $B^{\circ}-\bar{B}^{\circ}$ mixing, which has been well measured in $B^{\circ} \rightarrow J / \psi K_{S}$ decays to be $\sin \varphi_{d}=0.67 \pm 0.02[271]$.

From (6.25) we see that $S_{K^{*} \gamma}$ is very sensitive to new phsyics in the chirality flipped operator $C_{7}^{\prime}$ and vanishes in the limit $C_{7}^{\prime} \rightarrow \mathrm{o}$. Consequently the sm prediction is suppressed by the ratio $m_{s} / m_{b}$ and is therefore very small [280],

$$
\begin{equation*}
S_{K^{*} \gamma}^{S \mathrm{M}}=(-2.3 \pm 1.6) \% . \tag{6.26}
\end{equation*}
$$

Measuring a sizable CP asymmetry $S_{K^{*} \gamma}$ would thus not only be a clear sign of physics beyond the $S M$, but unambiguously indicate the presence of new right handed currents. The present experimental constraint [271,282,283],

$$
\begin{equation*}
S_{\mathrm{K}^{*} \gamma}^{\exp }=-16 \% \pm 22 \% \tag{6.27}
\end{equation*}
$$

is still subject to large uncertainties but already puts strong constraints on NP in $b \rightarrow s$ transitions [281]. A significant improvement is expected soon from LhCb, and the next generation $B$ factories will reduce the uncertainty even further.

### 6.5 Semileptonic B decays

Semileptonic $B$ decays such as $B \rightarrow X_{s} \mu^{+} \mu^{-}$and $B \rightarrow K^{*} \mu^{+} \mu^{-}$offer an interesting opportunity to not only look for deviations from the SM, but also to identify the pattern of NP contributions and therewith distinguish various NP scenarios. These decays receive contributions from semileptonic four-fermion operators $(\bar{s} b)(\bar{\mu} \mu)$ in addition to the magnetic dipole operators discussed earlier. While the dipole operators receive rS contributions first at the one-loop level as required by gauge invariance, the four fermion operators are already affected at tree level by the exchange of the $Z$ boson and the heavy electroweak KK gauge bosons.

In this section we discuss the effective Hamiltonian for $b \rightarrow s \mu^{+} \mu^{-}$transitions. Subsequently we will review a number of benchmark observables that are relevant for the study of RS contributions.

### 6.5.1 Effective Hamiltonian for $b \rightarrow s \mu^{+} \mu^{-}$transitions

The effective Hamiltonian for $b \rightarrow s \mu^{+} \mu^{-}$reads

$$
\begin{align*}
\mathcal{H}_{\mathrm{eff}}=\mathcal{H}_{\mathrm{eff}}(b \rightarrow s \gamma)-\frac{G_{F}}{\sqrt{2}} V_{t s}^{*} V_{t b}[ & C_{9 V}(\mu) Q_{9 V}(\mu)+C_{9 V}^{\prime}(\mu) Q_{9 V}^{\prime}(\mu) \\
& \left.+C_{10 \mathrm{~A}}(\mu) Q_{\mathrm{oA}}(\mu)+C_{10 A}^{\prime}(\mu) Q_{10 \mathrm{~A}}^{\prime}(\mu)\right]+ \text { h.c. }, \tag{6.28}
\end{align*}
$$

where we neglect the terms proportional to $V_{u s}^{*} V_{u b}$, and

$$
\begin{array}{ll}
Q_{9 V}=2\left(\bar{s} \gamma_{\mu} P_{L} b\right)\left(\bar{\mu} \gamma^{\mu} \mu\right) & Q_{9}^{\prime}{ }_{V}=2\left(\bar{s} \gamma_{\mu} P_{R} b\right)\left(\bar{\mu} \gamma^{\mu} \mu\right) \\
Q_{10 A}=2\left(\bar{s} \gamma_{\mu} P_{L} b\right)\left(\bar{\mu} \gamma^{\mu} \gamma_{5} \mu\right) & Q_{1 O A}^{\prime}=2\left(\bar{s} \gamma_{\mu} P_{R} b\right)\left(\bar{\mu} \gamma^{\mu} \gamma_{5} \mu\right) .
\end{array}
$$

In the Sm only the unprimed Wilson coefficients are relevant. At the scale $M_{W}$ they are given by

$$
\begin{equation*}
C_{9 V}^{S M}\left(M_{W}\right)=\frac{\alpha}{2 \pi}\left[\frac{Y_{\circ}\left(x_{t}\right)}{\sin ^{2} \theta_{W}}-4 Z_{\circ}\left(x_{t}\right)\right] \quad C_{10 A}^{S M}\left(M_{W}\right)=-\frac{a}{2 \pi} \frac{Y_{\circ}\left(x_{t}\right)}{\sin ^{2} \theta_{W}} \tag{6.31}
\end{equation*}
$$

where $x_{t}=m_{t}^{2} / M_{W}^{2}$ and the dimensionless loop functions $Y_{\circ}\left(x_{t}\right) \approx 0.94$ and $Z_{\circ}\left(x_{t}\right) \approx 0.65$ are explicitly written in (3.27) and (3.28) of [135].

While $C_{7}^{(\prime)}$ and $C_{8}^{(\prime)}$ receive the loop-level RS contributions calculated in Section 6.3, $C_{9 V}^{(\prime)}$ and $C_{10 A}^{(\prime)}$ are corrected at tree level from the new flavor-changing couplings to the $Z$ boson and the exchange of neutral electroweak gauge boson KK modes. In this analysis we only keep the leading contribution to each of these operators, i.e. we consider $\Delta C_{7 \gamma, 8 G}^{(1)}$ at one loop and $\Delta C_{9 V, 10 A}^{(\prime)}$ at tree level. Strictly speaking, such an approach leads to an inconsistent perturbative expansion, but it is reasonable to expect that the one loop corrections to the latter Wilson coefficients are sub-dominant with respect to the tree level contributions, and by considering only the RS tree level contribution one should still capture the dominant np effects.

Explicit expressions for $\Delta C_{9 V}^{(\prime)}$ and $\Delta C_{10 A}^{(\prime)}$ can be straightforwardly obtained from [234]. These expressions can be written in terms of RG invariants $\Delta Y^{(/)}$and $\Delta Z^{(/)}$and the coupling $a$, which itself is only very weakly scale dependent above $M_{W}$. Thus one may use these expressions to directly write the rs contributions at the scale $M_{W}$,

$$
\begin{align*}
\Delta C_{9 V} & =\frac{\alpha}{2 \pi}\left[\frac{\Delta Y_{s}}{\sin ^{2} \theta_{W}}-4 \Delta Z_{s}\right]  \tag{6.32}\\
\Delta C_{9 V}^{\prime} & =\frac{\alpha}{2 \pi}\left[\frac{\Delta Y_{s}^{\prime}}{\sin ^{2} \theta_{W}}-4 \Delta Z_{s}^{\prime}\right]  \tag{6.33}\\
\Delta C_{10 A} & =-\frac{a}{2 \pi} \frac{\Delta Y_{s}}{\sin ^{2} \theta_{W}}  \tag{6.34}\\
\Delta C_{10 A}^{\prime} & =-\frac{a}{2 \pi} \frac{\Delta Y_{s}^{\prime}}{\sin ^{2} \theta_{W}} \tag{6.35}
\end{align*}
$$

The functions $\Delta Y^{(\prime)}$ and $\Delta Z^{(\prime)}$ are given by

$$
\begin{align*}
\Delta Y_{s} & =-\frac{1}{V_{t s}^{*} V_{t b}} \sum_{X} \frac{\Delta_{L}^{\mu \mu}(X)-\Delta_{R}^{\mu \mu}(X)}{4 M_{X}^{2} g_{S M}^{s}} \Delta_{L}^{b s}(X)  \tag{6.36}\\
\Delta Y_{s}^{\prime} & =-\frac{1}{V_{t s}^{*} V_{t b}} \sum_{X} \frac{\Delta_{L}^{\mu \mu}(X)-\Delta_{R}^{\mu \mu}(X)}{4 M_{X}^{2} g_{S M}^{2}} \Delta_{R}^{b s}(X)  \tag{6.37}\\
\Delta Z_{s} & =\frac{1}{V_{t s}^{*} V_{t b}} \sum_{X} \frac{\Delta_{R}^{\mu \mu}(X)}{8 M_{X}^{2} g_{S M}^{2} \sin ^{2} \theta_{W}} \Delta_{L}^{b s}(X)  \tag{6.38}\\
\Delta Z_{s}^{\prime} & =\frac{1}{V_{t s}^{*} V_{t b}} \sum_{X} \frac{\Delta_{R}^{\mu \mu}(X)}{8 M_{X}^{2} g_{S M}^{2} \sin ^{2} \theta_{W}} \Delta_{R}^{b s}(X) \tag{6.39}
\end{align*}
$$

Here the summation runs over $X=Z, Z^{(1)}, A^{(1)}$ in the minimal model and over $X=Z, Z_{H}, Z^{\prime}, A^{(1)}$ in the custodial model. The flavor violating 4 D fermion gauge boson couplings $\Delta_{L, R}^{i j}(X)$ depend on the overlap of the fermion profile with the corresponding gauge boson profile. Their explicit form depends on both the fermion and gauge boson mixing matrices. The explicit expressions are complicated and unilluminating, hence we do not quote them here but refer the reader to Appendix A of [234]. Furthermore

$$
\begin{equation*}
g_{S M}^{2}=\frac{G_{F}}{\sqrt{2}} \frac{a}{2 \pi \sin ^{2} \theta_{W}} \tag{6.40}
\end{equation*}
$$

The tree level contributions to $b \rightarrow s \mu^{+} \mu^{-}$transitions in the minimal RS model are evaluated in [208] without making the approximations of taking into account only the first kK modes or treating the Higgs vacuum expectation value as a perturbation. In this paper we are mainly interested in the effects of $\sim 2.5 \mathrm{TeV}$ KK modes. As these are ruled out in the minimal model by precision electroweak constraints, we focus on the phenomenological effects of the custodial rs model on these transitions.

For the study of observables related to $b \rightarrow s \mu^{+} \mu^{-}$, it is useful to introduce the effective Wilson coefficients at the scale $\mu_{b}$ that include the effects of operator mixing,

$$
\begin{align*}
C_{7}^{\mathrm{eff}} & =\left(C_{7}^{\mathrm{eff}}\right)_{S M}+\Delta C_{7}\left(\mu_{b}\right), & & C_{7}^{\prime \text { eff }}=\left(C_{7}^{\prime \text { eff }}\right)_{S M}+\Delta C_{7}^{\prime}\left(\mu_{b}\right),  \tag{6.41}\\
C_{9 V}^{\mathrm{eff}}\left(q^{2}\right) & =\left(C_{9 V}^{\mathrm{eff}}\right)_{S M}\left(q^{2}\right)+\frac{2 \pi}{a} \Delta C_{9 V}, & & C_{9 V}^{\prime \text { eff }}=\frac{2 \pi}{a} \Delta C_{9 V}^{\prime}, \\
C_{10 A}^{\mathrm{eff}} & =\left(C_{10 A}^{\mathrm{eff}}\right)_{S M}+\frac{2 \pi}{a} \Delta C_{10 A}, & & C_{10 A}^{\prime \text { eff }}=\frac{2 \pi}{a} \Delta C_{10 A}^{\prime} . \tag{6.42}
\end{align*}
$$

The sm values of the effective Wilson coefficients can be found in Table 2 of [284], which also gives the $q^{2}$ dependence of $\left(C_{9 V}^{\text {eff }}\right)_{S M}\left(q^{2}\right)$ in terms of a linear combination of the other Wilson coefficients. While in principle all contributions have to be taken at the scale $\mu_{b}$, the NP contributions to $C_{g V, 10 A}^{(\prime)}$ are invariant under renormalization group evolution.

With these effective Wilson coefficients at the $B$ scale, we are now equipped to study observables in $b \rightarrow s \mu^{+} \mu^{-}$ transitions. While this system offers a plethora of observables for study, a detailed analysis of all of them is beyond the scope of this paper, and we concentrate on studying a few benchmark observables that are particularly relevant for Rs physics. A numerical analysis is presented in Section 6.6.

In passing we would like to remark on the pattern of contributions to $C_{9 V, 10 A}^{(\prime)}$ in the custodial model, as pointed out in [234]. Due to the suppression of flavor violating $Z d_{L}^{i} \cdot \bar{d}_{L}^{j}$ couplings by the discrete $P_{L R}$ symmetry, the main contributions arise in the primed Wilson coefficients $C^{\prime}{ }^{\prime} V, 10 A$, which are absent in the SM. Since the right-handed $b$ quark, localized significantly further away from the IR brane than the left-handed one, is far less sensitive to flavor violating effects introduced by the RS KK modes, the RS effects in $Y_{s}^{(\prime)}, Z_{s}^{(\prime)}$ turn out to be rather small (typically below $10 \%$ ). This pattern is very different from the minimal model, where the $P_{L R}$ suppression mechanism is absent and large tree level flavor violating $Z$ couplings to left-handed down-type quarks are present.

### 6.5.2 Benchmark observables

$\mathrm{BR}\left(B \rightarrow X_{s} \mu^{+} \mu^{-}\right)$
For very low lepton invariant mass $q^{2} \rightarrow 0$, the $B \rightarrow X_{s} \mu^{+} \mu^{-}$transition is completely dominated by the photon pole and doesn't provide any new insight with respect to the $B \rightarrow X_{s} \gamma$ decay discussed above. Furthermore, in the intermediate region $6 \mathrm{GeV}^{2}<q^{2}<14.4 \mathrm{GeV}^{2}$ the sensitivity to NP is very small, as the decay rate in this region is completely dominated by charm resonances. Hence one usually restricts oneself to either the low $q^{2}$ region $1 \mathrm{GeV}^{2}<q^{2}<6 \mathrm{GeV}^{2}$, or the high $q^{2}$ region $q^{2}>14.4 \mathrm{GeV}^{2}$. In what follows we will consider only the low $q^{2}$ region. While the high $q^{2}$ region is potentially interesting since it exhibits a small tension between SM prediction [285] and experimental data [286,287], it is far less sensitive to NP in $C_{7}^{(\prime)}$, which is the main focus of this study. In the custodial RS model, the tension in the high $q^{2}$ region cannot be resolved since the new contributions to $C_{9 V, 10 A}^{(\prime)}$ are generally small [234]. In addition, the high $q^{2}$ region is subject to larger theoretical uncertainties.

In the low $q^{2}$ region, adapting the formulae of [288] to the more general case of complex NP contributions, we find

$$
\begin{equation*}
\operatorname{Br}\left(B \rightarrow X_{s} \mu^{+} \mu^{-}\right)^{\text {low } q^{2}}=\operatorname{Br}\left(B \rightarrow X_{s} \mu^{+} \mu^{-}\right)_{\mathrm{SM}}^{\mathrm{low} q^{2}}+\Delta \operatorname{Br}\left(B \rightarrow X_{s} \mu^{+} \mu^{-}\right)^{\text {low } q^{2}} \tag{6.44}
\end{equation*}
$$

with the NNLL prediction [289]

$$
\begin{equation*}
\operatorname{Br}\left(B \rightarrow X_{s} \mu^{+} \mu^{-}\right)_{\mathrm{SM}}^{\mathrm{low} q^{2}}=(15.9 \pm 1.1) \cdot 10^{-7} \tag{6.45}
\end{equation*}
$$

and the NP contribution [288]

$$
\begin{align*}
\Delta \operatorname{Br}\left(B \rightarrow X_{s} \mu^{+} \mu^{-}\right)^{\mathrm{low} q^{2}} \simeq 10^{-7} \cdot[ & -0.517 \operatorname{Re}\left(\Delta C_{7}\left(\mu_{b}\right)\right)-0.680 \operatorname{Re}\left(\Delta C_{7}^{\prime}\left(\mu_{b}\right)\right) \\
& +2.663 \operatorname{Re}\left(\delta C_{9 V}\right)-4.679 \operatorname{Re}\left(\delta C_{10 A}\right) \\
& +27.776\left(\left|\Delta C_{7}\left(\mu_{b}\right)\right|^{2}+\left|\Delta C_{7}^{\prime}\left(\mu_{b}\right)\right|^{2}\right) \\
& +0.534\left(\left|\delta C_{9 V}\right|^{2}+\left|\delta C_{9 V}^{\prime}\right|^{2}\right) \\
& +0.543\left(\left|\delta C_{10 A}\right|^{2}+\left|\delta C_{10 A}^{\prime}\right|^{2}\right) \\
& \left.+4.920 \operatorname{Re}\left(\Delta C_{7}\left(\mu_{b}\right) \delta C_{9 V}^{*}+\Delta C_{7}^{\prime}\left(\mu_{b}\right) \delta C_{9 V}^{\prime *}\right)\right] \tag{6.46}
\end{align*}
$$

where we defined

$$
\begin{equation*}
\delta C_{i}=\frac{2 \pi}{a} \Delta C_{i} \tag{6.47}
\end{equation*}
$$

Note that we dropped all interference terms between unprimed and primed contributions since they are suppressed by a factor $m_{s} / m_{b}$ and therefore small. The only exception is the term linear in $\Delta C_{7}^{\prime}$, which receives a large numerical enhancement factor, and is therefore non-negligible; hence we keep it in our analysis.

The measurements of BaBar [286] and Belle [287] yield the averaged value

$$
\begin{equation*}
\operatorname{Br}\left(B \rightarrow X_{s} \mu^{+} \mu^{-}\right)_{\exp }^{\mathrm{low} q^{2}}=(16.3 \pm 5.0) \cdot 10^{-7} \tag{6.48}
\end{equation*}
$$

As LHCb is not well suited for performing inclusive measurements, a significant reduction of uncertainties will only be feasible at the next generation $B$ factories Belle-II and SuperB [290-293].
$B \rightarrow K^{\circ *}(\rightarrow \pi K) \mu^{+} \mu^{-}$
While the inclusive $B \rightarrow X_{s} \mu^{+} \mu^{-}$mode is theoretically very clean, such measurements are experimentally challenging, and competitive results (in particular for angular distributions) will not be available before the Belle II and SuperB era [290-293]. For this reason, exclusive decay modes have received well-deserved attention. An especially interesting decay is
$B \rightarrow K^{*}(\rightarrow K \pi) \mu^{+} \mu^{-}$, where a plethora of angular observables can be studied thanks to the four-body final state [281, 284, 294-299]. These can provide detailed information on the operator and flavor structure of the underlying NP scenario.

The downside is that many $B \rightarrow K^{*}(\rightarrow K \pi) \mu^{+} \mu^{-}$observables, such as the branching ratio and differential decay distribution, are plagued by large theoretical uncertainties in the determination of the $B \rightarrow K^{*}$ matrix elements governed by long-distance non-perturbative QCD dynamics. These matrix elements are most conveniently described by a set of seven form factors. Presently, the best predictions for these form factors at large final state meson $K^{*}$ energies, i.e. small lepton invariant mass $q^{2}$, stem from QCD sum rules at the light cone [300]. Furthermore, non-factorizable corrections are calculated using QCD factorization, which is only valid in the low $q^{2}$ regime. ${ }^{3}$ On the other hand, as mentioned above, at very low $q^{2}<1 \mathrm{GeV}^{2}$ the $b \rightarrow s \mu^{+} \mu^{-}$transition is dominated by the $C_{7}^{(\prime)}$ contributions due to the infrared photon pole and therefore does not provide any insight beyond what is already obtained from $b \rightarrow s \gamma$. Consequently, we henceforth restrict our attention to the range $1 \mathrm{GeV}^{2} \leq q^{2} \leq 6 \mathrm{GeV}^{2}$.

Fortunately, it is possible to partly circumvent the theoretical uncertainties by studying angular observables that are less dependent on the form factors in question. Detailed analyses of their NP sensitivity and discovery potential have been performed by various groups, both model-independently and within specific NP scenarios [281,284, 294-297]. We leave such a detailed analysis in the context of rs models for future work. We focus instead on two benchmark observables, the forward backward asymmetry $A_{\mathrm{FB}}$, which is experimentally well constrained, and the transverse asymmetry $A_{T}^{(2)}$, which offers unique sensitivity to NP in the primed Wilson coefficients.

We note that the recently measured CP asymmetry $A_{9}[305,306]$, as defined in [284,307], is also very sensitive to NP in $C_{7}^{\prime}$ and therefore is in principle an interesting observable to look for RS effects. Because it is sensitive to the phase of $C_{7}^{\prime}$, it yields partly complementary information with respect to the CP conserving transverse asymmetry $A_{T}^{(2)}$. Although this CP asymmetry is theoretically very clean, contrary to those studied in [296], we leave a detailed study within RS for future work.

Forward backward asymmetry The forward-backward asymmetry $A_{\mathrm{FB}}$ in $B \rightarrow K^{*} \mu^{+} \mu^{-}$decays is defined by

$$
\begin{equation*}
A_{\mathrm{FB}}\left(q^{2}\right)=\frac{1}{d \Gamma / d q^{2}}\left(\int_{0}^{1} d\left(\cos \theta_{\mu}\right) \frac{d^{2} \Gamma}{d q^{2} d\left(\cos \theta_{\mu}\right)}-\int_{-1}^{\circ} d\left(\cos \theta_{\mu}\right) \frac{d^{2} \Gamma}{d q^{2} d\left(\cos \theta_{\mu}\right)}\right), \tag{6.49}
\end{equation*}
$$

where $\theta_{\mu}$ is the angle between the $K^{*}$ momentum and the relative momentum of $\mu^{+}$and $\mu^{-}$. $A_{\mathrm{FB}}$ has recently received a lot of attention as data from BaBar, Belle, and the Tevatron seem to indicate a deviation from the SM, albeit with low statistical significance [305,308,309]. On the other hand, recent LHCb data [?] show excellent agreement with the SM prediction, and as uncertainties are presently dominated by statistics, an improved measurement should be available soon.

A precise theoretical determination of $A_{\mathrm{FB}}$ is appealing since it offers a sensitive probe of the helicity of NP contributions. To leading order, the forward backward asymmetry is proportional to [307]

$$
\begin{equation*}
A_{\mathrm{FB}}\left(q^{2}\right) \propto \operatorname{Re}\left[\left(C_{9 V}\left(q^{2}\right)+\frac{2 m_{b}^{2}}{q^{2}} C_{7}\right) C_{10 A}^{*}-\left(C_{9 V}^{\prime}+\frac{2 m_{b}^{2}}{q^{2}} C_{7}^{\prime}\right) C_{10 A}^{\prime *}\right], \tag{6.50}
\end{equation*}
$$

where we dropped the superscript "eff" for the effective Wilson coefficients at the scale $\mu_{b}$, (6.41-6.43). From (6.50) we can see explicitly that $A_{\mathrm{FB}}$ does not receive contributions from the interference of different chirality operators (unprimed and primed). Consequently, with the SM contribution being the dominant effect, potential non-standard effects in $A_{\text {FB }}$ arise mainly from NP in $C_{7}$ and $C_{9} V$. On the other hand, $A_{F B}$ is rather insensitive to NP in the primed Wilson coefficients $C_{7 \gamma, 9 V, 10 A}^{\prime}$.
$A_{\mathrm{FB}}$ has been studied in the context of the minimal rs model considering only tree level contributions and omitting loop level dipole contributions to $C_{7}^{(1)}$ [208], where small positive contributions to $A_{\mathrm{FB}}$ were found. While $A_{\mathrm{FB}}$ is very sensitive to NP effects in $C_{7}$, the RS dipole contributions we calculated predict rather small contributions to this Wilson coefficient. On the other hand, $A_{\mathrm{FB}}$ is insensitive to $C_{7}^{\prime}$, where RS effects are expected to be more pronounced over the SM. Thus the overall prediction of small deviations of $A_{\mathrm{FB}}$ from the SM obtained in [208] remains consistent with our calculations. Note that the restriction to tree level Rs effects is not necessarily a good approximation for observables sensitive to $C_{7}^{\prime}$, such as $F_{L}$, which was also studied in [208]. A detailed study including one-loop contributions to the dipole operators would therefore be desirable but lies beyond the scope of the present analysis.

[^8]In the custodial RS model, due to the protection of the $Z d_{L}^{i} \bar{d}_{L}^{j}$ vertex [231], the RS contributions to $C_{9 V, 10 A}$ are highly suppressed, and only the new contributions to the primed operators are relevant. As $A_{\text {FB }}$ is insensitive to the latter Wilson coefficients, it remains very close to the SM prediction.

We conclude that RS effects in the forward backward asymmetry $A_{\text {FB }}$ are generally small, so the recent data from LHCb do not pose any stringent constraint on the minimal or custodial model, the latter being even more insensitive to RS contributions.

Transverse asymmetry $A_{T}^{(2)}$ The asymmetries $A_{T}^{(i)}$, which are introduced in [295,310], offer a particularly good probe of NP in $b \rightarrow s \mu^{+} \mu^{-}$transitions since at leading order they are free of any hadronic uncertainties and are given in terms of calculable short distance physics. In this paper we will restrict ourselves to the study of the asymmetry

$$
\begin{equation*}
A_{T}^{(2)}=\frac{\left|A_{\perp}\right|^{2}-\left|A_{\|}\right|^{2}}{\left|A_{\perp}\right|^{2}+\left|A_{\|}\right|^{2}} \tag{6.51}
\end{equation*}
$$

Here $A_{\perp}$ and $A_{\|}$are the transversity amplitudes [310] describing the polarization of the $K^{*}$ and the $\mu^{+} \mu^{-}$pair; both are transverse with linear polarization vectors perpendicular $(\perp)$ or parallel $(\|)$ to each other. In the limit of heavy quark $\left(m_{B} \rightarrow \infty\right)$ mass and large $K^{*}$ energy (small $\left.q^{2}\right)$, this asymmetry takes a particularly simple form [296]

$$
\begin{equation*}
A_{T}^{(2)}\left(q^{2}\right)=\frac{2\left[\operatorname{Re}\left(C_{10 A}^{\prime} C_{10 A}^{*}\right)+F^{2} \operatorname{Re}\left(C_{7}^{\prime} C_{7}^{*}\right)+F \operatorname{Re}\left(C_{7}^{\prime} C_{9 V}^{*}\right)\right]}{\left|C_{10 A}\right|^{2}+\left|C_{10 A}^{\prime}\right|^{2}+F^{2}\left(\left|C_{7}\right|^{2}+\left|C_{7}^{\prime}\right|^{2}\right)+\left|C_{9 V}\right|^{2}+2 F \operatorname{Re}\left(C_{7} C_{9 V}^{*}\right)} \tag{6.52}
\end{equation*}
$$

with $F=2 m_{b} m_{B} / q^{2}$, and we have again dropped the superscript "eff" from the Wilson coefficients. In this limit it is clear that $A_{T}^{(2)}$ is independent of form factors and is governed only by calculable short distance physics, making this observable theoretically clean. Second, we notice that since the primed Wilson coefficients are highly suppressed in the SM, $\left(A_{T}^{(2)}\right)_{\text {SM }}$ is very small. $A_{T}^{(2)}$ therefore offers unique sensitivity to $N P$ entering dominantly in the primed operators $C_{7 \gamma, 9 V, 10 A}^{\prime}$. This asymmetry is thus a benchmark observable for discovering Rs physics in $B \rightarrow K^{*} \mu^{+} \mu^{-}$decays. We investigate the possible size of RS contributions to this channel in our numerical analysis in the next section.

A first measurement of $A_{T}^{(2)}$ by CDF [305] is still plagued by large uncertainties. LHCb has recently put more stringent constraints on this asymmetry, and more precise measurements will be possible in the near future [306]. With $10 \mathrm{fb}^{-1}$ of data, LHCb is expected to reach a sensitivity of about $\pm 0.16$.

### 6.6 NUMERICAL ANALYSIS

### 6.6.1 STRATEGY

In this section we present a numerical analysis of the observables introduced in the previous sections. To this end we follow the following strategy:

1. The first goal is to understand the generic pattern of effects induced by rs penguins on flavor observables. We generate a set of parameter points that satisfy the known experimental constraints from quark masses and CKM parameters. However, we do not yet impose any additional flavor bounds so as not to be biased by their impact. With these points we evaluate the new RS contributions to the Wilson coefficients $\Delta C_{7}^{(\prime)}$ and $\Delta C_{8}^{(\prime)}$ at the KK scale for both the minimal and the custodial model. Subsequently we calculate the new contributions to the branching ratios of $B \rightarrow X_{s, d} \gamma$ and analyze the constraints.
Note that the same set of parameter points is used for the minimal and the custodial model in this case, in order to minimize the sampling bias on the results obtained.
2. The second goal is to understand the effect of the RS penguins on the existing parameter space for realistic RS models. We restrict our attention to the custodial model, which can be made consistent with electroweak precision tests for KK scales as low as $M_{\mathrm{KK}} \simeq 2.5 \mathrm{TeV}$. In addition to quark masses and CKM parameters, we now also impose constraints from $\Delta F=2$ observables which are analyzed at length in [231]. After evaluating the size of the effects in the


Figure 6.6.1: RS contributions to the $b \rightarrow s$ Wilson coefficients $C_{7}\left(M_{\mathrm{KK}}\right)$ (upper left), $C_{7}^{\prime}\left(M_{\mathrm{KK}}\right)$ (upper right), $C_{8}\left(M_{\mathrm{KK}}\right)$ (lower left) and $C_{8}^{\prime}\left(M_{\mathrm{KK}}\right)$ (lower right) in the minimal (red, dashed) and custodial (blue, solid) models, and from the misalignment contribution alone (black, dotted).
$B \rightarrow X_{s, d} \gamma$ branching ratios and their constraint on the model, we study the benchmark observables outlined above, namely the CP asymmetry in $B \rightarrow K^{*} \gamma$, the branching ratio $\operatorname{Br}\left(B \rightarrow X_{s} \mu^{+} \mu^{-}\right)$, and the transverse asymmetry $A_{T}^{(2)}$ in $B \rightarrow K^{*} \mu^{+} \mu^{-}$decays.

Throughout our analysis we restrict ourselves to $1 / R^{\prime}=1 \mathrm{TeV}$, so that the lowest KK gauge bosons have a mass of $M_{\mathrm{KK}} \simeq 2.5 \mathrm{Tev}$. We note that in the minimal model such low KK masses are already excluded due to unacceptably large corrections to electroweak precision observables. However, we use the same mass scale for both the minimal and custodial models to enable a straightforward comparison of the two sets of results. Furthermore, we restrict the fundamental Yukawa couplings to lie in their perturbative regime, i. e. $\left|Y_{i j}\right| \leq 3$. More details on the parameter scan can be found in [231].

### 6.6.2 GENERAL PATTERN OF RS CONTRIBUTIONS

This part of the numerical analysis is dedicated to determining the size of NP effects generated by the RS KK modes in the dipole operators $C_{7}, C_{7}^{\prime}$ and $C_{8}, C_{8}^{\prime}$ mediating the $b \rightarrow(s, d) \gamma$ and $b \rightarrow(s, d) g$ transitions respectively. We advise caution when interpreting the density of points since these distributions are influenced by the details of the parameter scan performed. The qualitative features in our plots should however remain unaffected by the scanning procedures.

The first row of Figure 6.6.1 shows the RS contributions to $C_{7}\left(M_{\mathrm{KK}}\right)$ and $C_{7}^{\prime}\left(M_{\mathrm{KK}}\right)$ in the


Figure 6.6.2: RS contributions to the $b \rightarrow s \gamma$ Wilson coefficients $C_{7}$ (left) and $C_{7}^{\prime}$ (right), evaluated at the scale $\mu_{b}=2.5 \mathrm{GeV}$. The minimal model distribution is shown in red (dashed), and the custodial one in blue (solid).
$b \rightarrow s$ system. Observe that the total RS contribution (dashed red and solid blue distributions, corresponding to the minimal and custodial model) to the primed Wilson coefficient is typically an order of magnitude larger than the corresponding effect in the unprimed Wilson coefficient. This matches the naive expectation that the $b_{L} \rightarrow s_{R}$ transition should be enhanced relative to $b_{R} \rightarrow s_{L}$ due to the hierarchy $f_{Q_{3}} \gg f_{b_{R}}$ of fermion localizations. Furthermore the custodial contribution is somewhat enhanced relative to the minimal one, due to the additional fermion modes running in the loop. Also shown, in black (dotted), is the contribution to $C_{7}\left(M_{\mathrm{KK}}\right)$ and $C_{7}^{\prime}\left(M_{\mathrm{KK}}\right)$ generated by only the misalignment term, which is equal for the minimal and the custodial models. Unlike the anarchic term, this contribution is generically comparable in both cases. This naively unexpected behavior is explained in Appendix 6.C. While it is subdominant but non-negligible in the case of $C_{7}\left(M_{\mathrm{KK}}\right)$, it turns out to be generally irrelevant in the case of $\mathrm{C}_{7}^{\prime}\left(M_{\mathrm{KK}}\right)$.

The second row of Fig. 6.6.1 shows the results for the gluonic penguin Wilson coefficients $C_{8}$ and $C_{8}^{\prime}$. The values at the KK scale are larger than the corresponding values of $C_{7}$ and $C_{7}^{\prime}$ by about an order of magnitude due to the large contribution from the diagram containing the non-Abelian $S U(3)_{c}$ vertex, which is absent in the $b \rightarrow s \gamma$ penguin. Other than that, the pattern of effects is qualitatively similar to that for $C_{7}^{(\prime)}$ : the primed Wilson coefficient is larger than the unprimed coefficient by about an order of magnitude, and the custodial model yields somewhat bigger effects than the minimal model. Furthermore, the misalignment contributions to the unprimed and primed Wilson coefficients are again roughly comparable; consequently, its effect is negligible in $C_{8}^{\prime}$ but can be sizable in $C_{8}$.

To facilitate comparison with other models of NP, Fig. 6.6.2 shows the rs contributions to the $b \rightarrow s \gamma$ Wilson coefficients $C_{7}$ (left) and $C_{7}^{\prime}$ (right) evaluated at the scale $\mu_{b}=2.5 \mathrm{GeV}$, i.e. taking into account the RG evolution and operator mixing with $C_{8}^{(\prime)}$. The rS contribution to $C_{7}$ turns out to be small and typically constitues less than a few percent of the SM value $C_{7}^{\prime}\left(\mu_{b}\right)^{S M}=-0.353$. On the other hand, $C_{7}^{\prime}$ is suppressed by $m_{s} / m_{b}$ in the $S M$, so the unsuppressed contribution from RS dominates, though its value is still typically smaller than $C_{7}\left(\mu_{b}\right)^{\text {SM }}$.

Next, we examine the relative importance of the various rs contributions to the effective $b \rightarrow s \gamma$ Wilson coefficients at the scale $\mu_{b}$. Fig. 6.6.3 shows the size of the two main anarchic contributions to $\Delta C_{8}^{(\prime)}\left(M_{\mathrm{KK}}\right)$ (see Fig. 6.3.2a for the relevant Feynman diagrams) normalized to the anarchic contribution to $\Delta C_{7}^{(\prime)}\left(M_{\mathrm{KK}}\right)$ (see Fig. 6.3.1a). For a straightforward comparison, we also include the relevant RG evolution factors from eq. (6.16). The ratio of the Higgs penguin contribution to $\Delta C_{7}^{(\prime)}\left(M_{\mathrm{KK}}\right)$ and $\Delta \mathrm{C}_{8}^{(\prime)}\left(M_{\mathrm{KK}}\right)$, shown by the black (dotted) peak, is constant and equal for both the minimal and custodial model. As the relevant diagrams depend on the same loop integral and the same combination of Yukawa couplings, their relative size at the кK scale is simply given by the electric charge $Q_{u}$ of the up-type quark coupled to the photon. After including the RG running down to the scale $\mu_{b}$, the Higgs penguin contribution to $C_{8}^{(\prime)}$ turns out to be roughly a $50 \%$


Figure 6.6.3: Relative sizes of anarchic contributions to the Wilson coefficients $C_{7}\left(\mu_{b}\right)$ (left) and $C_{7}^{\prime}\left(\mu_{b}\right)$ (right) from the RG evolution and operator mixing of $\Delta C_{8}^{\left({ }^{\prime}\right)}$ from $M_{\mathrm{KK}}$ to $\mu_{b}$, normalized to the Higgs penguin contribution to $\Delta C_{7}^{(\prime)}\left(M_{\mathrm{KK}}\right)$, with relevant RG evolution factors included. The black (dotted) peak shows the ratio of the Higgs penguin contribution to $\Delta C_{8}^{(\prime)}\left(M_{\mathrm{KK}}\right)$. The red (dashed) and blue (solid) distributions show the ratio of the gluon penguin to $\Delta C_{8}^{(\prime)}\left(M_{\mathrm{KK}}\right)$ for the minimal and custodial model respectively.
correction to the effect of the anarchic $\Delta C_{7}^{(\prime)}\left(M_{\mathrm{KK}}\right)$ contribution.
The effect of the gluon penguin diagram in $\Delta C_{8}^{(/)}\left(M_{\mathrm{KK}}\right)$ depends on a different loop integral and a different combination of Yukawa couplings than the Higgs diagram in $\Delta C_{7}^{(\prime)}\left(M_{\mathrm{KK}}\right)$. Consequently its relative size, again including the relevant RG factors, varies considerably within the minimal (shown in red, dashed line) and the custodial (shown in blue, solid line) model. Observe that the distribution for the minimal model is rather symmetric and peaked around 1 , implying that the rs $b \rightarrow s g$ loop generally contributes as much as the rs $b \rightarrow s \gamma$ loop in low energy observables, even yielding the dominant RS contribution in parts of the parameter space. This is in contrast with the SM case, where the $C_{8}$ contribution only gives a few percent correction to the dominant $C_{7}$ contribution. In the custodial model the gluon penguin contribution becomes even more important, so that the peak of the distribution gets shifted above 1. Since, as opposed to the Higgs penguin, the additional custodial gluon penguin diagram shown in Fig. 6.3.3 carries the same Yukawa spurion as the minimal model diagram, they simply add constructively, further enhancing the effect of the gluonic penguin contribution. Neglecting these contributions or even the $\mathrm{C}_{8}^{(\prime)}$ contribution as a whole, as sometimes done in the literature, would therefore be a rather poor approximation. Note that the relative importance of the gluon penguin diagrams depends crucially on the matching of the ${ }_{5} \mathrm{D}$ to the 4 D strong gauge coupling. Invoking one loop level matching rather than tree level matching as done here whould reduce their relative size by roughly a factor of four. On the other hand the presence of brane kinetic terms could further enhance the gluonic penguin contribution.

For the sake of completeness Fig. 6.6.4 shows the Wilson coefficients for the $b \rightarrow d$ system, in analogy to Fig. 6.6.1. The pattern of effects is very similar to the case of the $b \rightarrow s$ system discussed above.

Figure 6.6.5 shows the predicted deviations from the SM in the $B \rightarrow X_{s, d} \gamma$ branching ratios in the minimal and custodial models. We observe that in both models these branching ratios typically obtain a moderate positive NP contribution well within the current experimental and theoretical uncertainties. Nevertheless, the decays in question put nontrivial constraints on parts of the rs parameter space and should be included in a complete analysis of rs flavor phenomenology. As expected from the size of the Wilson coefficients, the custodial model induces somewhat larger effects than the minimal model.

Interestingly, this pattern of effects is very different from that of the ADD model of a universal extra dimension [311], where the KK excitations affect mainly the Wilson coefficient $C_{7}$, while the opposite-chirality Wilson coefficient $C_{7}^{\prime}$ remains very small $[312,313]$. Since the ADD contribution interferes destructively with the SM contribution, a rather pronounced


Figure 6.6.4: RS contributions to the $b \rightarrow d$ Wilson coefficients $C_{7}\left(M_{\mathrm{KK}}\right)$ (upper left), $C_{7}^{\prime}\left(M_{\mathrm{KK}}\right)$ (upper right), $C_{8}\left(M_{\mathrm{KK}}\right)$ (lower left) and $C_{8}^{\prime}\left(M_{\mathrm{KK}}\right)$ (lower right) in the minimal model (red, dashed), the custodial model (blue, solid), and from the misalignment contribution alone (black, dotted).


Figure 6.6.5: Rs contribution to $\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)$ (left) and $\left\langle\operatorname{Br}\left(B \rightarrow X_{d} \gamma\right)\right\rangle$ in the minimal (red, dashed) and custodial (blue, solid) model. The experimental constraints according to (6.13) and (6.14) are displayed as grey bands.


Figure 6.6.6: CP asymmetry $S_{K^{*} \gamma}$ as a function of $\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)$. The black-and-white dot indicates the central SM prediction, while the dashed lines show the experimental central values. The grey bands display the experimental $1 \sigma$ and $2 \sigma$ ranges for $S_{K^{*}} \gamma$. We omit showing the uncertainty in $\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)$ as it covers the whole range.
suppression of $\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)$ is predicted, which was used in [314] to derive the bound $1 / R>600 \mathrm{GeV}$ on the radius $R$ of the extra dimension.

We also investigated the dependence of the size of the rs contribution to $\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)$ and $\left\langle\operatorname{Br}\left(B \rightarrow X_{d} \gamma\right)\right\rangle$ on the average Yukawa coupling $Y_{*}$, but did not find any significant correlation within our parameter scan. These findings at first sight seem to contradict the analytic estimate in section 6.4.4. Recall that these estimates have been performed in the fully anarchic limit where $Y_{*}$ is the only free parameter in the Yukawa sector. On the other hand, our scan varies all independent parameters in the flavor sector, so that $\mathcal{O}(1)$ deviations from the fully anarchic ansatz are intrinsic. The dependence on these additional parameters fully hides the dependence on $Y_{*}$; note also that the latter only varies over an $\mathcal{O}(1)$ range.

### 6.6.3 Effects on benchmark observables

We now restrict our attention to the custodial model and consider only parameter points that agree with the existing constraints from $\Delta F=2$ transitions, as analyzed in [231]. We also impose the bounds from the $B \rightarrow X_{s, d} \gamma$ decays as approximated in (6.13-6.14), so that all points displayed in the plots lie within the experimentally allowed region.

Since the dipole operators depend on a different combination of rs flavor parameters from the tree level contributions to $\Delta F=2$ processes [231] and $\Delta F=1$ rare decays [234], observables related to the various sectors are essentially uncorrelated; hence we do not show any numerical results here.

Figure 6.6 .6 shows the correlation between the time-dependent CP asymmetry $S_{K^{*} \gamma}$ and the branching ratio of $B \rightarrow X_{s} \gamma$. Observe that $S_{K^{*} \gamma}$ can receive large enhancements relative to its tiny SM value. While non-standard effects in $S_{K^{*} \gamma}$ are possible for any value of $\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)$, large effects are more likely with enhanced values of the branching ratio. This is related to the fact that RS contributions dominantly affect $C_{7}^{\prime}$. While the sm prediction for $B \rightarrow X_{s} \gamma$ is in good agreement with data, it lies below the central value, and an enhancement of this branching ratio is preferred. One can also see that large enhancements are possible in $S_{K^{*} \gamma}$, and that the present experimental $2 \sigma$ range excludes only a small fraction of the rS parameter space.

The decay $B \rightarrow X_{s} \mu^{+} \mu^{-}$poses strong constraints on various extensions of the sm, hence it is worth studying it in the custodial RS model. Figure 6.6 .7 shows the custodial Rs branching ratio $\operatorname{Br}\left(B \rightarrow X_{s} \mu^{+} \mu^{-}\right)$in the low $q^{2}$ region as a function


Figure 6.6.7: Correlation between $\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)$ and $\operatorname{Br}\left(B \rightarrow X_{s} \mu^{+} \mu^{-}\right)$for $q^{2} \in[1,6] \mathrm{GeV}^{2}$. The black-andwhite dot indicates the central Sm prediction, while the dashed lines show the experimental central values. We omit showing the experimental and theoretical uncertainties as they cover the whole range.


Figure 6.6.8: Transverse asymmetry $A_{T}^{(2)}$ as a function of $q^{2}$, for a few parameter points. The SM prediction is indicated by the thick black line, while each blue line corresponds to an Rs parameter point.


Figure 6.6.9: Correlation between $S_{K^{*} \gamma}$ and $A_{T}^{(2)}\left(q^{2}={ }_{1 \mathrm{GeV}^{2}}\right)$. The black-and-white dot indicates the central sm prediction, while the dashed line shows the experimental central value. The grey bands display the experimental $1 \sigma$ and $2 \sigma$ ranges for $S_{K^{*} \gamma}$.
of $\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)$. We observe that the enhancement in the custodial rs model is rather small, typically below $10 \%$. Due to the experimental and theoretical uncertainties involved, this channel does not put any significant constraint on the model.

Observables far more sensitive to NP in $\mathrm{C}_{7}^{\prime}$ can be constructed from the angular distribution of $B \rightarrow K^{*} \mu^{+} \mu^{-}$. Of particular interest is the transverse asymmetry $A_{T}^{(2)}$, whose $q^{2}$ dependence is shown in Figure 6.6.8. Observe that large enhancements relative to the small sm value are possible, in particular in the very small $q^{2}$ region $<2 \mathrm{GeV}^{2}$. This pattern can be understood from (6.52): the $C_{7}^{\prime}$ contribution is enhanced at small $q^{2}$ due to a $1 / q^{2}$ factor, see also [296,315]. Also note that the custodial RS model predicts a zero crossing for $A_{T}^{(2)}$ at $q^{2} \sim 2.7 \mathrm{GeV}^{2}$. The differential asymmetry would exhibit a very different shape if the dominant NP contribution appeared in $C_{10 A}^{\prime}$. This underlines the model-discriminating power of the $A_{T}^{(2)}$ asymmetry-in the custodial RS model a deviation from the SM is most likely to be observed for small $q^{2}$, whereas other models that dominantly affect $C_{10 A}^{\prime}$ predict large effects for larger $q^{2}$. This pattern is particularly interesting in light of LHCb and the next generation $B$ factories, which will soon be able to measure this asymmetry.

Finally, one may consider a possible correlation between $S_{K^{*} \gamma}$ and $A_{T}^{(2)}$. Both observables are mostly affected by a large $C_{7}^{\prime}$, hence some nontrivial correlation can be expected. On the other hand, $S_{K^{*} \gamma}$ is CP violating while $A_{T}^{(2)}$ is CP conserving, so the phase of $C_{7}^{\prime}$ can wash out such correlations. Figure 6.6 .9 shows $A_{T}^{(2)}\left(q^{2}=1 \mathrm{GeV}^{2}\right)$ as a function of $S_{K^{*} \gamma}$, where a nontrivial linear anti-correlation is seen between the two observables in question. However, this correlation is visibly weakened by the impact of the phase of $C_{7}^{\prime}$, as expected.

### 6.7 Conclusions

In this paper we have performed an explicit ${ }_{5} \mathrm{D}$ calculation of the dominant contributions to the Wilson coefficients $\mathrm{C}_{7}, \mathrm{C}_{7}^{\prime}$, and $C_{8}, C_{8}^{\prime}$ that mediate the $b \rightarrow s, d \gamma$ and $b \rightarrow s, d g$ transitions respectively, in the Rs setup with bulk fermions and gauge bosons and an IR-brane localized Higgs. We have evaluated the relevant diagrams for both the minimal scenario with only the SM gauge group in the bulk, and for the custodial model with the electroweak gauge group extended by $\mathrm{SU}(2)_{\mathrm{R}}$ and a discrete $P_{\text {LR }}$ symmetry. Our main findings from this analysis can be summarized as follows:

- The rs contributions to $C_{7}^{\prime}$ typically exceed those to $C_{7}$ by an order of magnitude, and the latter remain a rather small correction to the SM value. This pattern can be understood by considering the bulk profiles of the quark fields
involved: the primed Wilson coefficient describes the decay of a left-handed $b$ quark, which, being localized towards the IR brane, is more sensitive to flavor violating effects than the right-handed $b$ quark entering $C_{7}$. Analogous comments apply regarding the hierarchy $C_{8} \ll C_{8}^{\prime}$.
- Contrary to the Sm, where $C_{8}<C_{7}$, RS contributions to the gluonic penguins are larger than the ones to the photonic penguins. This results from the large contributions from the diagram containing the non-abelian triple gluon (kK gluon) vertex, which is absent in $C_{7}^{(\prime)}$ and does not change flavor in the SM. In addition, the renormalization group mixing of $C_{7}^{(\prime)}$ and $C_{8}^{(\prime)}$ is more pronounced due to the large separation of the $M_{\mathrm{KK}}$ and $m_{b}$ scales. Consequently, gluonic penguin contributions have a significant impact on $b \rightarrow s, d \gamma$, comparable to or larger than the photonic penguin contribution. This is in contrast to the SM, where they yield only a few percent correction to the photonic Wilson coefficients at the $m_{b}$ scale.
- In all cases, the dominant effect comes from the anarchic contributions, which are not aligned with the sm quark mass matrices. However, the unprimed (right to left) operators pick up appreciable contributions from misalignment diagrams, which are proportional to the SM quark mass matrices up to a dependence on the bulk spectrum. This is because, in contrast to the anarchic diagrams, the misalignment diagrams are not suppressed by the $b_{R}$ wave function relative to the $b_{L}$ wavefunction, as explained in Appendix 6.C.
- The impact on the Wilson coefficients in question is somewhat larger in the custodial model than in the minimal model, since the extended fermion content that was introduced to reconcile the model with the $Z b \bar{b}$ constraint yields additional contributions.

For a study of the phenomenological implications of these new contributions, we restricted our attention to the custodial model since the minimal model is not consistent with electroweak precision constraints for low KK masses $M_{\mathrm{KK}}=2.5 \mathrm{TeV}$. To this end, following [231] we performed a parameter scan of the 5 D bulk masses and fundamental Yukawa coupling matrices, imposing constraints from quark masses and CKM parameters and from meson-antimeson mixing. We studied the bounds provided by the branching ratios $\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)$ and $\left\langle\operatorname{Br}\left(B \rightarrow X_{d} \gamma\right)\right\rangle$ and the effects in a number of benchmark observables, namely the time-dependent CP asymmetry $S_{K^{*} \gamma}$, the inclusive branching ratio $\operatorname{Br}\left(B \rightarrow X_{s} \mu^{+} \mu^{-}\right)$and the forward-backward asymmetry $A_{\mathrm{FB}}$ and the transverse asymmetry $A_{T}^{(2)}$ in $B \rightarrow K^{*} \mu^{+} \mu^{-}$, where we found the following patterns:

- The branching ratios of the radiative inclusive $B \rightarrow X_{s, d} \gamma$ decays provide a non-negligible constraint on RS models and exclude roughly $15 \%$ of the parameter points generated for the custodial model that were in agreement with bounds from $\Delta F=2$ observables. A complete phenomenological study should therefore take these constraints into account. However, since the major part of parameter space survives, no useful bound on the KK scale can be derived.
- Due to more precise data and sm theory prediction, $\operatorname{Br}\left(B \rightarrow X_{s} \gamma\right)$ generally puts a stronger constraint on the RS parameter space than $\left\langle\operatorname{Br}\left(B \rightarrow X_{d} \gamma\right)\right\rangle$. The latter observable is still useful as it yields complementary information on the allowed parameter space.
- As the RS contributions enter dominantly through the primed operators, a modest enhancement of the $B \rightarrow X_{s, d} \gamma$ branching ratios can be expected, although a slight suppression is not rigorously excluded. Such an enhancement would be welcome in $B \rightarrow X_{s} \gamma$, where the data lie somewhat above the sm value, albeit still in good agreement. On the other hand, for $B \rightarrow X_{d} \gamma$ the central values of the SM and the data are in excellent agreement and the uncertainties are sizable, and no prefered sign for the NP contribution can be deduced.
- The inclusive branching ratio $\operatorname{Br}\left(B \rightarrow X_{s} \mu^{+} \mu^{-}\right)$and the forward backward asymmetry $A_{\mathrm{FB}}$ in $B \rightarrow K^{*} \mu^{+} \mu^{-}$receive very small corrections from RS physics and remain in good agreement with recent data. While we restricted our analysis to the low $q^{2}$ region, these statements also apply to the high $q^{2}$ region since the latter region is mostly sensitive to NP in the electroweak Wilson coefficients $C_{9 V, 10 A}^{(/)}$, which remain SM-like in the custodial model.
- We identify the time-dependent CP asymmetry $S_{K^{*} \gamma}$ in $B \rightarrow K^{*} \gamma$ decays and the transverse asymmetry $A_{T}^{(2)}$ in the low $q^{2}$ region of $B \rightarrow K^{*} \mu^{+} \mu^{-}$as promising benchmark observables to look for large effects generated by the custodial RS model. Both observables are known to be very sensitive to the primed Wilson coefficients, in particular $C_{7}^{\prime}$, which is dominantly affected by RS contributions. Furthermore, studying the $q^{2}$ dependence of $A_{T}^{(2)}$ allows for a clear distinction of models such as the custodial rs model that dominantly affect $C_{7}^{\prime}$ from models that predict large NP effects in the electroweak Wilson coefficient $C_{10 A}^{\prime}$.

In summary, our analysis shows that radiative and semileptonic $B$ decays offer intriguing possibilities to find deviations from the SM generated by rS KK modes and anarchic Yukawa structure. If such effects are found at the LHCb and the next generation $B$ factories, it will be particularly interesting to study the plethora of observables provided by these decay modes in a correlated manner, which offers the ability to distinguish RS with custodial symmetry from other NP scenarios that predict a different pattern of effects.

## 6.A Dimensionless Integrals for Leading Diagrams

This appendix defines the dimensionless integrals associated with the leading contributions to the $a$ and $b$ terms of the dipole Wilson coefficients $C_{7,8}$ in Section 6.3. Details of the derivation of these integrals are found in the appendix of [2]. In the mass insertion approximation the Standard Model contribution appears as an infrared pole, which we subtract.

## 6.A. 1 Propagator functions

We use dimensionless integration variables $x \equiv k_{E} z \in[w y, y]$ and $y \equiv k_{E} R^{\prime} \in[\mathrm{o}, \infty]$, where $k_{E}$ is the Euclidean loop momentum and $w=\left(R / R^{\prime}\right)$ is the warp factor. The integrals are expressed with respect to the functions that appear in the mixed position-Euclidean momentum space fermion propagator,

$$
\Delta\left(k_{E}, x, x^{\prime}\right) \equiv i \frac{R^{\prime}}{w^{4}} \overline{\mathcal{D}} \tilde{F}_{y}^{x x^{\prime}}=\left(\begin{array}{ll}
y \tilde{D}_{-} \tilde{F}_{-} & \sigma^{\mu} y_{\mu} \tilde{F}_{+}  \tag{6.53}\\
\bar{\sigma}^{\mu} y_{\mu} \tilde{F}_{-} & y \tilde{D}_{+} \tilde{F}_{+}
\end{array}\right), \quad \quad \tilde{D}_{ \pm} \equiv \pm\left(\partial_{x}-\frac{2}{x}\right)+\frac{c}{x} .
$$

where the $\tilde{F}$ functions are defined for $x>x^{\prime}$ (i.e. $z>z^{\prime}$ ) by

$$
\begin{array}{ll}
\tilde{F}_{-}^{L}=\frac{\left(x x^{\prime}\right)^{5 / 2}}{y^{5}} \frac{S_{c_{L}}\left(x_{-}, y_{-}\right) S_{c_{L}}\left(x_{-}^{\prime}, w y_{-}\right)}{S_{c_{L}}\left(y_{-}, w y_{-}\right)} & \tilde{F}_{+}^{L}=-\frac{\left(x x^{\prime}\right)^{5 / 2}}{y^{5}} \frac{T_{c_{L}}\left(x_{+}, y_{-}\right) T_{c_{L}}\left(x_{+}^{\prime}, w y_{-}\right)}{S_{c_{L}}\left(y_{-}, w y_{-}\right)} \\
\tilde{F}_{-}^{R}=-\frac{\left(x x^{\prime}\right)^{5 / 2}}{y^{5}} \frac{T_{c_{R}}\left(x-, y_{+}\right) T_{c_{R}}\left(x_{-}^{\prime}, w y_{+}\right)}{S_{c_{R}}\left(y_{+}, w y_{+}\right)} & \tilde{F}_{+}^{R}=\frac{\left(x x^{\prime}\right)^{5 / 2}}{y^{5}} \frac{S_{c_{R}}\left(x_{+}, y_{+}\right) S_{c_{R}}\left(x_{+}^{\prime}, w y_{+}\right)}{S_{c_{R}}\left(y_{+}, w y_{+}\right)}
\end{array}
$$

The analogous functions for $x<x^{\prime}$ are given by replacing $x \leftrightarrow x^{\prime}$ in the above formulas. $S$ and $T$ function are products of Bessel functions,

$$
\begin{align*}
& S_{c}\left(x_{ \pm}, x_{ \pm}^{\prime}\right)=I_{c \pm 1 / 2}(x) K_{c \pm 1 / 2}\left(x^{\prime}\right)-I_{c \pm 1 / 2}\left(x^{\prime}\right) K_{c \pm 1 / 2}(x)  \tag{6.56}\\
& S_{c}\left(x_{ \pm}, x_{\mp}^{\prime}\right)=I_{c \pm 1 / 2}(x) K_{c \mp 1 / 2}\left(x^{\prime}\right)-I_{c \mp 1 / 2}\left(x^{\prime}\right) K_{c \pm 1 / 2}(x)  \tag{6.57}\\
& T_{c}\left(x_{ \pm}, x_{\mp}^{\prime}\right)=I_{c \pm 1 / 2}(x) K_{c \mp 1 / 2}\left(x^{\prime}\right)+I_{c \mp 1 / 2}\left(x^{\prime}\right) K_{c \pm 1 / 2}(x) . \tag{6.58}
\end{align*}
$$

Similarly, the mixed position-Euclidean momentum space vector propagators are $-i \eta^{\mu \nu} G$ and $i \bar{G}$ for the 4 -vector and scalar parts respectively. For $x<x^{\prime}$, the $G$ functions are,

$$
\begin{align*}
& G_{k}\left(z, z^{\prime}\right)=\frac{\left(R^{\prime}\right)^{2}}{R} G_{y}\left(x, x^{\prime}\right)=\frac{\left(R^{\prime}\right)^{2}}{R} \frac{x x^{\prime}}{y} \frac{T_{10}(x, y) T_{10}\left(x^{\prime}, w y\right)}{S_{\circ \circ}(w y, y)},  \tag{6.59}\\
& G_{5 k}\left(z, z^{\prime}\right)=\frac{\left(R^{\prime}\right)^{2}}{R} \bar{G}_{y}\left(x, x^{\prime}\right)=\frac{\left(R^{\prime}\right)^{2}}{R} \frac{x x^{\prime}}{y} \frac{S_{\circ \circ}(x, y) S_{\circ \circ}\left(x^{\prime}, w y\right)}{S_{\circ \circ}(w y, y)}, \tag{6.60}
\end{align*}
$$

where

$$
\begin{align*}
T_{i j}(x, y) & =I_{i}(x) K_{j}(y)+I_{j}(y) K_{i}(x)  \tag{6.61}\\
S_{i j}(x, y) & =I_{i}(x) K_{j}(y)-I_{j}(y) K_{i}(x) . \tag{6.62}
\end{align*}
$$

For $z<z^{\prime}$ the above formula is modified by $x \leftrightarrow x^{\prime}$.

## 6.A. $2 C_{7}$ INTEGRALS

We label vertices such that the external fermion legs attach to vertices 1 and 3 , and the photon or gluon is emitted at vertex 2 . Propagators attached to the brane $x=y$ signify Yukawa couplings or mass insertions, which may change the fermion flavor as labeled by its bulk mass, $c$. We have left this $c$ dependence implicit in the following expressions.

$$
\begin{align*}
I_{C_{7 a}}= & \int d y d x y^{2}\left(\frac{y}{x}\right)^{4}\left[-2 \tilde{F}_{+, y}^{L y x} \tilde{F}_{+, y}^{L x y} \tilde{F}_{-, y}^{R y y} \frac{y^{2}}{y^{2}+\left(M_{W} R^{\prime}\right)^{2}}\right. \\
& +\tilde{F}_{+, y}^{L y x} \tilde{F}_{+, y}^{L x y} \tilde{F}_{-, y}^{R y y} \frac{y^{4}}{\left(y^{2}+\left(M_{W} R^{\prime}\right)^{2}\right)^{2}}-\frac{1}{2}\left(y \partial_{k_{E}} \tilde{F}_{+, y}^{L y x}\right) \tilde{F}_{+, y}^{L x y} \tilde{F}_{-, y}^{R y y} \frac{y^{2}}{y^{2}+\left(M_{W} R^{\prime}\right)^{2}} \\
& -\frac{1}{2}\left(y \partial_{k_{E}} \tilde{D}_{-} \tilde{F}_{-, y}^{L y x}\right) \tilde{D}_{+} \tilde{F}_{+, y}^{L x y} \tilde{F}_{-, y}^{R y y} \frac{1}{y^{2}+\left(M_{W} R^{\prime}\right)^{2}}+2 \tilde{F}_{+, y}^{L y y} \tilde{D}_{+} \tilde{F}_{+, y}^{R y x} \tilde{D}_{-} \tilde{F}_{-, y}^{R x y} \frac{1}{y^{2}+\left(M_{W} R^{\prime}\right)^{2}} \\
& -\tilde{F}_{+, y}^{L y y} \tilde{D}_{+} \tilde{F}_{+, y}^{R y x} \tilde{D}_{-} \tilde{F}_{-, y}^{R x y} \frac{y^{2}}{\left(y^{2}+\left(M_{W} R^{\prime}\right)^{2}\right)^{2}}+\frac{1}{2}\left(y \partial_{k_{E}} \tilde{F}_{+, y}^{L y y}\right) \tilde{D}_{+} \tilde{F}_{+, y}^{R y x} \tilde{D}_{-} \tilde{F}_{-, y}^{R x y} \frac{1}{y^{2}+\left(M_{W} R^{\prime}\right)^{2}} \\
& +\tilde{F}_{+, y}^{L y y} \tilde{F}_{-, y}^{R y x} \tilde{F}_{-, y}^{R x y} \frac{y^{2}}{y^{2}+\left(M_{W} R^{\prime}\right)^{2}}+\frac{1}{2}\left(y \partial_{k_{E}} \tilde{F}_{+, y}^{L y y}\right) \tilde{F}_{-, y}^{R y x} \tilde{F}_{-, y}^{R x y} \frac{y^{2}}{y^{2}+\left(M_{W} R^{\prime}\right)^{2}} \\
& +\frac{1}{2} \tilde{F}_{+, y}^{L y y}\left(y \partial_{k_{E}} \tilde{F}_{-, y}^{R y y}\right) \tilde{F}_{-, y}^{R x y} \frac{y^{2}}{y^{2}+\left(M_{W} R^{\prime}\right)^{2}} \\
& \left.+\frac{1}{2} \tilde{F}_{+, y}^{L y y}\left(y \partial_{k_{E}} \tilde{D}_{+} \tilde{F}_{+, y}^{R y x}\right) \tilde{D}_{-} \tilde{F}_{-, y}^{R x y} \frac{1}{y^{2}+\left(M_{W} R^{\prime}\right)^{2}}\right] . \tag{6.63}
\end{align*}
$$

The $C_{7 b}$ integral is the sum of two parts corresponding to diagrams with an internal gluon $(G)$ or scalar gluon $\left(G_{5}\right)$ in the loop,

$$
\begin{equation*}
I_{\mathrm{C}_{7 b}}=I_{\mathrm{C}_{7 b}}^{(G)}+I_{\mathrm{C}_{7 b}}^{\left(\mathrm{G}_{5}\right)} \tag{6.64}
\end{equation*}
$$

Each of these terms include diagrams with a single mass insertion, either on the incoming, internal, or outgoing fermion line.

$$
\begin{align*}
& I_{C_{7 b}}=\int d y d x_{1} d x_{2} d x_{3} y\left(\frac{y}{x_{2}}\right)^{4} \partial_{k_{E}} G^{31} \\
& \left\{\frac{1}{2}\left(\frac{y}{x_{1}}\right)^{2+c_{L}}\left(\frac{y}{x_{3}}\right)^{4} \tilde{D}_{+} \tilde{F}_{+,\left(m_{b} R^{\prime}\right)}^{L\left(x_{3} m_{b} R^{\prime} / y\right)\left(m_{b} R^{\prime}\right)}\left(\tilde{D}_{-} \tilde{F}_{-, y}^{L_{12}} \tilde{F}_{-, y}^{L_{23}}+\tilde{F}_{+, y}^{L_{12}} \tilde{D}_{-} \tilde{F}_{-, y}^{L_{23}}\right)\right. \\
& +\frac{1}{2}\left(\frac{y}{x_{1}}\right)^{4}\left(\frac{y}{x_{3}}\right)^{2-c_{R}} \tilde{D}_{+} \tilde{F}_{+,\left(m_{b} R^{\prime}\right)}^{R\left(m_{b} R^{\prime}\right)\left(x_{1} m_{b} R^{\prime} / y\right)}\left(\tilde{D}_{-} \tilde{F}_{-, y}^{R 12} \tilde{y}_{-, y}^{R 23}+\tilde{F}_{+, y}^{R 12} \tilde{D}_{-} \tilde{F}_{-, y}^{R 23}\right) \\
& +\left(\frac{y}{x_{1}}\right)^{2+c_{L}}\left(\frac{y}{x_{3}}\right)^{2-c_{R}}\left(-\tilde{D}_{+} \tilde{F}_{+, y}^{R_{32}} \tilde{D}_{-} \tilde{F}_{-, y}^{R_{22}} \tilde{F}_{+, y}^{L_{y 1}}+y^{2} \tilde{F}_{-, y}^{R_{32}} \tilde{F}_{-, y}^{R_{22}} \tilde{F}_{+, y}^{L_{y 1}}\right. \\
& \left.\left.-\tilde{D}_{-} \tilde{F}_{-, y}^{L L_{2}} \tilde{D}_{+} \tilde{F}_{+, y}^{L_{21}} \tilde{F}_{-, y}^{R_{3 y}}+y^{2} \tilde{F}_{-, y}^{R_{3 y}} \tilde{F}_{+, y}^{L_{2}} \tilde{F}_{+, y}^{L 21}\right)\right\} \tag{6.65}
\end{align*}
$$

$$
\begin{align*}
& I_{C_{7 b}}^{\prime}=\int d y d x_{1} d x_{2} d x_{3} \frac{1}{2}\left(\frac{y}{x_{2}}\right)^{4} \\
& \left\{\left(\frac{y}{x_{1}}\right)^{2+c_{L}}\left(\frac{y}{x_{3}}\right)^{4} \tilde{D}_{+} \tilde{F}_{+,\left(m_{b} R^{\prime}\right)}^{L\left(x_{3} m_{b} R^{\prime} / y\right)\left(m_{b} R^{\prime}\right)} \times\right. \\
& \left(\tilde{F}_{-, y}^{L 12} \tilde{D}_{+} \tilde{F}_{+, y}^{L 23}\left(y \partial_{k_{E}} G_{s}^{31}+4 G_{s}^{31}\right)+y G_{s}^{31}\left(\tilde{D}_{+} \tilde{F}_{+, y}^{L 23} \partial_{k_{E}} \tilde{F}_{-, y}^{L 12}-\tilde{F}_{+, y}^{L 23} \partial_{k_{E}} \tilde{D}_{+} \tilde{F}_{+, y}^{L 2}\right)\right) \\
& +\left(\frac{y}{x_{1}}\right)^{4}\left(\frac{y}{x_{3}}\right)^{2-c_{R}} \tilde{D}_{+} \tilde{F}_{+,\left(m_{b} R^{\prime}\right)}^{R\left(m_{b} R^{\prime}\right)\left(x_{2} m_{b} R^{\prime} / y\right)} \times \\
& \left(\tilde{F}_{-, y}^{R 12} \tilde{D}_{+} \tilde{F}_{+, y}^{R 23}\left(y \partial_{k_{E}} G_{s}^{31}+4 G_{s}^{31}\right)+y G_{s}^{31}\left(\tilde{D}_{+} \tilde{F}_{+, y}^{R 23} \partial_{k_{E}} \tilde{F}_{-, y}^{R 12}-\tilde{F}_{+, y}^{R 23} \partial_{k_{E}} \tilde{D}_{+} \tilde{F}_{+, y}^{R 12}\right)\right) \\
& +\left(\frac{y}{x_{1}}\right)^{2+c_{L}}\left(\frac{y}{x_{3}}\right)^{2-c_{R}} \times \\
& \left(\tilde{D}_{+} \tilde{F}_{+, y}^{L 12}\left(4+y \partial_{k_{E}}\right)\left(\tilde{F}_{+, y}^{L 2 y} \tilde{D}_{+} \tilde{F}_{+, y}^{R y_{3}} G_{s}^{13}\right)-y \tilde{F}_{-, y}^{L 2} \partial_{k_{E}}\left(\tilde{D}_{+} \tilde{F}_{+, y}^{L 2 y} \tilde{D}_{+} \tilde{F}_{+, y}^{R y_{3}}\right) G_{s}^{13}\right. \\
& \left.\left.+\tilde{D}_{+} \tilde{F}_{+, y}^{L 1 y} \tilde{D}_{+} \tilde{F}_{+, y}^{R y 2}\left(4+y \partial_{k_{E}}\right)\left(\tilde{F}_{+, y}^{R 23} G_{5}^{13}\right)-y \tilde{D}_{+} \tilde{F}_{+, y}^{L 1 y} \tilde{F}_{-, y}^{R y 2} G_{5}^{13} \partial_{k_{E}} \tilde{D}_{+} \tilde{F}_{+, y}^{R 23}\right)\right\} \tag{6.66}
\end{align*}
$$

## 6.A. $3 C_{8}$ INTEGRALS

The $C_{8 a}$ integral contains a piece identical to the $C_{7 a}$ integral associated with the charged Higgs loop as well as gluon loop diagrams with three mass insertions,

$$
\begin{equation*}
I_{C_{8 a}}=I_{\mathrm{C}_{8 \alpha}}^{(1)}+2 I_{\mathrm{C}_{8 a}}^{(2)}+I_{\mathrm{C}_{8} a}^{(3)} . \tag{6.67}
\end{equation*}
$$

The gluon loops are labeled by the number of internal mass insertions, so that $I_{C_{8 a}}^{(1)}$ is associated with the diagram with an external mass insertion on each leg, and the factor of two on $I_{C_{s a}}^{(2)}$ accounts for the two possible placements of the external mass insertion ${ }^{4}$.

$$
\begin{align*}
I_{C_{s a}}^{(1)}= & \int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{4}\left(\frac{y}{x_{2}}\right)\left(\frac{y}{x_{3}}\right)^{4} \times \\
& \tilde{D}_{+} \tilde{F}_{+, y_{s}}^{R y_{1}} \tilde{D}_{-} \tilde{F}_{-, y}^{R 1} \tilde{D}_{-} \tilde{F}_{-, y}^{L_{3}} \tilde{D}_{+} \tilde{F}_{+, y_{b}}^{L_{3 y}}\left\{-\frac{5}{2} y \partial_{k_{E}}\left(G_{y}^{12} G_{y}^{23}\right)+10 G_{y}^{12} G_{y}^{23}\right\},  \tag{6.68}\\
I_{C_{s a}}^{(2)}= & \int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{2+C_{L}}\left(\frac{y}{x_{2}}\right)\left(\frac{y}{x_{3}}\right)^{4} y^{3} \times \\
& \tilde{F}_{+, y}^{L_{1} 1} \tilde{F}_{-, y}^{R y y} \tilde{D}_{-} \tilde{F}_{-, y}^{L_{3}{ }_{3}} \tilde{D}_{+} \tilde{F}_{+,\left(m_{b} R^{\prime}\right)}^{L\left(x_{3} m_{b} R^{\prime} y\right)\left(m_{b} R^{\prime}\right)} \partial_{k_{E}}\left(G_{y}^{12} G_{y}^{23}\right)  \tag{6.69}\\
I_{C_{s a}}^{(3)}= & \int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{2+C_{L}}\left(\frac{y}{x_{2}}\right)\left(\frac{y}{x_{3}}\right)^{2-c_{R}} y^{2} \times \\
& \tilde{F}_{+, y}^{L_{1 y}} \tilde{F}_{-, y}^{R y_{3}} \tilde{F}_{+, y}^{L y y} \tilde{F}_{-, y}^{R y y}\left\{-\frac{5}{2} y \partial_{k_{E}}\left(G_{y}^{12} G_{y}^{23}\right)+10 G_{y}^{12} G_{y}^{23}\right\} . \tag{6.70}
\end{align*}
$$

For $C_{8 b}$, the only dominant diagram is the gluon loop with an internal mass insertion. All other analogous diagrams (e.g. mass insertion on an external leg, or loops with $G^{5}$ ) contain no zero modes and hence give negligible contributions after alignment.

$$
\begin{gather*}
I_{C_{s b}}=\int d y d x_{1} d x_{2} d x_{3}\left(\frac{y}{x_{1}}\right)^{2+c_{L}}\left(\frac{y}{x_{2}}\right)\left(\frac{y}{x_{3}}\right)^{2-c_{R}} y^{2} \times \\
\tilde{F}_{+, y}^{L_{1 y}^{1 y}} \tilde{F}_{-, y}^{R y_{3}}\left\{-\frac{5}{2} y \partial_{k_{E}}\left(G_{y}^{12} G_{y}^{23}\right)+10 G_{y}^{12} G_{y}^{23}\right\} . \tag{6.71}
\end{gather*}
$$

[^9]
## 6.B Charged Higgs diagram calculation

As an example of how to calculate diagrams in the mixed position/momentum formalism, we present the calculation of the leading contribution to the anarchic piece of the $C_{7}$ operator coming from the charged Higgs diagram in Figure 6.3.1a. As discussed in Section 6.3.2, it is sufficient to compute the coefficient of the $p_{\mu}$ term in the amplitude. This allows us to directly write the finite physical contribution to the amplitude without worrying about regularization of potentially divergent terms. In addition to the bulk fermion propagators in mixed position/momentum space, $\Delta\left(p, z, z^{\prime}\right)$, which are given in Appendix 6.A.1, the relevant Feynman rules are given by


A derivation of the propagators and a more complete set of Feynman rules is given the appendix of [2]. The amplitude for the diagram with $a b$ of momentum $p$ decaying into a photon of momentum $-q$ and as of momentum $p^{\prime}$ is

$$
\begin{equation*}
\mathcal{M}^{\mu}=\frac{e v}{\sqrt{2}} \frac{R^{8}}{R^{\prime 6}} Y_{s k} Y_{k \ell}^{\dagger} Y_{\ell b} \int \frac{d^{4} k}{(2 \pi)^{4}} \int_{R}^{R^{\prime}} d z\left(\frac{R}{z}\right)^{4} \bar{u}_{Q s}\left(p^{\prime}\right) f_{Q_{s}}\left[G^{\mu}\right]_{k \ell} f_{D_{b}} u_{D_{b}}(p) \Delta_{H}(k-p) \tag{6.72}
\end{equation*}
$$

where $k$ and $\ell$ index the flavors of the internal fermions and $\Delta_{H}$ is the 4 D Higgs propagator. Writing $k^{\prime}=k+q$, the Dirac structure $G^{\mu}$ for the diagram with the mass insertion before (a) or after (b) the photon emisison is

$$
\begin{align*}
& {\left[G_{(a)}^{\mu}\right]_{k \ell}=\Delta_{D_{k}}\left(k^{\prime}, R^{\prime}, z\right) \gamma^{\mu} \Delta_{D_{k}}\left(k, z, R^{\prime}\right) \Delta_{Q_{\ell}}\left(k, R^{\prime}, R^{\prime}\right),}  \tag{6.73}\\
& {\left[G_{(b)}^{\mu}\right]_{k \ell}=\Delta_{D_{k}}\left(k^{\prime}, R^{\prime}, R^{\prime}\right) \Delta_{Q_{\ell}}\left(k, R^{\prime}, z\right) \gamma^{\mu} \Delta_{Q_{\ell}}\left(k, z, R^{\prime}\right) .} \tag{6.74}
\end{align*}
$$

We may now expand the fermion propagators in terms of scalar functions $F$, which are the Minkowski space versions of the $\tilde{F}$ functions defined in Appendix 6.A.1 to simplify the Dirac structure and write the integrand in the form

$$
\begin{equation*}
\bar{u}_{\mathrm{Q}}\left(\bar{g}_{k \ell}^{(n)} \gamma^{\mu} k+g_{k \ell}^{(n)} k^{\prime} \gamma^{\mu}\right) P_{R} u_{D_{b}} \Delta_{H}(k-p) \quad n \in\{a, b\} \tag{6.75}
\end{equation*}
$$

where $g^{(n)}$ is a scalar function that takes the form

$$
\begin{align*}
g_{k \ell}^{(a)}\left(z, k, k^{\prime}\right) & =k^{2}\left[F_{D_{k}}^{-}\left(k^{\prime}, R^{\prime}, z\right)\right]\left[F_{D_{k}}^{-}\left(k, z, R^{\prime}\right)\right]\left[F_{Q_{l}}^{+}\left(k, R^{\prime}, R^{\prime}\right)\right]  \tag{6.76}\\
g_{k \ell}^{(b)}\left(z, k, k^{\prime}\right) & =\left[F_{D_{k}}^{-}\left(k^{\prime}, R^{\prime}, R^{\prime}\right)\right]\left[\tilde{D}_{-} F_{Q_{\ell}}^{-}\left(k^{\prime}, R^{\prime}, z\right)\right]\left[\tilde{D}_{+} F_{Q_{\ell}}^{+}\left(k, z, R^{\prime}\right)\right] . \tag{6.77}
\end{align*}
$$

The derivative operators $\tilde{D}_{ \pm}$are defined in (6.53). $\bar{g}^{(n)}$ has a similar definition but, as we show below, drops out of the final expression.

To identify the $p^{\mu}$ coefficient, which in turn determines the coefficient of the $C_{7}$ effective operator, Taylor expand in $p$ and $q$ and perform the integral. It is sufficient to take only the leading order terms since higher terms are suppressed by the ratio of the external fermion masses to the characteristic loop energy scale (e.g. $m_{H}$ or $1 / R^{\prime}$ ). The terms proportional to $g^{(a)}$ and $g^{(b)}$ thus can be expanded as

$$
\begin{equation*}
g\left(z, k, k^{\prime}\right) k^{\prime} \gamma^{\mu} \Delta_{H}(k-p)=\left.\left(g+\frac{\partial g}{\partial k^{\prime}} \frac{k \cdot q}{k}\right)\right|_{k^{\prime}=k}(k+\notin) \gamma^{\mu}\left(\Delta_{H}(k)+2 k \cdot p \Delta_{H}^{2}(k)\right) \tag{6.78}
\end{equation*}
$$

The $\bar{g}$ terms yield expressions proportional to $\gamma^{\mu} \not p$ and $\gamma^{\mu} \beta^{\prime}$. By using the Clifford algebra and the equations of motion for the external particles one can show that these terms are proportional to $m_{b} \gamma^{\mu}$ and $2 p^{\prime \mu}-m_{s} \gamma^{\mu}$ respectively. Thus these terms can be ignored since these do not contribute to the $p^{\mu}$ coefficient. The $g$ terms, on the other hand, contribute expressions of the form

$$
\begin{equation*}
\left(k^{2} g \Delta_{H}^{2}(k)-\frac{1}{2} k \frac{\partial g}{\partial k^{\prime}} \Delta_{H}(k)-2 g \Delta_{H}(k)\right) p^{\mu}, \tag{6.79}
\end{equation*}
$$

where we write $k=\sqrt{k_{\mu} k^{\mu}}$ and $g$ is evaluated at $q=0$, i.e. $k^{\prime}=k$.
Finally, the coefficient $a_{k \ell}$ of the amplitude (6.5) can be written with respect to the Wick-rotated integral of the prefactor multiplying $p^{\mu}$,

$$
\begin{equation*}
a_{k \ell}=-2 i \sum_{n=a, b} \int_{0}^{\infty} d y \int_{0}^{y} d x y^{2}\left(\frac{y}{x}\right)^{4}\left\{y^{2} g^{(n)} \Delta_{H}+\frac{y}{2} \frac{\partial g^{(n)}}{\partial y^{\prime}}+2 g^{(n)}\right\} \Delta_{H}, \tag{6.80}
\end{equation*}
$$

where we have defined the dimensionless integration variables $x=-i k z, y=-i k R^{\prime}$, and $y^{\prime}=-i(k+q) R^{\prime}$. This is equivalent to replacing the Minkowski space functions $F\left(k R^{\prime}, z R^{\prime}, z^{\prime} R^{\prime}\right)$ with the Euclidean space functions $\tilde{F}\left(y, x, x^{\prime}\right)$ defined in Appendix 6.A.1. The $g$ and $\Delta_{H}$ functions are evaluated at $k \rightarrow i y$ and $m_{H} \rightarrow m_{H} R^{\prime}$. These are now completely scalar expressions that can be evaluated numerically. The explicit form of the integrand is given in (6.63).

Other diagrams are calculated following a similar algorithm with the caveat that diagrams with bulk gauge bosons have ${ }_{5} \mathrm{D}$ propagators, which carry additional space integrals over the extra dimension.

## 6.C Estimating the size of the misalignment contribution

In this appendix we clarify a subtlety in the size of the anarchic contributions $\left(\Delta C_{7,8 a}^{(\prime)}\right)$ versus the misalignment contributions $\left(\Delta C_{7,8 b}^{(\prime)}\right)$ to the Wilson coefficients, as defined in Section 6.3.2. For the anarchic contributions the relative sizes of the right-to-left (unprimed) coefficients to the left-to-right (primed) coefficients are given by the relative size of the $f_{b_{L}}$ and $f_{b_{R}}$ wavefunctions on the IR brane. On the other hand, the misalignment contributions for the two chiral transitions do not follow this pattern and are, in fact, of the same order of magnitude. We show here that this apparent inconsistency can be understood by accounting for cancelations coming from the rotation to the SM fermion mass basis.

For simplicity, consider the $2 \times 2$ matrix of misalignment diagrams $q_{j}^{R} \rightarrow q_{i}^{L}$ where we only consider the second and third generations. This transition is given by the $b_{i j}$ term in ( 6.5 ), which we may parameterize as

$$
\text { (misalignment term) }{ }_{i j} \sim\left(\begin{array}{ll}
(b-c-d) y_{11} & (b-c+d) y_{12}  \tag{6.81}\\
(b+c-d) y_{21} & (b+c+d) y_{22}
\end{array}\right) .
$$

Here we have written $b$ as an average scale for the $b_{i j}$ matrix, and $y_{i j}=f_{Q_{i}} Y_{d i j}^{\dagger} f_{D_{j}}$. The $c \sim 10^{-1}$ and $d \sim 10^{-2}$ terms represent deviations from the average. In particular, the $c$ deviations account for the effect of an internal $b_{L}$ (whose zero mode profile is very different from that of the light quarks) while the $d$ deviations account for the smaller effect of an internal $b_{R}$.

In order to pass to the physical basis, one must apply to this matrix the same rotation that diagonalizes the Sm mass matrix, which is proportional to $y$. The off-diagonal terms of the rotated misalignment matrix give the $C_{7}$ and $C_{7}^{\prime}$ coefficients (the argument for $C_{8}$ is identical),

$$
\left(\begin{array}{rr}
1 & \delta  \tag{6.82}\\
\delta^{\prime} & 1
\end{array}\right) \text { (misalignment term) }\left(\begin{array}{cc}
1 & \gamma \\
\gamma^{\prime} & 1
\end{array}\right) \sim\left(\begin{array}{cc} 
& C_{7 b} \\
C_{7 b}^{\prime \dagger} &
\end{array}\right)
$$

The parameters $\delta$ and $\gamma$ are ratios of the left- and right-handed zero mode wavefunctions on the brane; the primed and unprimed parameters are related by a minus sign.

We focus on order of magnitude estimates, so we introduce a numerical parameter $\varepsilon \sim 10^{-1}$. Normalizing the Yukawa to
$y_{22}=1$, our parameters are approximately

$$
\begin{equation*}
c \sim \varepsilon \quad d \sim \varepsilon^{2} \quad y_{11} \sim \varepsilon^{3} \quad y_{12} \sim \varepsilon^{2} \quad y_{21} \sim \varepsilon \quad \delta^{(\prime)} \sim \varepsilon^{2} \quad \gamma^{(\prime)} \sim \varepsilon \tag{6.83}
\end{equation*}
$$

Note that $\varepsilon$ is merely a fiducial quantity, not an expansion parameter of the model. We now apply the rotation (6.82) and study the order of magnitude of the off-diagonal terms. By construction the terms proportional to $b$ are completely diagonalized. We consider the terms proportional to $c\left(f_{b_{L}}\right)$ and $d\left(f_{b_{R}}\right)$ separately.

## 6.C. 1 Misalignment from $f_{b_{L}}$

First consider the terms proportional to $c$, which are split by the relative size of $f_{b_{L}}$ versus $f_{s_{L}}$ from internal zero mode propagators. The part of the $C_{7 b}^{\prime \dagger}$ term proportional to $c$ goes like

$$
\begin{equation*}
\left.C_{7 b}^{\prime \dagger}\right|_{c} \sim\left(y_{21}+\gamma^{\prime} y_{22}\right)-\delta^{\prime}\left(\gamma^{\prime} y_{12}+y_{11}\right) \tag{6.84}
\end{equation*}
$$

Naively the first term is of $\mathcal{O}(\varepsilon)$ and appears to dominate the expression. This, however, does not account for relations coming from alignment. Observe that the minus sign here comes from the choice of parameterization in $(6.81)$. Further, observe that changing the relative sign in (6.84) is equivalent to changing the sign of $c$ in the top row of (6.81). In this case, however, the $c$ matrix would be completely aligned with the SM mass matrix and the off diagonal term (6.84) would vanish. Thus the first and second terms in (6.84) must be of the same order of magnitude in order for them to cancel when the relative sign is swapped—in other words, $\left(y_{21}+\gamma^{\prime} y_{22}\right) \sim \varepsilon^{5}$ in order to match the naive order of magnitude of the second term. We thus have

$$
\begin{equation*}
\left.c C_{7 b}^{\prime}\right|_{c} \sim \varepsilon^{6} \tag{6.85}
\end{equation*}
$$

This observation reflects the key cancelation that causes the relative size of the primed and unprimed misalignment terms to differ from that of the anarchic terms of the amplitude.

The contribution to the $C_{7 b}$ term proportional to $c$ is

$$
\begin{equation*}
\left.C_{7 b}\right|_{c} \sim \delta\left(\gamma y_{21}+y_{22}\right)-\left(\gamma y_{11}+y_{12}\right) \tag{6.86}
\end{equation*}
$$

Unlike $C_{7 b}^{\prime}$, both terms in the above expression are dominated by their $\mathcal{O}\left(\varepsilon^{2}\right)$ components and we find

$$
\begin{equation*}
\left.c C_{7 b}\right|_{c} \sim \varepsilon \varepsilon^{2}=\varepsilon^{3} \tag{6.87}
\end{equation*}
$$

as expected from a naive estimate.

## 6.C. 2 Misalignment from $f_{b_{R}}$

We perform the same analysis on the terms proportional to $d$, which implicitly encode the split between terms that carry factors of $f_{b_{R}}$ versus $f_{s_{R}}$ from internal propagators. For $C_{7 b}$ we have

$$
\begin{equation*}
\left.C_{7 b}\right|_{d} \sim\left(y_{12}+\delta y_{22}\right)-\gamma\left(y_{11}+\delta y_{21}\right) \tag{6.88}
\end{equation*}
$$

Following the argument that the terms should cancel when the sign is swapped and using this to estimate the size of each bracketed term, one finds $\left.d C_{7 b}\right|_{d} \sim \varepsilon^{6}$, so that the net contribution of the $d$ term is subdominant to (6.87).

On the other hand, the $f_{b_{R}}$ misalignment in the $C_{7 b}^{\prime}$ term cannot be neglected,

$$
\begin{equation*}
\left.C_{7 b}^{\prime}\right|_{d} \sim \gamma^{\prime}\left(\delta^{\prime} y_{12}+y_{22}\right)-\left(\delta^{\prime} y_{11}+y_{21}\right) \tag{6.89}
\end{equation*}
$$

Here both terms are $\mathcal{O}(\varepsilon)$ so that the total contribution is

$$
\begin{equation*}
\left.d C_{7 b}^{\prime \dagger}\right|_{d} \sim \varepsilon^{3} \tag{6.90}
\end{equation*}
$$

which dominates over the term proportional to $c$ in (6.85).

## 6.C. 3 Size of misalignment coefficients

Thus the final order of magnitude estimate for the $C_{7 b}$ and $C_{7 b}^{\prime}$ coefficients are

$$
\begin{aligned}
& \left.C_{7 b} \sim c C_{7 b}\right|_{c} \sim \varepsilon^{3} \\
& \left.C_{7 b}^{\prime} \sim d C_{7 b}^{\prime \dagger}\right|_{d} \sim \varepsilon^{3},
\end{aligned}
$$

so that unlike the anarchic contribution, the right-to-left (unprimed) and left-to-right (primed) Wilson coefficients are of the same order of magnitude.

## 6.D COMMENTS ON 5 D DIPOLE THEORY UNCERTAINTIES

Finite ${ }_{5}$ D loop effects carry subtleties associated with cutoffs and UV sensitivity ${ }^{5}$. While the one loop contribution discussed in this paper is manifestly finite, higher loops are potentially divergent and require explicit calculations. Here we focus on the sensitivity of the finite loop-level result to UV physics at, for example, the strong coupling scale where the ${ }_{5}$ D theory is expected to break down. In [241] it was pointed out that the naive dimensional analysis (NDA) for a brane and a bulk Higgs differ due to the dimension of the Yukawa coupling-the NDA two-loop contribution for the former gives an $\mathcal{O}(1)$ correction relative to the one loop result, whereas this is not expected for the latter. In this appendix we comment on subtleties coming from ${ }_{5}$ D Lorentz invariance that may plausibly avoid this 'worst case' NDA estimate. Indeed, the NDA for the one-loop contribution to these dipole operators is logarithmically divergent; one may understand the correct one-loop finiteness as coming from ${ }_{5} \mathrm{D}$ Lorentz symmetry.

These comments are meant to demonstrate non-trivial points in these calculations that require particular care when drawing conclusions about UV sensitivity in these processes; a more careful investigation with explicit calculations of these effects is beyond the scope of this work.

Note that the general features of the phenomenological picture presented in Section 6.6 are unchanged even if there are $\mathcal{O}(1)$ corrections to the Wilson coefficients.

## 6.D. 1 KK DECOMPOSITION

${ }_{5} \mathrm{D}$ Lorentz invariance imposes that in the кк reduced theory, the 4 D loop momentum cutoff should be matched to the number of kK modes in the effective theory. This was mentioned in [2] to motivate a manifestly ${ }_{5} \mathrm{D}$ calculation by pointing out that naively taking the finite 4 D loop cutoff to infinity drops terms of the form $\left(n M_{\mathrm{KK}} / \Lambda\right)^{2}$, where $n M_{\mathrm{KK}}$ is approximately the mass of the $n^{\text {th }}$ KK mode. Indeed, from the 4 D perspective this may appear to suggest a non-decoupling effect where the dominant contribution comes from heavy кК states so that the calculation seems to be sensitive to UV physics.

However, as demonstrated in Figure 6.D.1, imposing 5D Lorentz invariance requires that each kK mode carries a different 4 D momentum cutoff. In particular, the $n^{\text {th }}$ KK mode carries a smaller 4D cutoff $\Lambda_{n}$ than that of the first KK mode, $\Lambda_{1}$ since the momentum integral must fall within the circle of radius $\Lambda$, the ${ }_{5} \mathrm{D}$ momentum space cutoff. Thus in 4 D the high кк modes are not sensitive to the same cutoff as lower KK modes. This gives a sense in which 4D decoupling can manifest itself while preserving ${ }_{5} \mathrm{D}$ Lorentz invariance. In this sense it is difficult to use this matching to diagnose UV sensitivity.

As a qualitative and demonstrative estimate, one can use the expression in Section 6.6 of [2] for a neutral Higgs diagram and impose a KK number dependent cutoff for each state in the loop so that ${ }_{5} \mathrm{D}$ Lorentz invariance is imposed as in Figure 6.D.1. One finds that, for example, in a sum of 200 kK modes, the highest 20 modes only contribute $\sim 20 \%$ to the total result.

[^10]

Figure 6.D.1: A sketch of the ${ }_{5} \mathrm{D}$ momentum space where the circle of radius $\Lambda$ represents the boundary of a ${ }_{5} \mathrm{D}$ Lorentz invariant loop momentum integration region. Marks on the $k_{z}$ axis show the masses of kK states. Dashed lines demonstrate that the $4_{4} \mathrm{D}$ loop cutoff which respects ${ }_{5} \mathrm{D}$ Lorentz invariance depends on the particular Kk mode.

## 6.D. 25 D cutoff



Figure 6.D.2: Plot of the charged Higgs integrand as a function of the dimensionless loop momentum in the position/momentum space picture. The dashed line is a heuristic 5 D cutoff $\Lambda$ representing the strong coupling scale. The shaded region represents the error from taking the loop momentum to infinity rather than $\Lambda$; the contribution of this shaded region is approximately $15 \%$ of the total integral.

Another way to diagnose UV sensitivity is to consider the effect of a cutoff in the ${ }_{5} \mathrm{D}$ picture, for example, by setting a cutoff at $\Lambda=5 \mathrm{Tev}$ representing the strong coupling scale at which the ${ }_{5} \mathrm{D}$ theory breaks down. Figure 6.D. 2 shows the dimensionless integral associated with the charged Higgs loop, where $y=R^{\prime} k_{E}$ is the dimensionless variable representing the loop momentum. Observe that the dominant contribution to the effect does not come from arbitrarily large $y$ but rather in the peak at low values of $y$. Cutting off the integral at $\Lambda=5 \mathrm{Tev}$ (dashed line) gives an error of approximately $15 \%$, which is
comparable to the subleading diagrams that were not included in this analysis.

Flip: Before I leave, do you know how to use BibTeX?
Csaba: Yes, of course.
[ 15 hours later, Flip's phone rings as he gets to his hotel]
Flip: Hey Csaba, what's up?
Csaba: Where are all the references??
7-8 April 2010

## Supersymmetry

Supersymmetry is one of the leading candidates for physics beyond the Standard Model. It is an extension of quantum field theory to quantum extra dimensions that anticommute. In this way it relates between 'force' particles to 'matter' particles.

### 7.1 Introduction to Supersymmetry

Around the same time that the Beatles released Sgt. Pepper's Lonely Hearts Club Band, Coleman and Mandula published their famous 'no-go' theorem which stated that the most general symmetry Lie group of an $S$-matrix in four dimensions is the direct product of the Poincaré group with an internal symmetry group [316]. In other words, there can be no mixing of spins within a symmetry multiplet.

Ignorance is bliss, however, and physicists continued to look for extensions of the Poincaré symmetry for some years without knowing about Coleman and Mandula's result. In particular, Golfand and Likhtman extended the Poincaré group using Grassmann operators [317], 'discovering' supersymmetry (susy) in physics. Independently, early string theorists where applying similar ideas in two dimensions to insert fermions into a budding theory of superstrings. In this chapter we explore how susy evades the Coleman-Mandula theorem and further highlight remarkable features of supersymmetric field theories. Key papers in the development of supersymmetry are found in [318], while personal descriptions of its discovery have been collected in [319].

Our treatment of susy will be very lopsided. For a working background in supersymmetric field theory, there is now a wealth of standard textbooks and review articles available. In this chapter we shall instead focus on front-loading our exploration with the foundational material and leave phenomenological details to the reader. In this spirit, most of this chapter will be devoted to the susy algebra and its immediate consequences. Appendices 7.A and 7.B review background material on the Lorentz and Poincaré algebras while simultaneously establishing our notations and conventions.

### 7.2 The Poincaré Algebra

The Poincare group describes the symmetries of Minkowski space and is composed of transformations of the form

$$
\begin{equation*}
x^{\mu} \rightarrow x^{\prime \mu}=\Lambda_{v}^{\mu} x^{v}+a^{\mu} \tag{7.1}
\end{equation*}
$$

where $a^{\mu}$ parameterizes translations and $\Lambda^{\mu}{ }_{v}$ parameterizes transformations of the Lorentz group containing rotations and boosts. We can write elements of the Poincaré group as $\{(\Lambda, a)\}$. A pure Lorentz transformation is thus $(\Lambda, o)$ while a pure translation is $(\mathbb{1}, a)$. Elements are multiplied according to the rule

$$
\begin{equation*}
\left(\Lambda_{2}, a_{2}\right) \cdot\left(\Lambda_{1}, a_{1}\right)=\left(\Lambda_{2} \Lambda_{1}, \Lambda_{2} a_{1}+a_{2} .\right) \tag{7.2}
\end{equation*}
$$

Note that these transformations do not commute,

$$
\begin{align*}
& (\Lambda, o) \cdot(\mathbb{1}, a)=(\Lambda, \Lambda a)  \tag{7.3}\\
& (\mathbb{1}, a) \cdot(\Lambda, \circ)=(\Lambda, a) \tag{7.4}
\end{align*}
$$

Thus the Poincaré group is not a direct product of the Lorentz group and the group of 4-translations, but rather a semi-direct product. Locally the Poincaré group is represented by the algebra

$$
\begin{align*}
{\left[M^{\mu v}, M^{\rho \sigma}\right] } & =i\left(M^{\mu \sigma} \eta^{\nu \rho}+M^{\nu \rho} \eta^{\mu \sigma}-M^{\mu \rho} \eta^{v \sigma}-M^{v \sigma} \eta^{\mu \rho}\right)  \tag{7.5}\\
{\left[P^{\mu}, P^{\nu}\right] } & =o  \tag{7.6}\\
{\left[M^{\mu v}, P^{\sigma}\right] } & =i\left(P^{\mu} \eta^{v \sigma}-P^{v} \eta^{\mu \sigma}\right)
\end{align*}
$$

The $M$ are the antisymmetric generators of the Lorentz group,

$$
\begin{equation*}
\left(M^{\mu v}\right)_{\rho \sigma}=i\left(\delta_{\rho}^{\mu} \delta_{\sigma}^{v}-\delta_{\sigma}^{\mu} \delta_{\rho}^{v}\right) \tag{7.8}
\end{equation*}
$$

and the $P$ are the generators of translations. We recognize in $(7.5)$ the usual $O(3)$ Euclidean symmetry by taking $\mu, v, \rho, \sigma \in\{1,2,3\}$ and noting that at most only one term on the right-hand side survives. This coincides with the algebra for angular momenta, J. Equation (7.6) says that translations commute, while (7.7) says that the generators of translations transform as vectors under the Lorentz group. The factors of $i$ are required so that the generators $P$ and $M$ are Hermitian. See Section 4.2.3 for an explicit discussion of Hermiticity. A convenient matrix representation of this algebra is:

$$
\left(\begin{array}{c:c}
M & P  \tag{7.9}\\
\hdashline 0 & 0
\end{array}: 1-1\right)
$$

The 'translation' part of the Poincaré algebra is trivial and requires no further elucidation. It is the Lorentz algebra that yields the interesting features of our fields under Poincaré transformations.

### 7.3 THE SUSY ALGEBRA

SUSY evades the Coleman-Mandula theorem by generalizing the symmetry from a Lie algebra to a graded Lie algebra. This has the property that if $\mathcal{O}_{a}$ are operators, then

$$
\begin{equation*}
\mathcal{O}_{a} \mathcal{O}_{b}-(-1)^{\eta_{a} \eta_{b}} \mathcal{O}_{b} \mathcal{O}_{a}=i C_{a b}^{e} \mathcal{O}_{e} \tag{7.10}
\end{equation*}
$$

where,

$$
\eta_{a}= \begin{cases}0 & \text { if } \mathcal{O}_{a} \text { is bosonic }  \tag{7.11}\\ 1 & \text { if } \mathcal{O}_{a} \text { is fermionic }\end{cases}
$$

The Poincaré generators $P^{\mu}, M^{\mu v}$ are both bosonic generators with $(A, B)=\left(\frac{1}{2}, \frac{1}{2}\right),(1,0) \oplus(0,1)$ respectively. In supersymmetry, on the other hand, we add fermionic generators, $Q_{a}^{A}, \bar{Q}_{\dot{\dot{d}}}^{B}$. Here $A, B=1, \cdots, \mathcal{N}$ label the number of supercharges and $\alpha, \dot{\alpha}=1,2$ are Weyl spinor indices. We primarily focus on simple supersymmetry where $\mathcal{N}=1$. The case $\mathcal{N}>1$ is referred to as extended supersymmetry.

Haag, Lopuszanski, and Sohnius [320] showed in 1974 that ( $\frac{1}{2}, \circ$ ) and ( $0, \frac{1}{2}$ ) are the only generators for supersymmetry. For example, it would be inconsistent to include generators $\tilde{Q}$ in the representation $\left(A, B=\left(\frac{1}{2}, 1\right)\right)$. The resulting algebra is

$$
\begin{align*}
{\left[M^{\mu \nu}, M^{\rho \sigma}\right] } & =i\left(M^{\mu \nu} \eta^{v \rho}+M^{\nu \rho} \eta^{\mu \sigma}-M^{\mu \rho} \eta^{v \sigma}-M^{v \sigma} \eta^{\mu \rho}\right)  \tag{7.12}\\
{\left[P^{\mu}, P^{\nu}\right] } & =0 \\
{\left[M^{\mu \nu}, P^{\sigma}\right] } & =i\left(P^{\mu} \eta^{v \sigma}-P^{v} \eta^{\mu \sigma}\right) \\
{\left[Q_{\alpha}, M^{\mu \nu}\right] } & =\left(\sigma^{\mu \nu}\right)_{\alpha}^{\beta} Q_{\beta}  \tag{7.15}\\
{\left[Q_{a}, P^{\mu}\right] } & =0  \tag{7.16}\\
\left\{Q_{a}, Q^{\beta}\right\} & =0  \tag{7.17}\\
\left\{Q_{a}, \bar{Q}_{\dot{\beta}}\right\} & =2\left(\sigma^{\mu}\right)_{\alpha \dot{\beta}} P_{\mu} \tag{7.18}
\end{align*}
$$

We're already familiar with (7.12) - (7.14) from the Poincare algebra. The remaining terms are motivated by their index and symmetry structure.

### 7.3.1 Commutators with Internal Symmetries

By the Coleman-Mandula theorem, we know that internal symmetry generators commute with the Poincaré generators. For example, the Standard Model gauge group commutes with the momentum, rotation, and boost operators. This carries over to the SUSY algebra. For an internal symmetry generator $T_{a}$,

$$
\begin{equation*}
\left[T_{a}, Q_{a}\right]=0 \tag{7.19}
\end{equation*}
$$

This is true with one exception. The SUSY generators come equipped with their own internal symmetry, called R-symmetry. There exists an automorphism of the supersymmetry algebra,

$$
\begin{align*}
& Q_{a} \rightarrow e^{i t} Q_{a}  \tag{7.20}\\
& \bar{Q}_{\dot{a}} \rightarrow e^{-i t} \bar{Q}_{\dot{\alpha}}, \tag{7.21}
\end{align*}
$$

for some transformation parameter $t$. This is a $U(1)$ internal symmetry. Applying this symmetry preserves the SUSY algebra. If $R$ is the generator of this $U(1)$, then its action on the SUSY operators is given by

$$
\begin{equation*}
Q_{a} \rightarrow e^{-i R t} Q_{a} e^{i R t} . \tag{7.22}
\end{equation*}
$$

By comparing the transformation of $Q$ under (7.22) and (7.22), we find the corresponding algebra,

$$
\begin{align*}
{\left[Q_{a}, R\right] } & =Q_{a}  \tag{7.23}\\
{\left[\bar{Q}_{\dot{a}}, R\right] } & =-\bar{Q}_{\dot{d}} . \tag{7.24}
\end{align*}
$$

## 7.4 $\mathcal{N}=1$ SUPERSYMMETRY

In Appendix 7.B we summarize the representations of the Poincaré group. We now explore what happens when this is extended by $\mathcal{N}=1$ supersymmetry. $C_{1}=P^{2}$ is still a Casimir operator. This means that all the particles in a susy multiplet have the same mass. Now, however, $C_{2}=W^{2}$ is no longer a Casimir. This is intuitive since we saw that the Pauli-Lubanski vector is associated with spin spin and supersymmetry mixes particles of different spins into a single irreducible representation. This is, of course, how it evades the Coleman-Mandula theorem.

In place of $C_{2}$, we can define another Casimir operator, $\tilde{C}_{2}$, in a somewhat oblique way:

$$
\begin{align*}
\tilde{C}_{2} & \equiv C_{\mu \nu} \nu^{\mu \nu}  \tag{7.25}\\
C_{\mu \nu} & \equiv B_{\mu} P_{\nu}-B_{\nu} P_{\mu}  \tag{7.26}\\
B_{\mu} & \equiv W_{\mu}-\frac{1}{4} \bar{Q}_{\dot{\alpha}}\left(\bar{\sigma}_{\mu}\right)^{\dot{\alpha} \alpha} Q_{\alpha} . \tag{7.27}
\end{align*}
$$

Thus our irreducible representations still have two labels, but the second one is no longer related directly to spin.

### 7.4.1 Massless Multiplets

As before we can boost into a frame where $p_{\mu}=(E, \circ, \circ, E)$. Explicit calculation shows that both Casimir operators vanish,

$$
\begin{equation*}
C_{1}=\tilde{C}_{2}=0 \tag{7.28}
\end{equation*}
$$

Now consider the now-familiar anticommutator of $Q$ and $\bar{Q}$ and write it out explicitly as

$$
\left\{Q_{\alpha}, \bar{Q}_{\dot{\beta}}\right\}=2\left(\sigma^{\mu}\right)_{\alpha \dot{\beta}} P_{\mu}=2 E\left(\sigma^{\circ}+\sigma^{3}\right)_{\alpha \dot{\beta}}=4 E\left(\begin{array}{ll}
1 & 0  \tag{7.29}\\
0 & 0
\end{array}\right)
$$

In components,

$$
\left\{Q_{1}, \bar{Q}_{i}\right\}=4 E \quad\left\{Q_{2}, \bar{Q}_{2}\right\}=0
$$

Recall that the $\bar{Q}$ is really short-hand for the complex conjugate of $Q$. Thus the product $\bar{Q}_{\dot{\alpha}} Q_{a}$ for $\dot{\alpha}=\alpha$ is something like $\left|Q_{a}\right|^{2}$ and is non-negative. Thus the second equation tells us that for any massless state $\left|p_{\mu}, \lambda\right\rangle$,

$$
Q_{2}\left|p_{\mu}, \lambda\right\rangle=\mathrm{o}
$$

To be explicit, one can write

$$
\begin{align*}
o & =\left\langle p_{\mu}, \lambda\right|\left\{Q_{2}, \bar{Q}_{i}\right\}\left|p^{\mu}, \lambda\right\rangle  \tag{7.32}\\
& =\left\langle p_{\mu}, \lambda\right| Q_{2} \bar{Q}_{i}\left|p^{\mu}, \lambda\right\rangle+\left\langle p^{\mu}, \lambda\right| \bar{Q}_{i} Q_{2}\left|p^{\mu}, \lambda\right\rangle  \tag{7.33}\\
& \left.\left.=\left|\bar{Q}_{\dot{2}}\right| p_{\mu}, \lambda\right\rangle\left.\right|^{2}+\left|Q_{2}\right| p_{\mu}, \lambda\right\rangle\left.\right|^{2}
\end{align*}
$$

from which each term on the right hand side must vanish and we obtain (7.31).
Using (7.30) we can define raising and lowering operators,

$$
\begin{equation*}
a \equiv \frac{Q_{1}}{2 \sqrt{E}} \quad a^{\dagger} \equiv \frac{\bar{Q}_{i}}{2 \sqrt{E}} \tag{7.35}
\end{equation*}
$$

These satisfy the relation $\left\{a, a^{\dagger}\right\}=1$. Consider the spin of a massless state after acting with these operators:

$$
\begin{align*}
J^{3} a\left|p_{\mu}, \lambda\right\rangle & =\left(a J^{3}-\left[a, J^{3}\right]\right)\left|p^{\mu}, \lambda\right\rangle  \tag{7.36}\\
& =\left(\lambda-\frac{1}{2}\right) a\left|p_{\mu}, \lambda\right\rangle \tag{7.37}
\end{align*}
$$

We have used the fact that $\left[J^{3}, Q_{1,2}\right]=\mp \frac{1}{2} Q_{1,2}$. We find that if we start with a state $\left|p_{\mu}, \lambda\right\rangle$ of helicity $\lambda$, then acting with
 helicity $\left(\lambda+\frac{1}{2}\right)$.

Evidently, we can generate the following states:

$$
\begin{align*}
\left|p_{\mu}, \lambda\right\rangle & \text { helicity } \lambda  \tag{7.38}\\
a\left|p_{\mu}, \lambda\right\rangle & \text { helicity }\left(\lambda-\frac{1}{2}\right)  \tag{7.39}\\
a^{\dagger}\left|p_{\mu}, \lambda\right\rangle & \text { helicity }\left(\lambda+\frac{1}{2}\right) \tag{7.40}
\end{align*}
$$

From this we can build a supermultiplet. We start with a state that is annihilated by the lowering operator, i.e. a state of minimum helicity $|\Omega\rangle=\left|p_{\mu}, \lambda\right\rangle$ such that $a|\Omega\rangle=0$. The next state we can construct comes from acting on $|\Omega\rangle$ with a creation operator,

$$
\begin{equation*}
a^{\dagger}|\Omega\rangle=\left|p_{\mu},\left(\lambda+\frac{1}{2}\right)\right\rangle \tag{7.41}
\end{equation*}
$$

From here, acting with another creation operator, $a^{\dagger} a^{\dagger}|\Omega\rangle$, vanishes since $a^{\dagger} a^{\dagger} \equiv o$ from the Grassmann nature of the SUSY generator. To exhaust our possibilities, $a a^{\dagger}|\Omega\rangle=\left(1-a^{\dagger} a\right)|\Omega\rangle=|\Omega\rangle$. Thus our massless $\mathcal{N}=1$ supersymmetry multiplet has only two states, $\left|p_{\mu}, \lambda\right\rangle$ and $\left|p_{\mu},\left(\lambda+\frac{1}{2}\right)\right\rangle$. We have paired a bosonic and a fermionic state, so we're happy that this is supersymmetric in an intuitive way. We haven't said anything about what the lowest helicity $\lambda$ is, and in fact we are free to choose this.

Let us note here that nature respects the discrete СРт symmetry. Thus if we construct a model of a massless supermultiplet that is not СРт self-conjugate, then we are obliged to also add a partner СРт-conjugate multiplet as well. For example, if $\lambda=\frac{1}{2}$, then our construction yields a multiplet with a fermion of helicity $\lambda=\frac{1}{2}$ and a vector partner with helicity $\lambda=1$. СPT invariance mandates that we must also have a fermion with helicity $\lambda=-\frac{1}{2}$ and a vector partner with helicity $\lambda=-1$. More generally, СРт compels us to fill in our massless multiplets with states $\left|p_{\mu}, \pm \lambda\right\rangle$ and $\left|p_{\mu}, \pm\left(\lambda+\frac{1}{2}\right)\right\rangle$.

### 7.4.2 Massive Multiplets

Having fleshed out the massless supermultiplet, do the same for the massive multiplets. In this case we boost to a particle's rest frame,

$$
\begin{equation*}
p_{\mu}=(m, \mathrm{o}, \mathrm{o}, \mathrm{o}) . \tag{7.42}
\end{equation*}
$$

The Casimir operators are given by

$$
\begin{equation*}
C_{1}=m^{2} \quad \tilde{C}_{2}=2 m^{4} Y^{i} Y_{i} \tag{7.43}
\end{equation*}
$$

where $Y=J_{i}-\frac{1}{4 m}\left(Q \sigma_{i} \bar{Q}\right)$ is the superspin. The nice feature of the superspin is that

$$
\begin{equation*}
\left[Y_{i}, Y_{j}\right]=i \varepsilon_{i j k} Y_{k}, \tag{7.44}
\end{equation*}
$$

that is they satisfy the same algebra as the angular momentum operators, $J_{i}$. Thus we can label a multiplet by its mass $m$ and superspin $y$, the root of the eigenvalue of $Y^{2}$. As before, we can work out the anticommutator of the sUSY generators acting on a state with $p_{\mu}=(m, \mathrm{o}, \mathrm{o}, \mathrm{o})$ :

$$
\left\{Q_{\alpha}, \bar{Q}_{\dot{\beta}}\right\}=2 m\left(\begin{array}{ll}
1 & 0  \tag{7.45}\\
0 & 1
\end{array}\right) .
$$

We now have two sets of raising and lowering operators,

$$
\begin{equation*}
a_{1,2}=\frac{1}{\sqrt{2 m}} Q_{1,2} \tag{7.46}
\end{equation*}
$$

$$
a_{1,2}^{\dagger}=\frac{1}{\sqrt{2 m}} \bar{Q}_{i, i}
$$

These satisfy the anticommutation relations

$$
\begin{equation*}
\left\{a_{p}, a_{q}^{\dagger}\right\}=\delta_{p q} \quad\left\{a_{p}, a_{q}\right\}=0 \quad\left\{a_{p}^{\dagger}, a_{q}^{\dagger}\right\}=0 \tag{7.47}
\end{equation*}
$$

As before we define a ground state $|\Omega\rangle$ that is annihilated by both $a_{1}$ and $a_{2}, a_{1,2}|\Omega\rangle=0$. Note that for the ground state, $\mathbf{Y}|\Omega\rangle=\mathbf{J}|\Omega\rangle$, and so we can label the ground state by

$$
\begin{equation*}
|\Omega\rangle=\left|m, y=j ; p_{\mu}, j_{3}\right\rangle . \tag{7.48}
\end{equation*}
$$

The spin in the $z$-direction, $j_{3}$, takes values from $-y$ to $y$ and so there are $(2 y+1)$ ground states.
We can now act on $|\Omega\rangle$ with creation operators. The resulting states are

$$
\begin{align*}
& a_{1}^{\dagger}|\Omega\rangle=\left|m, j=y+\frac{1}{2} ; p_{\mu}, j_{3}\right\rangle  \tag{7.49}\\
& a_{2}^{\dagger}|\Omega\rangle=\left|m, j=y-\frac{1}{2} ; p_{\mu}, j_{3}\right\rangle . \tag{7.50}
\end{align*}
$$

We see that $a_{1}^{\dagger}|\Omega\rangle$ has $2\left(y+\frac{1}{2}\right)+1=2 y+2$ states while $a_{2}^{\dagger}|\Omega\rangle$ has $2\left(y-\frac{1}{2}\right)+1=2 y$ states. This can be understood group theoretically, since

$$
\begin{equation*}
\frac{1}{2} \otimes j=\left(j-\frac{1}{2}\right) \oplus\left(j+\frac{1}{2}\right) \tag{7.51}
\end{equation*}
$$

We're going to want to keep track of these to make sure that our bosonic and fermionic degrees of freedom match.
Unlike the massless case, we can now form a state with two creation operators,

$$
\begin{equation*}
a_{1}^{\dagger} a_{2}^{\dagger}|\Omega\rangle=-a_{2}^{\dagger} a_{1}^{\dagger}|\Omega\rangle=\left|m, j=y ; p_{\mu}, j_{3}\right\rangle=\left|\Omega^{\prime}\right\rangle . \tag{7.52}
\end{equation*}
$$

This state looks very similar to the base state $\Omega$, but the two are not equivalent: $\left.\Omega^{\prime}\right\rangle$ is annihilated by the $a^{\dagger}$ s rather than the as:

$$
\begin{align*}
& a_{1,2}^{\dagger}\left|\Omega^{\prime}\right\rangle=0  \tag{7.53}\\
& a_{1,2}|\Omega\rangle=\mathrm{o}
\end{align*}
$$

(7.54)

The $a_{p}^{\dagger}$ and $a_{p}$ are related by a parity transformation:

$$
\begin{equation*}
\underbrace{a_{1,2}^{\dagger}}_{\left(0, \frac{1}{2}\right)} \leftrightarrow \underbrace{a_{1,2}}_{\left(\frac{1}{2}, 0\right)} \tag{7.55}
\end{equation*}
$$

and so the above equation suggests that $|\Omega\rangle$ and $\left|\Omega^{\prime}\right\rangle$ are also related by parity. Then we can define parity eigenstates

For $y=o$ the $|+\rangle$ is a scalar while $|-\rangle$ is a pseudoscalar. We can see that the fermionic and bosonic states have the same number of degrees of freedom. $|\Omega\rangle$ and $\left|\Omega^{\prime}\right\rangle$ each have $2 y+1$ states, while $a_{1,2}^{\dagger}|\Omega\rangle$ give $(2 y+1) \pm 1$ states. Hence there sums are each $4 y+2$, and hence the number of fermionic and bosonic states are equal.

In summary, for $y>0$, we have the states

$$
\begin{align*}
|\Omega\rangle & =\left|m, j=y ; p_{\mu}, j_{3}\right\rangle  \tag{7.57}\\
\left|\Omega^{\prime}\right\rangle & =\left|m, j=y ; p_{\mu}, j_{3}\right\rangle  \tag{7.58}\\
a_{1}^{\dagger}|\Omega\rangle & =\left|m, j=y+\frac{1}{2} ; p_{\mu}, j_{3}\right\rangle  \tag{7.59}\\
a_{2}^{\dagger}|\Omega\rangle & =\left|m, j=y-\frac{1}{2} ; p_{\mu}, j_{3}\right\rangle . \tag{7.60}
\end{align*}
$$

For $y=0$, we have the states

$$
\begin{align*}
|\Omega\rangle & =\left|m, j=o ; p_{\mu}, j_{3}\right\rangle  \tag{7.61}\\
\left|\Omega^{\prime}\right\rangle & =\left|m, j=o ; p_{\mu}, j_{3}\right\rangle  \tag{7.62}\\
a_{1}^{\dagger}|\Omega\rangle & =\left|m, j=\frac{1}{2} ; p_{\mu}, j_{3}= \pm \frac{1}{2}\right\rangle . \tag{7.63}
\end{align*}
$$

### 7.4.3 Equality of Fermionic and Bosonic States

In any susy multiplet, the number $n_{B}$ of bosons equals the number $n_{F}$ of fermions. To show this, make use of the operator $(-)^{F}$, which assigns a 'parity' to a state depending on whether it is a boson $(|B\rangle)$ or fermion $(|F\rangle)$ :

$$
\begin{equation*}
(-)^{F}|B\rangle=|B\rangle \quad(-)^{F}|F\rangle=-|F\rangle \tag{7.64}
\end{equation*}
$$

Observe that this operator anticommutes with the susy generators since

$$
\begin{equation*}
(-)^{F} Q_{\alpha}|F\rangle=(-)^{F}|B\rangle=|B\rangle=Q_{\alpha}|F\rangle=-Q_{\alpha}(-)^{F}|F\rangle \tag{7.65}
\end{equation*}
$$

Let us now calculate the following curious-looking trace:

$$
\begin{align*}
\operatorname{Tr}\left\{(-)^{F}\left\{Q_{a}, \bar{Q}_{\dot{\beta}}\right\}\right\} & =\operatorname{Tr}\left\{(-)^{F} Q_{a} \bar{Q}_{\dot{\beta}}+(-)^{F}{\left.\overline{Q_{\dot{\beta}}} Q_{a}\right\}}\right.  \tag{7.66}\\
& =\operatorname{Tr}\{\underbrace{-Q_{a}(-)^{F} \bar{Q}_{\dot{\beta}}}_{\text {Using anticommutator }}+\underbrace{Q_{a}(-)^{F} \bar{Q}_{\dot{\beta}}}_{\text {Using cyclicity of trace }}\}  \tag{7.67}\\
& =0 . \tag{7.68}
\end{align*}
$$

But since $\left\{Q_{\alpha}, \bar{Q}_{\dot{\beta}}\right\}=2\left(\sigma^{\mu}\right)_{\alpha \dot{\beta}} P_{\mu}$, the above trace is

$$
\begin{equation*}
\operatorname{Tr}\left\{(-)^{F} 2\left(\sigma^{\mu}\right)_{\alpha \dot{\beta}} P_{\mu}\right\}=2\left(\sigma^{\mu}\right)_{\alpha \dot{\beta}} P_{\mu} \operatorname{Tr}\left((-)^{F}\right), \tag{7.69}
\end{equation*}
$$

and hence $\operatorname{Tr}\left((-)^{F}\right)=0$. This trace is called the Witten index and will plays a central role in susy breaking. The Witten index can be written more explicitly as a sum over bosonic and fermionic states,

$$
\begin{align*}
\operatorname{Tr}\left((-)^{F}\right) & =\sum_{B}\langle B|(-)^{F}|B\rangle+\sum_{F}\langle F|(-)^{F}|F\rangle  \tag{7.70}\\
& =\sum_{B}\langle B \mid B\rangle-\sum_{F}\langle F \mid F\rangle  \tag{7.71}\\
& =n_{B}-n_{F} . \tag{7.72}
\end{align*}
$$

Thus the vanishing of the Witten index implies that $n_{B}=n_{F}$, or that there are an equal number of bosonic and fermionic states.

### 7.5 Superspace

We would like to transfer our algebraic understanding of supersymmetry to a geometric (or 'field-theoretical') understanding of the transformation of fields in susy multiplets. Superspace, developed by Strathdee and Salam in 1974 [321,322], is a convenient way to do this. Our main goal will be to identify the elements of the susy algebra (the generators of supersymmetry) with differential operators acting on fields.


Figure 7.5.1: Illustration of a coset space.

### 7.5.1 Coset spaces

Lie groups are also manifolds. For example, for $U(1)$, we may write $g=e^{i a}$ with $\alpha \in[0,2 \pi]$. Thus the manifold associated with $G$ is a circle, $M_{U(1)}=S^{1}$. Similarly, one finds that the manifold associated with $S U(2)$ is a 3 -sphere, $M_{S U(2)}=S^{3}$. Cosets, $G / H$ ("elements of $G$ that aren't in $H$ "), can be used to define more general manifolds. A coset is composed of equivalence classes,

$$
\begin{equation*}
g \equiv g h, \quad \forall h \in H \tag{7.73}
\end{equation*}
$$

This coset can be used to define submanifolds of $G$. For example $S^{2}$ is given by $\operatorname{SU}(2) / U(1)$. We may draw this heuristically: Here the $x$-and $y$-axes represent the transformation parameters for the $S U(2)$ generators. The manifold for $S U(2)$ is represented by the light green square. The dotted red line represents a section of $U(1)$ that we would like to identify as part of the equivalence class for a point $g$. The solid blue line represents the coset $S U(2) / U(1)$. More generally, we may write $S^{n}=S O(n+1) / S O(n)$.

We would like to use a cosets space to define superspace through supersymmetry (or 'super Poincare' symmetry). As an illustrative example, we may define Minkowski space as the coset space 'Poincaré/Lorentz', or $P / S O(3,1)^{\uparrow}$ where $P$ is the Poincaré group.This is an intuitive statement since one can map the generators of translations with points on Minkowski space,

$$
\begin{equation*}
g_{P}=e^{i\left(\omega_{\mu \nu} M^{\mu v}+a_{\mu} P^{\mu}\right)}, \tag{7.74}
\end{equation*}
$$

while the generators of the Lorentz group take the form

$$
\begin{equation*}
g_{L}=e^{i\left(\omega_{\mu \nu} M^{\mu \nu}\right)} \tag{7.75}
\end{equation*}
$$

One can thus identify the coset manifold with the translation parameters,

$$
\begin{equation*}
M_{\text {Poincaré/Lorentz }}=\left\{a^{\mu}\right\} . \tag{7.76}
\end{equation*}
$$

Multiplication of group elements correspond to successive translations on the Minkowski manifold. This is, of course, a bit of overkill for the rather trivial case of Minkowski space.

We now generalize this idea to an (arguably) non-trivial case: the coset space ( $\mathcal{N}=1$ super-Poincaré)/Lorentz, or $S P / S O(3,1)^{\uparrow}$. We call the resulting manifold $\mathcal{N}=1$ superspace. The generators of the super-Poincaré group take the form

$$
\begin{equation*}
g_{S P}=e^{i\left(\omega_{\mu \nu} M^{\mu \nu}+a_{\mu} P^{\mu}+\theta^{a} Q_{a}+\bar{\theta}_{\bar{a}} \bar{Q}^{\dot{a}}\right)}, \tag{7.77}
\end{equation*}
$$

were $\omega_{\mu \nu}$ and $a_{\mu}$ are the usual commuting ' $c$-number'parameters for the Poincaré group while $\theta$ and $\bar{\theta}$ are anticommuting Grassmann parameters. Thus we may write coordinates for $\mathcal{N}=1$ superspace as

$$
\begin{equation*}
\left\{a^{\mu}, \theta^{a}, \bar{\theta}_{\dot{\alpha}}\right\} \tag{7.78}
\end{equation*}
$$

In this sense, supersymmetry is a fermionic extra dimension. The products $\theta Q$ and $\bar{\theta} \bar{Q}$ are commuting objects, and so we may write the susy algebra using commutators,

$$
\begin{equation*}
\left\{Q_{\alpha}, \bar{Q}_{\dot{\alpha}}\right\}=2\left(\sigma^{\mu}\right)_{\alpha \dot{\alpha}} P_{\mu} \Rightarrow\left[\theta^{a} Q_{\alpha}, \bar{\theta}^{\dot{\beta}} \bar{Q}_{\dot{\beta}}\right]=2 \theta^{a}\left(\sigma^{\mu}\right)_{\alpha \dot{\beta}} \bar{\theta}^{\dot{\beta}} P_{\mu} . \tag{7.79}
\end{equation*}
$$

This will allow us to apply useful results from non-graded Lie algebras, such as the Baker-Campbell-Hausdorff formula for the product of exponentiated generators.

Armed with this spacetime extended by Grassmann coordinates, we may proceed to define superfields as a generalization of the usual fields that live on Minkowski space. These fields contain entire susy multiplets of component Minkowski-space fields.

### 7.5.2 Grassmann calculus

Now that we've generalized Minkowski space to superspace, we would like to write Lagrangian densities on superspace such that the action is given by an integration over $d^{4} x d^{2} \theta d^{2} \bar{\theta}$. In order to do this recall the calculus of Grassmann variables [323].

Superspace extends Minkowski space with two spinor degrees of freedom, $\theta_{\alpha}$ and $\theta_{\dot{\alpha}}$. It is useful to review conventions for the calculus of fermionic Weyl spinor variables.

We may define differentiation in the usual way,

$$
\begin{equation*}
\frac{\partial}{\partial \theta^{\alpha}} \theta^{\beta}=\delta_{a}^{\beta} \quad \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} \bar{\theta}_{\dot{\beta}}=\delta_{\dot{\alpha}}^{\dot{\beta}} \tag{7.80}
\end{equation*}
$$

Note that $\partial / \partial \theta^{\alpha}$ transforms as a lower-index left-handed spinor (i.e. $\psi_{\alpha}$-type) and $\partial / \partial \bar{\theta}_{\dot{\alpha}}$ transforms as an upper-index right-handed spinor (i.e. $\bar{\chi}^{\dot{a}}$-type). This is completely analogous to the case of vector derivatives where $\partial / \partial x^{\mu}$ transforms as a lower-index object. One may further check that

$$
\begin{equation*}
\frac{\partial}{\partial \theta^{\alpha}}(\theta \theta)=2 \theta_{\alpha} \quad \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}}(\overline{\theta \theta})=2 \bar{\theta}_{\dot{\alpha}} \tag{1}
\end{equation*}
$$

Following the convention of ( 7.80 ), however, we run into an immediate issue of consistency that requires some care. Suppose we naively defined the $\partial / \partial \theta_{a}$ and $\partial / \partial \bar{\theta}^{\dot{a}}$ partial derivatives in the same way. Then we'd run into problems since (ignoring the index height on the Kronecker $\delta$ ),

$$
\begin{equation*}
\frac{\partial}{\partial \theta^{\alpha}} \theta^{\beta}=\delta_{\alpha}^{\beta} \stackrel{?}{=} \frac{\partial}{\partial \theta_{\alpha}} \theta_{\beta}, \tag{7.82}
\end{equation*}
$$

while we also have, from (7.202),

$$
\begin{equation*}
\frac{\partial}{\partial \theta^{a}} \theta^{\beta}=-\frac{\partial}{\partial \theta_{\alpha}} \theta_{\beta} \tag{7.83}
\end{equation*}
$$

The only way for (7.82) and (7.83) to be consistent is if both types of derivatives are identically zero... which would make for a pitifully trivial theory indeed. Thus our naive guess in (7.82) must not be correct. A more careful analysis shows

$$
\begin{align*}
\frac{\partial}{\partial \theta_{\alpha}} \theta_{\beta} & =\varepsilon^{\alpha \beta} \frac{\partial}{\partial \theta^{\beta}} \varepsilon_{\alpha \gamma} \theta^{\gamma}  \tag{7.84}\\
& =-\frac{\partial}{\partial \theta^{a}} \theta^{\alpha} . \tag{7.85}
\end{align*}
$$

Since we've already defined at the variable $\theta$ has its index raised and lowered like a usual left-handed Weyl spinor, we are led to the following definitions for raising and lowering indices on spinor derivatives,

$$
\begin{equation*}
\frac{\partial}{\partial \theta_{\alpha}}=-\varepsilon^{\alpha \beta} \frac{\partial}{\partial \theta^{\beta}} \quad \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}=-\varepsilon_{\dot{\alpha} \dot{\beta}} \frac{\partial}{\partial \bar{\theta}_{\dot{\beta}}} . \tag{7.86}
\end{equation*}
$$

We define the two-dimensional integral as

$$
\begin{equation*}
\int d^{2} \theta \equiv \frac{1}{2} \int d \theta^{1} d \theta^{2} \tag{7.87}
\end{equation*}
$$

such that

$$
\begin{equation*}
\int d^{2} \theta(\theta \theta)=1 . \tag{7.88}
\end{equation*}
$$

We use the same normalization for the right-handed superspace coordinates, and can thus write the integral over both $\theta$ and $\bar{\theta}$ as

$$
\begin{equation*}
\int d^{2} \theta \int d^{2} \bar{\theta}(\theta \theta)(\overline{\theta \theta})=\int d^{4} \theta(\theta \theta)(\overline{\theta \theta})=1 \tag{7.89}
\end{equation*}
$$

where we have defined measure $d^{4} \theta=d^{2} \theta d^{2} \bar{\theta}$.
Finally, we can introduce an inner product for superfields,

$$
\begin{equation*}
\langle F(x, \theta, \bar{\theta}), G(x, \theta, \bar{\theta})\rangle=\int d^{4} x d^{4} \theta F^{*}(x, \theta, \bar{\theta}) G(x, \theta, \bar{\theta}) \tag{7.90}
\end{equation*}
$$

This means that we can also define a superspace Hermitian conjugation operation, ${ }^{\dagger}$. For example, using integration by parts the Hermitian conjugate of the (Minkowski) spacetime derivative behaves as

$$
\begin{equation*}
\partial_{\mu}^{\dagger}=-\partial_{\mu} . \tag{7.91}
\end{equation*}
$$

This Hermitian conjugation is antilinear (i.e. it "represents an involutive anti-homomorphism"), for complex coefficients $a, b$ and superfields $\varphi, \psi$,

$$
\begin{align*}
(a \varphi+b \psi)^{\dagger} & =\varphi^{\dagger} a^{*}+\psi^{\dagger} b^{*}  \tag{7.92}\\
(\varphi \psi)^{\dagger} & =\psi^{\dagger} \varphi^{\dagger} . \tag{7.93}
\end{align*}
$$

### 7.5.3 GENERAL SUPERFIELDS

We can now define superfields as scalar functions of superspace. These superfields are complete susy multiplets containing fields of different spins. We may Taylor expand a superfield $S\left(x^{\mu}, \theta^{\alpha}, \bar{\theta}_{\dot{\alpha}}\right)$ in its Grassmann variables,

$$
\begin{align*}
S\left(x^{\mu}, \theta^{\alpha}, \bar{\theta}_{\dot{\alpha}}\right)= & a(x)+\theta^{\alpha} b_{\alpha}(x)+\bar{\theta}_{\dot{\alpha}} \bar{c}^{\dot{a}}(x)+\theta \theta d(x)+\overline{\theta \theta} e(x) \\
& +\theta^{\alpha} f_{\alpha \dot{\beta}}(x) \bar{\theta}^{\dot{\beta}}+\theta \theta \bar{\theta}_{\dot{\alpha}} \bar{g}^{\dot{g}}(x)+\overline{\theta \theta} \theta^{a} h_{\alpha}(x)+\theta \theta \overline{\theta \theta} j(x), \tag{7.94}
\end{align*}
$$

where we've used the fact that the spinor expansion is finite. The components $a(x), b_{\alpha}(x), \cdots$ are fields on Minkowski space and we see that the Taylor expansion requires them to have certain spins. Terms like $\theta^{a} s_{a}{ }^{\beta}(x) \theta_{\beta}$ can be written as a contribution to $d(x)$. Further, by (7.207), we may write the field $f_{\alpha \dot{\beta}}(x)$ as a vector,

$$
\begin{equation*}
f_{\alpha \dot{\beta}}(x)=V_{\mu}(x)\left(\sigma^{\mu}\right)_{\alpha \dot{\beta}} . \tag{7.95}
\end{equation*}
$$

Thus let us rewrite our superfield expansion using a standard notation,

$$
\begin{align*}
S\left(x^{\mu}, \theta^{\alpha}, \bar{\theta}_{\dot{\alpha}}\right)= & \phi(x)+\theta \psi(x)+\bar{\theta} \bar{\chi}(x)+\theta \theta M(x)+\overline{\theta \theta} N(x) \\
+ & \left(\theta \sigma^{\mu} \bar{\theta}\right) V_{\mu}(x)+\theta \theta \overline{\theta \lambda}(x)+\overline{\theta \theta} \theta \rho(x)+\theta \theta \overline{\theta \theta} D(x), \tag{7.96}
\end{align*}
$$

where we have suppressed spinor indices using our convention for the contraction of those indices.

### 7.6 SUSY DIFFERENTIAL OPERATORS

To be a 'true' superfield, $S(x, \theta, \bar{\theta})$ must transform properly under susy. Let us refresh our memory with the transformation of non-susy fields on Minkowski space in non-supersymmetric field theory. Recall that a Minkowski-space field $\varphi(x)$ transforms under translations,

$$
\begin{equation*}
\varphi \rightarrow e^{-i a^{\mu} P_{\mu}} \varphi e^{i a^{\mu} P_{\mu}} \tag{7.97}
\end{equation*}
$$

where $P_{\mu}$ is the abstract generator of translations and $\varphi$ is being thought of as an operator. Alternately, we can think of $\varphi$ as a function that transforms under translations via the differential operator $\mathcal{P}$,

$$
\begin{equation*}
\varphi(x) \rightarrow e^{i a^{\mu} \mathcal{P}_{\mu}} \varphi(x)=\varphi(x+a) . \tag{7.98}
\end{equation*}
$$

By comparing both transformations for infinitesimal parameter $a$, we find that

$$
\begin{equation*}
\delta \varphi=i\left[\varphi, a^{\mu} P_{\mu}\right]=i a^{\mu} \mathcal{P}_{\mu} \varphi=a^{\mu} \partial_{\mu} \varphi, \tag{7.99}
\end{equation*}
$$

that is: we may write the differential operator as

$$
\begin{equation*}
\mathcal{P}=-i \frac{\partial}{\partial x^{\mu}} \tag{7.100}
\end{equation*}
$$

We now perform the same analysis for the susy generators $Q_{\alpha}$ and $\bar{Q}_{\dot{\alpha}}$. As an operator, a superfield $S$ transforms under infinitesimal parameters $\varepsilon_{\alpha}$ and $\bar{\varepsilon}_{\dot{\alpha}}$ as

$$
\begin{equation*}
S(x, \theta, \bar{\theta}) \rightarrow e^{-i(\varepsilon Q+\bar{\varepsilon} \bar{Q})} S(x, \theta, \bar{\theta}) e^{i(\varepsilon Q+\bar{\varepsilon} \bar{Q})} \tag{7.101}
\end{equation*}
$$

Alternately, we may define superspace differential operators $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ so that the superfields transform as

$$
\begin{equation*}
S(x, \theta, \bar{\theta}) \rightarrow e^{i(\varepsilon \mathcal{Q}+\bar{\varepsilon} \bar{Q})} S(x, \theta, \bar{\theta})=S(x+\delta x, \theta+\varepsilon, \bar{\theta}+\bar{\varepsilon}) \tag{7.102}
\end{equation*}
$$

We've written in a motion in Minkowski space, $\delta x$, with the foresight that supersymmetry transformations are a "square root" of translations so we ought to provide for the sUSY differential operators also having some Minkowski space component. The most general form that $\delta x$ can take given the parameters $\varepsilon_{a}$ and $\bar{\varepsilon}_{\dot{q}}$ is

$$
\begin{equation*}
\delta x^{\mu}=-i c\left(\varepsilon \sigma^{\mu} \bar{\theta}\right)+i c^{*}\left(\theta \sigma^{\mu} \bar{\varepsilon}\right) \tag{7.103}
\end{equation*}
$$

where we have demanded that $\delta x \in \mathbb{R}$ and $c$ is a constant that we would like to determine. From an analogous argument as that for $\mathcal{P}$, we can look at infinitesimal transformations to determine the susy differential operators:

$$
\begin{equation*}
\delta S=i[S, \varepsilon \mathcal{Q}+\bar{\varepsilon} \overline{\mathbb{Q}}]=i(\varepsilon \mathcal{Q}+\bar{\varepsilon} \overline{\mathcal{Q}}) S, \tag{7.104}
\end{equation*}
$$

from which we find

$$
\begin{align*}
& \varepsilon^{\alpha} \mathcal{Q}_{\alpha}=-i \varepsilon^{\alpha} \frac{\partial}{\partial \theta^{\alpha}}-c \varepsilon^{\alpha}\left(\sigma^{\mu}\right)_{\alpha \dot{\alpha}} \bar{\theta}^{\dot{\theta}} \frac{\partial}{\partial x^{\mu}}  \tag{7.105}\\
& \bar{\varepsilon}_{\dot{\alpha}} \overline{\mathcal{Q}}^{\dot{\alpha}}=-i \bar{\varepsilon}_{\dot{\alpha}} \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}}+c^{*} \theta^{\alpha}\left(\sigma^{\mu}\right)_{\alpha \dot{\alpha}} \dot{\varepsilon}^{\dot{\varepsilon}} \frac{\partial}{\partial x^{\mu}} . \tag{7.106}
\end{align*}
$$

We would like to 'peel off' the transformation parameters $\varepsilon$ and $\bar{\varepsilon}$. This is straightforward for the first equation since the $\varepsilon$ appears with the same index height and on the left of the spinor structure for every term,

$$
\begin{equation*}
\mathcal{Q}_{\alpha}=-i \frac{\partial}{\partial \theta^{a}}-c\left(\sigma^{\mu} \bar{\theta}\right)_{\alpha} \frac{\partial}{\partial x^{\mu}} . \tag{7.107}
\end{equation*}
$$

Technically we should say that (7.105) holds for any value of $\varepsilon^{\alpha}$, thus (7.107) must hold. However, we have to do a bit of work to remove the $\bar{\varepsilon}_{\dot{d}}$ from (7.106) and then subsequently lower the index on $\mathcal{Q}^{\dot{\alpha}}$,

$$
\begin{align*}
\varepsilon_{\dot{\alpha}} \overline{\mathcal{Q}}^{\dot{\alpha}} & =-i \varepsilon_{\dot{\alpha}} \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}}+c^{*}\left(\theta \sigma^{\mu}\right)_{\dot{\gamma}} \varepsilon^{\dot{\dot{\alpha}} \dot{\alpha}} \varepsilon_{\dot{\alpha}} \frac{\partial}{\partial x^{\mu}}  \tag{7.108}\\
& =-i \varepsilon_{\dot{\alpha}} \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}}-c^{*} \varepsilon_{\dot{\alpha}}\left(\theta \sigma^{\mu}\right)_{\dot{\gamma}} \dot{\varepsilon}^{\dot{\gamma} \dot{\alpha}} \frac{\partial}{\partial x^{\mu}}  \tag{7.109}\\
\overline{\mathcal{Q}}^{\dot{\alpha}} & =-i \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}}-c^{*}\left(\theta \sigma^{\mu}\right)_{\dot{\gamma}} \varepsilon^{\dot{\gamma} \dot{\alpha}} \frac{\partial}{\partial x^{\mu}} . \tag{7.110}
\end{align*}
$$

To lower the index we must remember that we pick up a minus sign on the spinor derivative, c.f. (7.86).

$$
\begin{align*}
\overline{\mathcal{Q}}_{\dot{\alpha}} & =i \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}-c^{*}\left(\theta \sigma^{\mu}\right)_{\dot{\varepsilon}} \dot{\varepsilon}^{\dot{\gamma} \dot{\beta}} \dot{\varepsilon}_{\dot{\alpha} \dot{\beta}} \frac{\partial}{\partial x^{\mu}}  \tag{7.111}\\
& =i \frac{\partial}{\partial \bar{\theta}^{\dot{ }}}+c^{*}\left(\theta \sigma^{\mu}\right)_{\dot{\gamma}} \frac{\partial}{\partial x^{\mu}} \tag{7.112}
\end{align*}
$$

where we've used $\varepsilon_{\dot{a} \dot{\beta}} \varepsilon^{\dot{\beta} \dot{\gamma}}=\delta_{\dot{\alpha}}^{\dot{\gamma}}$. In order to satisfy the SUSY anticommutation relation $\left\{\mathcal{Q}_{\alpha}, \overline{\mathcal{Q}}_{\dot{\beta}}\right\}=2\left(\sigma^{\mu}\right)_{\alpha \dot{\beta}}=\mathcal{P}_{\mu}$, one must have $\operatorname{Re} c=1$. We shall choose $c=1$. In summary, the differential operators associated with our susy generators are given by,

$$
\begin{align*}
\mathcal{P} & =-i \frac{\partial}{\partial x^{\mu}}  \tag{7.113}\\
\mathcal{Q}_{\alpha} & =-i \frac{\partial}{\partial \theta^{a}}-\left(\sigma^{\mu} \bar{\theta}\right)_{\alpha} \frac{\partial}{\partial x^{\mu}}  \tag{7.114}\\
\overline{\mathcal{Q}}_{\dot{\alpha}} & =i \frac{\partial}{\partial \bar{\theta}^{\dot{a}}}+\left(\theta \sigma^{\mu}\right)_{\dot{\alpha}} \frac{\partial}{\partial x^{\mu}} . \tag{7.115}
\end{align*}
$$

A few remarks are in order:

- These differential operators satisfy the susy algebra in the sense of a (graded) Lie derivative.
- The operator $(\varepsilon \mathcal{Q}+\bar{\varepsilon} \overline{\mathcal{Q}})$ is Hermitian in the sense of the inner product on superspace (7.90). This is self-consistent since we work with unitary representations of our coset.
- These operators are equivalent to left-multiplication on the supersymmetric coset space.

One may form additional differential operators from right-multiplication on the supersymmetric coset space. These objects are identified with the susy covariant derivatives. These are the horizontal lifts in the susy coset space that should be
identified with susy parallel transport. They are:

$$
\begin{align*}
& \mathcal{D}_{\alpha}=\frac{\partial}{\partial \theta^{\alpha}}+i\left(\sigma^{\mu}\right)_{\alpha \dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_{\mu}  \tag{7.116}\\
& \overline{\mathcal{D}}_{\dot{\alpha}}=-\frac{\partial}{\partial \theta^{\dot{\alpha}}}-i \theta^{\beta}\left(\sigma^{\mu}\right)_{\beta \dot{\alpha}} \partial_{\mu} \tag{7.117}
\end{align*}
$$

The differential operators satisfy

$$
\begin{equation*}
\left\{\mathcal{D}_{a}, \mathcal{Q}_{\beta}\right\}=\left\{\mathcal{D}_{a}, \overline{\mathcal{Q}}_{\dot{\beta}}\right\}=\left\{\overline{\mathcal{D}}_{\dot{\alpha}}, \mathcal{Q}_{\beta}\right\}=\left\{\overline{\mathcal{D}}_{\dot{\alpha}}, \overline{\mathcal{Q}}_{\dot{\beta}}\right\}=\mathrm{o} \tag{7.118}
\end{equation*}
$$

In particular, this implies that the SUSY covariant derivatives commute with a supersymmetry transformation,

$$
\begin{equation*}
[\mathcal{D}, \varepsilon \mathcal{Q}+\bar{\varepsilon} \overline{\mathcal{Q}}]=[\overline{\mathcal{D}}, \varepsilon \mathcal{Q}+\bar{\varepsilon} \overline{\mathcal{Q}}]=\mathrm{o} \tag{7.119}
\end{equation*}
$$

and so $\mathcal{D}_{a} S$ and $\overline{\mathcal{D}}_{\dot{\alpha}} S$ are superfields.

### 7.7 TRANSFORMATION OF SUPERFIELDS

The action of susy on particles can best be seen how the susy differential operators act upon superfields, $(i \varepsilon \mathcal{Q}+i \bar{\varepsilon} \mathcal{Q}) S(x, \theta, \bar{\theta})$. Writing $i \mathcal{S}=i(\varepsilon \mathcal{Q}+\bar{\varepsilon} \overline{\mathcal{Q}})$,

$$
\begin{align*}
i S \cdot \phi(x) & =-i\left(\varepsilon \sigma^{\mu} \bar{\theta}\right) \partial_{\mu} \phi+i\left(\theta \sigma^{\mu} \bar{\varepsilon}\right) \partial_{\mu} \phi \\
i S \cdot \theta \psi(x) & =\varepsilon \psi(x)-i\left(\varepsilon \sigma^{\mu} \bar{\theta}\right) \theta \partial_{\mu} \psi(x)+i\left(\theta \sigma^{\mu} \bar{\varepsilon}\right) \theta \partial_{\mu} \psi(x) \\
i \mathcal{S} \cdot \bar{\theta} \bar{\chi}(x) & =-i\left(\varepsilon \sigma^{\mu} \bar{\theta}\right) \bar{\theta} \partial_{\mu} \bar{\chi}(x)+\overline{\varepsilon \chi}(x)+i\left(\theta \sigma^{\mu} \bar{\varepsilon}\right) \bar{\theta} \partial_{\mu} \bar{\chi}(x) \\
i S \cdot \theta \theta M(x) & =2 \varepsilon \theta M(x)-i\left(\varepsilon \sigma^{\mu} \bar{\theta}\right) \theta \theta \partial_{\mu} M(x) \\
i \mathcal{S} \cdot \overline{\theta \theta} N(x) & =2 \bar{\varepsilon} \bar{\theta} N(x)+i\left(\theta \sigma^{\mu} \bar{\varepsilon}\right) \overline{\theta \theta} \partial_{\mu} N(x) \\
i S \cdot\left(\theta \sigma^{\mu} \bar{\theta}\right) V_{\mu}(x) & =\left(\varepsilon \sigma^{\mu} \bar{\theta}\right) V_{\mu}(x)-i\left(\varepsilon \sigma^{\nu} \bar{\theta}\right)\left(\theta \sigma^{\mu} \bar{\theta}\right) \partial_{\nu} V_{\mu}(x) \\
& +\left(\theta \sigma^{\mu} \bar{\varepsilon}\right) V_{\mu}(x)+i\left(\theta \sigma^{\nu} \bar{\varepsilon}\right)\left(\theta \sigma^{\mu} \bar{\theta}\right) \partial_{\nu} V_{\mu}(x)  \tag{7.125}\\
i S \cdot \theta \theta \overline{\theta \lambda}(x) & =2 \varepsilon \theta \overline{\theta \lambda}(x)-i\left(\varepsilon \sigma^{\mu} \bar{\theta}\right) \theta \theta \bar{\theta} \partial_{\mu} \bar{\lambda}(x)+\theta \theta \bar{\varepsilon} \bar{\lambda}(x)  \tag{7.126}\\
i S \cdot \overline{\theta \theta} \theta \rho(x) & =\overline{\theta \theta} \varepsilon \rho(x)+2 \bar{\varepsilon} \theta \theta(x)+i\left(\theta \sigma^{\mu} \bar{\varepsilon}\right) \overline{\theta \theta} \theta \partial_{\mu} \rho(x)  \tag{7.127}\\
i S \cdot \theta \theta \overline{\theta \theta} D(x) & =2 \varepsilon \theta \overline{\theta \theta} D(x)+2 \theta \theta \bar{\varepsilon} \bar{\theta} D(x) . \tag{7.128}
\end{align*}
$$

The sum of all of these terms is the change in the superfield $S(x, \theta, \bar{\theta})$. On the right-hand sides of these equations we have terms with different powers of $\theta$ and $\bar{\theta}$. These correspond to the transformation of the component Minkowski-space field with the appropriate power of the Grassmann variables. For example, the terms on the right-hand side that have no powers of $\theta$ or $\bar{\theta}$ are contributions to $\delta \phi(x)$, the terms linear in $\theta$ with no $\bar{\theta}$ are contributions to $\delta \psi(x)$, and so forth.

After carefully using spinor identities to simplify the transformations, one arrives at the component transformations:

$$
\begin{aligned}
\delta \phi(x) & =\varepsilon \psi(x)+\bar{\varepsilon} \chi(x) \\
\delta \psi(x) & =2 \varepsilon M(x)+\sigma^{\mu} \bar{\varepsilon}\left(i \partial_{\mu} \phi(x)+V_{\mu}(x)\right) \\
\delta \bar{\chi}(x) & =2 \bar{\varepsilon} M(x)-\varepsilon \sigma^{\mu}\left(i \partial_{\mu} \phi(x)-V_{\mu}\right) \\
\delta M(x) & =\bar{\varepsilon} \bar{\lambda}(x)-\frac{i}{2} \partial_{\mu} \psi(x) \sigma^{\mu} \bar{\varepsilon} \\
\delta N(x) & =\varepsilon \rho(x)+\frac{i}{2} \varepsilon \sigma^{\mu} \partial_{\mu} \bar{\chi}(x) \\
\delta V_{\mu}(x) & =\varepsilon \sigma_{\mu} \bar{\lambda}(x)+\rho(x) \sigma_{\mu} \bar{\varepsilon}+\frac{i}{2}\left(\partial^{v} \psi(x) \sigma_{\mu} \bar{\sigma}_{v} \varepsilon-\bar{\varepsilon} \sigma_{v} \sigma_{\mu} \partial^{v} \bar{\chi}(x)\right) \\
\delta \bar{\lambda}(x) & =2 \varepsilon D(x)+\frac{i}{2}\left(\bar{\sigma}^{\nu} \sigma^{\mu} \bar{\varepsilon}\right) \partial_{\mu} V_{v}(x)+i\left(\bar{\sigma}^{\mu} \varepsilon\right) \partial_{\mu} M(x) \\
\delta \rho(x) & =2 \varepsilon D(x)-\frac{i}{2}\left(\sigma^{\nu} \bar{\sigma}^{\mu} \varepsilon\right) \partial_{\mu} V_{v}(x)+i\left(\sigma^{\mu} \bar{\varepsilon}\right) \partial_{\mu} N(x) \\
\delta D(x) & =\frac{i}{2} \partial_{\mu}\left(\varepsilon \sigma^{\mu} \bar{\lambda}(x)-\rho(x) \sigma^{\mu} \bar{\varepsilon}\right) .
\end{aligned}
$$

And there we go! That's it. The susy transformation of a general (scalar) superfield. Take a good look at them, bask in their glory, tattoo them to the insides of your eyelids. That's supersymmetry for you.

What can we glean from this? Well, first of all, you should notice that, as expected, susy transformations swaps bosons and fermions. The transformation for a fermionic field is composed of a linear combination of bosonic fields, and vice-versa. The factors of $\varepsilon, \bar{\varepsilon}, \sigma^{\mu}$, and $\partial_{\mu}$ float around to make sure indices match on the left- and right-hand sides. This is all self-consistent and we can pat ourselves on the back for doing such a good job of keeping indices in order, but it's not particularly insightful

There's something else in these equations, however, that's staring you right in the face. The transformation of $D(x)$ is a total derivative. It's the only component of the general superfield that is a total derivative. This is precisely the property required for a susy invariant term in the Lagrangian.

### 7.8 Irreducible superfields

While the susy transformation of a general superfield is a useful expression, we prefer to build our theories out of irreducible representations. One can construct irreducible representations by applying various constraints to the general superfield. For example:

1. chiral superfield: $\Phi(x, \theta, \bar{\theta})$ such that $\overline{\mathcal{D}} \Phi=0$.
2. antichiral superfield: $\bar{\Phi}(x, \theta, \bar{\theta})$ such that $\mathcal{D} \bar{\Phi}=0$.
vector (real) superfield: $V(x, \theta, \bar{\theta})$ such that $V=V^{\dagger}$.
linear superfield $L(x, \theta, \bar{\theta})$ such that $\mathcal{D}^{2} L=0$ and $L=L^{\dagger}$.

### 7.9 PUTTING TOGETHER THE INGREDIENTS OF A SUPERSYMMETRIC THEORY

At this point one may pick one's favorite review article or textbook on supersymmetry for the construction of supersymmetric Lagrangians. For these topics, then, we will be terse and only highlight how key points come together.

The basic building blocks of a realistic Tev-scale supersymmetric extension of the Standard Model are chiral and vector superfields. The chiral superfields furnish Weyl fermions so that we may associate each Standard Model matter field with a chiral superfield. The vector superfields, on the other hand, encode gauge bosons.

In order to write down a supersymmetric Lagrangian, one must identify objects which are invariant under the supersymmetry transformation up to a total derivative. Any residual total derivative transformation vanishes in the spacetime integral of the action. We have already identified the $D$-term above as a candidate action. This turns out to be a component of
a vector superfield called the Kähler potential. It encodes the kinetic terms of a theory. The form for the kinetic terms of chiral superfields $\Phi$ with charge $q$ under a gauge field with vector superfield $V$ is

$$
\begin{equation*}
\mathcal{L}=\int d^{4} \theta K=\Phi^{\dagger} e^{q V} \Phi \tag{7.138}
\end{equation*}
$$

Note that $K$ is forced by the vector superfield condition to be real. We may add higher order real terms that obey the symmetries of the theory. These will correspond to non-renormalizable derivative interactions.

The restriction on chiral superfield can be recast into a restriction onto a holomorphic slice of superspace. To see this, define the superspace coordinate

$$
\begin{equation*}
y^{\mu}=x^{\mu}+i \theta \sigma^{\mu} \bar{\theta} \tag{7.139}
\end{equation*}
$$

The chiral superfield restriction $\overline{\mathcal{D}} \Phi=\mathrm{o}$ then takes the form

$$
\begin{equation*}
\Phi(y, \theta)=\phi(y)+\sqrt{2} \theta \psi(y)+\theta \theta F(y) . \tag{7.140}
\end{equation*}
$$

In other words, $\Phi$ depends only on $y$ and $\theta$, not $\bar{\theta}$. What is even more intriguing is that the transformation of the $F(y)$ component under susy is

$$
\begin{equation*}
\delta F=i \sqrt{2} \bar{\varepsilon} \bar{\sigma}^{\mu} \partial_{\mu} \psi \tag{7.141}
\end{equation*}
$$

This is a total derivative and is another candidate for a susy Lagrangian term. Since it only depends on one superspace coordinate, it is holomorphic. Such a contribution is called a superpotential and is denoted $W$,

$$
\begin{equation*}
\mathcal{L}=\int d^{2} \theta W+\text { h.c. } . \tag{7.142}
\end{equation*}
$$

Since the product of any chiral superfields is also a chiral superfield, $W$ can be a product of any number of chiral superfields but no other types of superfields. The 'h.c.' is the Hermitian conjugate built out of antichiral superfields and is required for the action to be real. The holomorphy of the superpotential provides a great degree of theoretical control over the theory. In particular, one may show that the terms in the superpotential are not perturbatively renormalized. This non-renormalization term protects the Higgs mass in supersymmetric extensions of the Standard Model and is a powerful tool for solving the Hierarchy problem. Finally, the kinetic term for the gauge bosons lives in a special chiral superfield whose lowest component is the spin- $1 / 2$ gaugino $\lambda_{\alpha}$. This takes the form

$$
\begin{equation*}
\mathcal{W}_{a}(y, \theta)=-\frac{1}{8} \overline{\mathcal{D}}^{2} e^{-2 V} \mathcal{D}_{a} e^{2 V} \tag{7.143}
\end{equation*}
$$

The corresponding Lagrangian term is

$$
\begin{equation*}
\mathcal{L}=\int d^{2} \theta \mathcal{W}^{2}+\text { h.c. } \tag{7.144}
\end{equation*}
$$

With these ingredients it is straightforward to construct the susy-respecting part of the minimal supersymmetric Standard Model (MSSM). What remains is to parameterize the 'soft' breaking of supersymmetry that can account for the non-observation of equal-mass superpartner particles while also preserving the cancellation of quadratic divergences in scalar masses.

## 7.A The Lorentz Group

## 7.A. 1 The Lorentz Group

Let us now explore the Lorentz group, which is sometimes called the homogeneous Lorentz group to disambiguate it from the Poincaré group which is sometimes called the inhomogeneous Lorentz group. The Lorentz group is composed of the
transformations that preserve the Minkowski inner product, $\left\langle x^{\mu}, x^{\nu}\right\rangle=x^{\mu} \eta_{\mu \nu} x^{v}=x^{\mu} x_{\mu}$. In particular, for $x^{\mu} \rightarrow x^{\prime \mu}=\Lambda^{\mu}{ }_{v} x^{\nu}$, we have

$$
\begin{equation*}
\left(\Lambda^{\mu}{ }_{\rho} x^{\rho}\right) \eta_{\mu \nu}\left(\Lambda_{\sigma}^{v} x^{\sigma}\right)=x^{\rho} \eta_{\rho \sigma} x^{\sigma} \tag{7.145}
\end{equation*}
$$

From this we may deduce that the fundamental transformations of the Lorentz group satisfy the relation

$$
\begin{equation*}
\Lambda^{\mu}{ }_{\rho} \eta_{\mu \nu} \Lambda_{\sigma}^{\nu}=\eta_{\rho \sigma} . \tag{7.146}
\end{equation*}
$$

For simplicity we may write this in matrix notation where write matrices with suppressed indices are written in boldface

$$
\boldsymbol{\Lambda}^{\mathbf{T}} \boldsymbol{\eta} \boldsymbol{\Lambda}=\eta
$$

## Generators of the Lorentz Group

Let's spell out the procedure for determining the generators of the Lorentz group. We will later follow an analogous procedure to determine the generators of supersymmetry. We start by writing out a finite Lorentz transformation as the exponentiation

$$
\begin{equation*}
\boldsymbol{\Lambda}=e^{i t \mathrm{~W}} \tag{7.148}
\end{equation*}
$$

where $t$ is a transformation parameter and $\mathbf{W}$ is the Hermitian generator we'd like to determine. Note that (7.148) only encodes transformations that are connected to the identity. We discuss below why this is the subgroup of present interest.

Plugging (7.148) into (7.147) and setting $t=0$, we obtain the relation

$$
\begin{equation*}
\boldsymbol{\eta} \mathbf{W}+\mathbf{w}^{T} \boldsymbol{\eta}=\mathrm{o}, \tag{7.149}
\end{equation*}
$$

or with explicit indices, $W_{\mu \nu}+W_{v \mu}=0$. Thus the generators $\mathbf{W}$ are $4 \times 4$ antisymmetric matrices characterized by six real transformation parameters so that there are six generators. Let us thus write the exponent of the finite transformation (7.148) as

$$
\begin{equation*}
\text { it } W^{\lambda \sigma}=i t \omega^{\mu \nu}\left(M_{\mu \nu}\right)^{\lambda \sigma} \tag{7.150}
\end{equation*}
$$

where $\omega^{\mu \nu}$ is an antisymmetric $4 \times 4$ matrix parameterizing the linear combination of the independent generators and $\left(M_{\mu \nu}\right)^{\lambda \sigma}$ are the Hermitian generators of the Lorentz group. The $\mu, v$ indices label the six generators, while the $\lambda, \sigma$ indices label the matrix structure of each generator. This verifies that ( 7.8 ) indeed furnishes a basis for the generators of the Lorentz group.

## Components of the Lorentz Group

The full Lorentz group has four disconnected parts. The definition (7.147) implies that

$$
\begin{equation*}
(\operatorname{det} \boldsymbol{\Lambda})^{2}=1 \tag{7.151}
\end{equation*}
$$

$$
\left(\Lambda_{\circ}^{\circ}\right)^{2}-\sum_{i}\left(\Lambda_{\circ}^{i}\right)^{2}=1,
$$

where the first equation comes from taking a determinant and the second equation comes from taking $\rho=\sigma=0$ in (7.146).
From these equations we see that

$$
\begin{equation*}
\operatorname{det} \boldsymbol{\Lambda}= \pm 1 \tag{7.152}
\end{equation*}
$$

$$
\Lambda_{\circ}^{\circ}= \pm \sqrt{1+\sum_{i}\left(\Lambda_{\circ}^{i}\right)^{2}}
$$

The choice of the signs on the right-hand sides of these equations labels the four components of the Lorentz group. There is no connected path in the space of Lorentz transformations that can change these signs.

The component of the Lorentz group with det $\boldsymbol{\Lambda}=+1$ contains the identity element and is a subgroup that preserves parity. In order to preserve the direction of time, we further choose $\Lambda^{\circ}{ }_{\circ} \geq 1$. We shall specialize to this subgroup, which is
called the orthochronous Lorentz group, $S O(3,1)_{+}^{\uparrow}$ which further satisfies

$$
\begin{equation*}
\operatorname{det} \boldsymbol{\Lambda}=+\mathbf{1} \tag{7.153}
\end{equation*}
$$

$$
\Lambda_{\circ}^{\circ} \geq 1
$$

Other parts of the Lorentz group can be obtained from $S O(3,1)_{+}^{\uparrow}$ by applying the transformations

$$
\Lambda_{P}=\operatorname{diag}(+,-,-,-) \quad \Lambda_{T}=\operatorname{diag}(-,+,+,+)
$$

Here $\Lambda_{P}$ and $\Lambda_{T}$ respectively refer to parity and time-reversal transformations. One may thus write the Lorentz group in terms of 'components' (not necessarily 'subgroups'),

$$
\begin{equation*}
S O(3,1)=S O(3,1)_{+}^{\uparrow} \oplus S O(3,1)_{-}^{\uparrow} \oplus S O(3,1)_{+}^{\downarrow} \oplus S O(3,1)_{-}^{\downarrow} \tag{7.155}
\end{equation*}
$$

where the up/down arrow refers to $\Lambda^{\circ}{ }_{\circ}$ greater/less than $\pm 1$, while the $\pm$ refers to the sign of det $\Lambda$. Again, only $S O(3,1)_{ \pm}^{\uparrow}$ form subgroups. In this document we exclusively work with the orthochronous Lorentz group so that we will drop the $\pm$ and write this as $S O(3,1)^{\uparrow}$. The fact that the Lorentz group is not simply connected is related to the existence of a 'physical' spinor representation, as we will mention below.

## The Lorentz Group is related to $S U(\mathbf{2}) \times S U(\mathbf{2})$

Locally the Lorentz group is 'related' to the group $S U(2) \times S U(2)$, so that one might suggestively write

$$
\begin{equation*}
S O(3,1) \approx S U(2) \times S U(2) \tag{7.156}
\end{equation*}
$$

One can explicitly separate the Lorentz generators $M^{\mu \nu}$ into the generators of rotations, $J_{i}$, and boosts, $K_{i}$ :

$$
\begin{equation*}
J_{i}=\frac{\mathbf{1}}{\mathbf{2}} \varepsilon_{i j k} M_{j k} \quad K_{i}=M_{\circ i} \tag{7.157}
\end{equation*}
$$

where $\mathcal{E}_{i j k}$ is the usual antisymmetric Levi-Civita tensor. $\mathbf{J}$ and $\mathbf{K}$ satisfy the algebra

$$
\begin{equation*}
\left[J_{i}, J_{j}\right]=i \varepsilon_{i j k} J_{k} \tag{7.158}
\end{equation*}
$$

$$
\left[K_{i}, K_{j}\right]=-i \varepsilon_{i j k} J_{k}
$$

$$
\left[J_{i}, K_{j}\right]=i \varepsilon_{i j k} K_{k}
$$

We can now define 'nice' combinations of these two sets of generators,

$$
\begin{equation*}
A_{i}=\frac{1}{2}\left(J_{i}+i K_{i}\right) \quad B_{i}=\frac{1}{2}\left(J_{i}-i K_{i}\right) \tag{7.159}
\end{equation*}
$$

The resulting algebra decouples into two $S U(2)$ algebras,

$$
\begin{equation*}
\left[A_{i}, A_{j}\right]=i \varepsilon_{i j k} A_{k} \quad\left[B_{i}, B_{j}\right]=i \varepsilon_{i j k} B_{k} \quad\left[A_{i}, B_{j}\right]=\mathrm{o} \tag{7.160}
\end{equation*}
$$

Note, however, that from (7.159) that these generators are not Hermitian (gasp!). Recall that a Lie group is generated by Hermitian operators. Thus we have been careful not to say that $S U(3,1)$ is either isomorphic or homomorphic to $S U(2) \times S U(2)$. For example, $S U(2) \times S U(2)$ is manifestly compact while the Lorentz group cannot be since the elements corresponding to boosts can be arbitrarily far from the origin. This is all traced back to the sign difference in the time-like component of the metric, i.e. the difference between $S O(4)$ and $S O(3,1)$. While rotations are Hermitian and generate unitary matrices, boosts are anti-Hermitian and generate anti-unitary matrices. At this level, then, our representations are non-unitary.

We needn't worry about the precise sense in which $S O(3,1)$ and $S U(2) \times S U(2)$ are related, the point is that we may label particle representations of $S O(3,1)$ by the quantum numbers of $S U(2) \times S U(2),(A, B)$. This is because we label particles by their transformation under the algebra of the group. For example, a Dirac spinor is in the $\left(\frac{1}{2}, \frac{1}{2}\right)=\left(\frac{1}{2}, 0\right) \oplus\left(0, \frac{1}{2}\right)$ representation, i.e. the direct sum of two Weyl reps. To connect back to nature, the physical meaning of all this is that we may write the spin of a representation as $J=A+B$.

So how are $S O(3,1)$ and $S U(2) \times S U(2)$ actually related?
We've been deliberately vague about the exact relationship between the Lorentz group and $S U(2) \times S U(2)$. The precise relationship between the two groups are that the complex linear combinations of the generators of the Lorentz algebra are isomorphic to the complex linear combinations of the Lie algebra of $S U(2) \times S U(2)$.

$$
\begin{equation*}
\mathcal{L}_{\mathbb{C}}(S O(3,1)) \cong \mathcal{L}_{\mathbb{C}}(S U(2) \times S U(2)) \tag{7.161}
\end{equation*}
$$

We are careful not to say that the Lie algebras of the two groups are identical, it is important to emphasize that only the complexified algebras are identifiable. The complexification of $S U(2) \times S U(2)$ is the special linear group, $S L(2, \mathbb{C})$. In the next subsection we identify $S L(2, \mathbb{C})$ as the universal cover of the Lorentz group. First, however, we show that the Lorentz group is isomorphic to $S L(2, \mathbb{C}) / \mathbb{Z}_{2}$. We discuss this topic from an orthogonal direction in Section 7.A.1.

## The Lorentz group is isomorphic to SL(2,C)/Z2

While the Lorentz group and $S U(2) \times S U(2)$ are neither related by isomorphism nor homomorphism, we can concretely relate the Lorentz group to $S L(2, \mathbb{C})$. More precisely, the Lorentz group is isomorphic to the coset space $S L(2, \mathbb{C}) / \mathbb{Z}_{2}$

$$
\begin{equation*}
S O(3,1) \cong S L(2, \mathbb{C}) / \mathbb{Z}_{2} \tag{7.162}
\end{equation*}
$$

Recall that four-vectors in Minkowski space can be equivalent represented as complex Hermitian $2 \times 2$ matrices via $V^{\mu} \rightarrow V_{\mu} \sigma^{\mu}$, where the $\sigma^{\mu}$ are the usual Pauli matrices,

$$
\sigma^{0}=\left(\begin{array}{ll}
1 & 0  \tag{7.163}\\
0 & 1
\end{array}\right) \quad \sigma^{1}=\left(\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right) \quad \sigma^{2}=\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right) \quad \sigma^{3}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) .
$$

$S L(2, \mathbb{C})$ is the group of complex $2 \times 2$ matrices with unit determinant. It is spanned precisely by these Pauli matrices. For simplicity we will not distinguish between the Lie group $(S L(2, \mathbb{C}))$ and its algebra $(\mathcal{L}[S L(2, \mathbb{C})])$. This is not worth the extra notational baggage since the meaning is clear in context. Explicitly, we associate a vector $\mathbf{x}$ with either a vector in Minkowski space $\mathbb{M}^{4}$ spanned by the unit vectors $e^{\mu}$,

$$
\begin{equation*}
\mathbf{x}=x^{\mu} e_{\mu}=\left(x^{0}, x^{1}, x^{2}, x^{3}\right) \tag{7.164}
\end{equation*}
$$

or with an equivalent matrix in $\operatorname{SL}(2, \mathbb{C})$,

$$
\mathbf{x}=x_{\mu} \sigma^{\mu}=\left(\begin{array}{ll}
x_{0}+x_{3} & x_{1}-i x_{2}  \tag{7.165}\\
x_{1}+i x_{2} & x_{0}-x_{3}
\end{array}\right) .
$$

Note the lowered indices on the components of $x_{\mu}$, i.e. $\left(x^{\circ}, x^{1}, x^{2}, x^{3}\right)=\left(x_{0},-x_{1},-x_{2},-x_{3}\right)$. The four-vector components are recovered from the $S L(2, \mathbb{C})$ matrices via

$$
\begin{equation*}
x_{\circ}=\frac{1}{2} \operatorname{Tr}(\mathbf{x}), \quad x_{i}=\frac{1}{2} \operatorname{Tr}\left(\mathbf{x} \sigma^{i}\right) . \tag{7.166}
\end{equation*}
$$

The latter of these is easy to show by expanding $\mathbf{x}=x_{\circ} \mathbb{1}^{\circ}+x_{i} \sigma^{i}$ and then noting that $\mathbb{1} \sigma^{i} \propto \sigma^{i},\left.\sigma^{j} \sigma^{i}\right|_{j \neq i} \propto \sigma_{k \neq i}^{k}$, and $\left.\sigma^{i} \sigma^{i}\right|_{\text {no sum }} \propto \mathbb{1}$. Thus only the $\sigma^{i} \sigma^{i}$ term of $\mathbf{x} \sigma^{i}$ has a trace, so that taking the trace projects out the other components.

For the Minkowski four-vectors, we already understand how a Lorentz transformation $\boldsymbol{\Lambda}$ acts on a [covariant] vector $x^{\mu}$ while preserving the vector norm,

$$
\begin{equation*}
|\mathbf{x}|^{2}=x_{0}^{2}-x_{1}^{2}-x_{2}^{2}-x_{3}^{2} . \tag{7.167}
\end{equation*}
$$

This is just the content of (7.147), which defines the Lorentz group.
For Hermitian matrices, there is an analogous transformation by the action of the group of invertible complex matrices of unitary determinant, $S L(2, \mathbb{C})$. For $\mathbf{N} \in S L(2, \mathbb{C}), \mathbf{N}^{\dagger} \mathbf{x} \mathbf{N}$ is also in the space of Hermitian $2 \times 2$ matrices. Such
transformations preserve the determinant of $\mathbf{x}$,

$$
\begin{equation*}
\operatorname{det} \mathbf{x}=x_{0}^{2}-x_{1}^{2}-x_{2}^{2}-x_{3}^{2} \tag{7.168}
\end{equation*}
$$

The equivalence of the right-hand sides of $s(7.167)$ and (7.168) are very suggestive of an identification between the Lorentz group $S O(3,1)$ and $S L(2, \mathbb{C})$. Indeed, $(7.168)$ implies that for each $S L(2, \mathbb{C})$ matrix $\mathbf{N}$, there exists a Lorentz transformation $\Lambda$ such that

$$
\begin{equation*}
\mathbf{N}^{\dagger} x^{\mu} \boldsymbol{\sigma}_{\mu} \mathbf{N}=(\Lambda x)^{\mu} \boldsymbol{\sigma}_{\mu} \tag{7.169}
\end{equation*}
$$

A key feature is now apparent: the map from $S L(2, \mathbb{C}) \rightarrow S O(3,1)$ is two-to-one. This is clear since the matrices $\mathbf{N}$ and $-\mathbf{N}$ yield the same Lorentz transformation, $\Lambda^{\mu}{ }_{v}$. Hence it is not $S O(3,1)$ and $S L(2, \mathbb{C})$ that are isomorphic, but rather $S O(3,1)$ and $S L(2, \mathbb{C}) / \mathbb{Z}_{2}$.

The point is that one will miss something if one only looks at representations of $S O(3,1)$ and not the representations of $S L(2, \mathbb{C})$. This 'something' is the spinor representation. How should we have known that $S L(2, \mathbb{C})$ is the important group? One way of seeing this is noting that $S L(2, \mathbb{C})$ is simply connected as a group manifold.

By the polar decomposition for matrices, any $g \in S L(2, \mathbb{C})$ can be written as the product of a unitary matrix $U$ times the exponentiation of a traceless Hermitian matrix $h$,

$$
\begin{equation*}
g=U e^{h} \tag{7.170}
\end{equation*}
$$

We may write these matrices explicitly in terms of real parameters $a, \cdots, g$;

$$
h=\left(\begin{array}{cc}
c & a-i b  \tag{7.171}\\
a+i b & -c
\end{array}\right) \quad U=\left(\begin{array}{cc}
d+i e & f+i g \\
-f+i g & d-i e
\end{array}\right)
$$

Here $a, b, c$ are unconstrained while $d, \cdots, g$ must satisfy

$$
\begin{equation*}
d^{2}+e^{2}+f^{2}+g^{2}=1 \tag{7.172}
\end{equation*}
$$

Thus the space of $2 \times 2$ traceless Hermitian matrices $\{h\}$ is topologically identical to $\mathbb{R}^{3}$ while the space of unit determinant $2 \times 2$ unitary matrices $\{U\}$ is topologically identical to the three-sphere, $S_{3}$. Thus we have

$$
\begin{equation*}
S L(2, \mathbb{C})=\mathbb{R}^{3} \times S_{3} \tag{7.173}
\end{equation*}
$$

As both of the spaces on the right-hand side are simply connected, their product, $S L(2, \mathbb{C})$, is also simply connected. This is a 'nice' property because we can write down any element of the group by exponentiating its generators at the identity. But even further, since $S L(2, \mathbb{C})$ is simply connected, its quotient space $S L(2, \mathbb{C}) / \mathbb{Z}_{2}=S O(3,1)^{\uparrow}$ is not simply connected.

## Universal cover of the Lorentz group

The fact that $S O(3,1)^{\uparrow}$ is not simply connected should bother you. In the back of your mind, your physical intuition should be unsatisfied with non-simply connected groups. This is because simply-connected groups have the very handy property of having a one-to-one correspondence between representations of the group and representations of its algebra; i.e. we can write any element of the group as the exponentiation of an element of the algebra about the origin.

In quantum field theory fields transform according to representations of a symmetry's algebra, not representations of the group. Since $S O(3,1)^{\uparrow}$ is not simply connected, the elements of the algebra at the identity that we used do not tell the whole story. They were fine for constructing finite elements of the Lorentz group that were connected to the identity, but they don't capture the entire algebra of $S O(3,1)^{\uparrow}$.

Now we're in a pickle. Given a group, we know how to construct representations of an algebra near the identity based on group elements connected to the identity. But this only characterizes the entire algebra if the group is simply connected. $S O(3,1)^{\uparrow}$ is not simply connected. Fortunately, there's a trick. It turns out that for any connected Lie group, there exists a unique 'minimal' simply connected group that is homeomorphic to it called the universal covering group.

Stated slightly more formally, for any connected Lie group G, there exists a simply connected universal cover $\tilde{G}$ such that there exists a homomorphism $\pi: \tilde{G} \rightarrow G$ where $G \cong \tilde{G} / \operatorname{ker} \pi$ and $\operatorname{ker} \pi$ is a discrete subgroup of the center of $\tilde{G}$. Phew, that was a mouthful. For the Lorentz group this statement is $S O(3,1)^{\uparrow} \cong S L(2, \mathbb{C}) / \mathbb{Z}_{2}$. Thus the key statement is:

The Lorentz group is covered by $\operatorname{SL}(2, \mathbb{C})$.
The point is that the homomorphism $\pi$ is locally one-to-one and thus $G$ and $\tilde{G}$ have the same Lie algebras. Thus we can determine the Lie algebra of $G$ away from the identity by considering the Lie algebra for $\tilde{G}$ at the identity. This universal covering group of $\operatorname{SO}(3,1)^{\uparrow}$ is often referred to as $\operatorname{Spin}(3,1)$. The name is no coincidence, it has everything to do with the spinor representation.

## Projective representations

It may not be clear why the above rigamarole is necessary or interesting. In some sense, the spinor representation is necessarily the 'most basic' representation of four-dimensional spacetime symmetry.

Recall that when we write unitary representations $U$ of a group $G$, we have $U\left(g_{1}\right) U\left(g_{2}\right)=U\left(g_{1} g_{2}\right)$ for $g_{1}, g_{2} \in G$. In quantum physics, however, physical states are invariant under phases, so we have the freedom to be more general with our multiplication rule for representations: $U\left(g_{1}\right) U\left(g_{2}\right)=U\left(g_{1} g_{2}\right) \exp \left(i \varphi\left(g_{1}, g_{2}\right)\right)$. Such 'representations' are called projective representations. In other words, quantum mechanics allows us to use projective representations rather than ordinary representations.

Not every group admits 'inherently' projective representations. In cases where such reps are not allowed, a representation that one tries to construct to be projective can have its generators redefined to reveal that it is actually an ordinary non-projective representation. The relevant mathematical result for our purposes is that groups which are not simply connected-such as the Lorentz group-do admit inherently projective representations.

The Lorentz group is doubly connected, i.e. going over any loop twice will allow it to be contracted to a point. This means that the phase in the projective representation must be $\pm_{1}$. One can consider taking a loop in the Lorentz group that corresponds to rotating by $2 \pi$ along the $\hat{z}$-axis. Representations with a projective phase +1 will return to their original state after a single rotation, these are the particles with integer spin. Representations with a projective phase -1 will return to their original state only after two rotations, and these correspond to fractional-spin particles, or spinors.

## LORENTZ REPRESENTATIONS ARE NON-UNITARY AND NON-COMPACT

We have shown that representations of the Lorentz group are not unitary. The generators of boosts are imaginary. This makes them anti-unitary rather than unitary. From the point of view of quantum mechanics this is the appears to be the kiss of death since we know that only unitary representations preserve probability.

This non-unitarity from the factor of $i$ associated with boosts, (7.159). This factor of $i$ is crucial since is is related to the non-compactness of the Lorentz group. Totations are compact since the rotation parameter lives on a circle ( $\theta=0$ and $\theta=2 \pi$ are identified) while boosts are non-compact since the rapidity can take on any value along the real line. The dreaded factor of $i$, then, is deeply connected to the structure of the group. In fact, it's precisely the difference between $S O(3,1)$ and $S O(4)$, i.e. the difference between space and time: a minus sign in the metric. To make the situation look even more grim, even if we were able to finagle a way out of the non-unitarity issue (and we can't), there is a theorem that unitary representations of non-compact groups are infinite-dimensional. There is nothing infinite-dimensional about the particles we hope to describe with the Lorentz group. This is looking like quite a pickle!

Up until now we have considered representations of the Lorentz group as if they would properly describe particles. The key is that one must look at full Poincaré group (incorporating translations as well as Lorentz transformations) to develop a consistent picture. Adding the translation generator $P$ to the algebra cures this apparent non-unitarity. The cost for these features, as mentioned above, is that the representations will become infinite dimensional, but this infinite dimensionality is well-understood physically: we can boost into any of a continuum of frames where the particle has arbitrarily boosted four-momentum.

This is why it is not troubling that the Lorentz group does not furnish satisfactory particle representations. They actually end up being rather useful for describing fields, where the non-unitarity of the Lorentz representations isn't a problem because the actual states in the quantum Hilbert space are the particles which are representations of the full Poincaré group.

## 7.A. 2 Spinors are the fundamental representation of $\operatorname{SL}(2, \mathbb{C})$

The fundamental representation of the universal cover of the Lorentz group, $\operatorname{SL}(2, \mathbb{C})$, are spinors. In four dimensions it is natural to work with two-component spinors; see [24] for a comprehensive review.

Let us start by defining the fundamentaland conjugate (or antifundamental) representations of $\operatorname{SL}(2, \mathbb{C})$. These are just the matrices $N_{a}{ }^{\beta}$ and $\left(N^{*}\right)_{\dot{\alpha}}^{\dot{\beta}}$. The dots on the indices are a book-keeping device to keep the fundamental and the conjugate indices from getting mixed up. One cannot contract a dotted with an undotted $S L(2, \mathbb{C})$ index; this would be like trying to contract spinor indices ( $\alpha$ or $\dot{\alpha}$ ) with vector indices ( $\mu$ ): they index two totally different representations.

We are particularly interested in the objects that these matrices act on. Let us thus define left-handed Weyl spinors, $\psi$, as those acted upon by the fundamental rep and right-handed Weyl spinors, $\bar{\chi}$, as those that are acted upon by the conjugate rep. Again, do not be startled with the extra jewelry that our spinors display. The bar on the right-handed spinor just serves to distinguish it from the left-handed spinor. Explicitly,

$$
\begin{equation*}
\psi_{\alpha}^{\prime}=N_{a}^{\beta} \psi_{\beta} \tag{7.174}
\end{equation*}
$$

$$
\bar{\chi}_{\dot{\alpha}}^{\prime}=\left(N^{*}\right)_{\dot{\alpha}}^{\dot{\beta}} \bar{\chi}_{\dot{\beta}} .
$$

## 7.A. 3 Invariant Tensors

We know that $\eta_{\mu \nu}$ is invariant under $S O(3,1)$ and can be used (along with the inverse metric) to raise and lower $S O(3,1)$ indices. For $S L(2, \mathbb{C})$, we can build an analogous tensor, the unimodular antisymmetric tensor

$$
\varepsilon^{a \beta}=\left(\begin{array}{cc}
0 & 1  \tag{7.175}\\
-1 & 0
\end{array}\right)
$$

Unimodularity (unit determinant) and antisymmetry uniquely define the above form up to an overall sign. The choice of sign $\left(\varepsilon^{12}=1\right)$ is a convention. As a mnemonic, $\varepsilon^{\alpha \beta}=i\left(\sigma^{2}\right)_{\alpha \beta}$, but note that this is not a formal equality since $\sigma^{2}$ does does not have the correct index structure. This tensor is invariant under $\operatorname{SL}(2, \mathbb{C})$ since

$$
\begin{equation*}
\varepsilon^{\prime \alpha \beta}=\varepsilon^{\rho \sigma} N_{\rho}{ }^{\alpha} N_{\sigma}{ }^{\beta}=\varepsilon^{a \beta} \operatorname{det} N=\varepsilon^{a \beta} . \tag{7.176}
\end{equation*}
$$

We can now use this tensor to raise undotted $S L(2 \mathbb{C})$ indices:

$$
\begin{equation*}
\psi^{a} \equiv \varepsilon^{\alpha \beta} \psi_{\beta} . \tag{7.177}
\end{equation*}
$$

To lower indices we can use an analogous unimodular antisymmetric tensor with two lower indices. For consistency, the overall sign of the lowered-indices tensor must be defined as

$$
\begin{equation*}
\varepsilon_{\alpha \beta}=-\varepsilon^{\alpha \beta}, \tag{7.178}
\end{equation*}
$$

so that raising and then lowering returns us to our original spinor:

$$
\begin{equation*}
\varepsilon_{\alpha \beta} \varepsilon^{\beta \gamma}=\delta_{\alpha}^{\gamma} . \tag{7.179}
\end{equation*}
$$

This is to ensure that the upper- and lower-index tensors are inverses, i.e. so that the combined operation of raising then lowering an index does not introduce a sign. Dotted indices indicate the complex conjugate representation, $\varepsilon_{\alpha \beta}^{*}=\varepsilon_{\dot{\alpha} \dot{\beta}}$. Since $\varepsilon$ is real we thus use the same sign convention for dotted indices as undotted indices,

$$
\begin{equation*}
\varepsilon^{\mathrm{i} 2}=\varepsilon^{12}=-\varepsilon_{\mathrm{ij}}=-\varepsilon_{12} . \tag{7.180}
\end{equation*}
$$

We may raise dotted indices in exactly the same way:

$$
\begin{equation*}
\bar{\chi}^{\dot{\alpha}} \equiv \varepsilon^{\dot{\beta}} \bar{\chi}_{\dot{\beta}} . \tag{7.181}
\end{equation*}
$$

In order to avoid sign errors, it is a useful mnemonic to always put the $\varepsilon$ tensor directly to the left of the spinor whose indices it is manipulating, this way the index closest to the spinor contracts with the spinor index. In other words, one needs to be careful since $\varepsilon^{a \beta} \psi_{\beta} \neq \psi_{\beta} \varepsilon^{\beta \alpha}$. In summary:

$$
\psi^{\alpha}=\varepsilon^{\alpha \beta} \psi_{\beta} \quad \psi_{\alpha}=\varepsilon_{\alpha \beta} \psi^{\beta} \quad \bar{\chi}^{\alpha}=\varepsilon^{\dot{\alpha} \dot{\beta}} \psi_{\dot{\beta}} \quad \bar{\chi}_{\dot{\alpha}}=\varepsilon_{\dot{\alpha} \dot{\beta}} \dot{\chi}^{\dot{\beta}}
$$

## 7.A. 4 CONTRAVARIANT REPRESENTATIONS

Now that we're familiar with the $\varepsilon$ tensor, we can tie up a loose end from Section 7.A.2. We introduced the fundamental and conjugate representations of $S L(2, \mathbb{C})$. What happened to the contravariant representations that transform under the inverse matrices $N^{-1}$ and $N^{*-1}$ ? For a general group, e.g. $G L(N, \mathbb{C})$, these are unique so that we have a total of four different representations.

It turns out that for $S L(2, \mathbb{C})$ these contravariant representations are group theoretically equivalent to the fundamental and conjugate representations presented above. Using the antisymmetric tensor $\varepsilon_{\alpha \beta}\left(\varepsilon^{12}=1\right)$ and the unimodularity of $N \in S L(2, \mathbb{C})$,

$$
\begin{align*}
\varepsilon_{\alpha \beta} N^{a}{ }_{\gamma} N^{\beta}{ }_{\delta} & =\varepsilon_{\gamma \delta} \operatorname{det} N  \tag{7.183}\\
\varepsilon_{\alpha \beta} N^{a}{ }_{\gamma} N^{\beta}{ }_{\delta} & =\varepsilon_{\gamma \delta}  \tag{7.184}\\
\left(N^{T}\right)_{\gamma}{ }_{\gamma}{ }^{\varepsilon_{\alpha \beta}} N^{\beta}{ }_{\delta} & =\varepsilon_{\gamma \delta}  \tag{7.185}\\
\varepsilon_{\alpha \beta} N^{\beta}{ }_{\delta} & =\left[\left(N^{T}\right)^{-1}\right]_{a}^{\gamma}{ }_{\varepsilon_{\gamma \delta}} \tag{7.186}
\end{align*}
$$

And hence by Schur's Lemma $N$ and $\left(N^{T}\right)^{-1}$ are equivalent. Similarly, $N^{*}$ and $\left(N^{\dagger}\right)^{-1}$ are equivalent. This is not surprising since we already knew that the antisymmetric tensor, $\varepsilon$, is used to raise and lower indices in $\operatorname{SL}(2, \mathbb{C})$. Thus the equivalence of these representations is no more 'surprising' than the fact that Lorentz vectors with upper indices are equivalent to Lorentz vectors with lower indices. Explicitly, then, the contravariant representationstransform as

$$
\begin{equation*}
\psi^{\prime \alpha}=\psi^{\beta}\left(N^{-1}\right)_{\beta}{ }^{a} \quad \quad \bar{\chi}^{\prime \dot{\alpha}}=\bar{\chi}^{\dot{\beta}}\left(N^{*-1}\right)_{\dot{\beta}}^{\dot{a}} \tag{7.187}
\end{equation*}
$$

We summarize the different representations for $S L(2, \mathbb{C})$ :

| Representation | Index Structure | Transformation |
| :--- | :--- | :--- |
| Fundamental | Lower | $\psi_{a}^{\prime}=N_{\alpha}{ }^{\beta} \psi_{\beta}{ }^{\prime}$ |
| Conjugate | Lower dotted | $\bar{\chi}_{\dot{\alpha}}^{\prime}=\left(N^{*}\right)_{\dot{\alpha}}^{\dot{\beta}} \bar{\chi}_{\dot{\beta}}{ }^{a}$ |
| Contravariant | Upper | $\psi^{\prime \alpha}=\psi^{\beta}\left(N^{-1}\right)_{\beta}{ }^{\circ}$ |
| Contra-conjugate | Upper dotted | $\bar{\chi}^{\prime \dot{\alpha}}=\bar{\chi}^{\dot{\beta}}\left(N^{*-1}\right)_{\dot{\beta}}{ }^{\dot{\alpha}}$ |

Occasionally textbooks will write the conjugate and contravariant-conjugate transformations in terms of Hermitian conjugates,

$$
\begin{equation*}
\bar{\chi}_{\dot{\alpha}}^{\prime}=\bar{\chi}_{\dot{\beta}}\left(N^{\dagger}\right)_{\dot{\alpha}}^{\dot{\beta}} \quad \bar{\chi}^{\prime \dot{\alpha}}=\left(N^{\dagger-1}\right)^{\dot{a}} \dot{\dot{\beta}} \bar{\chi}^{\dot{\beta}} . \tag{7.188}
\end{equation*}
$$

We do not advocate this notation since Hermitian conjugates are a bit delicate notationally in quantum field theories. For more details about the representations of $S L(2, \mathbb{C})$, see the appendix of Wess and Bagger [324].

## 7.A. 5 Stars and Daggers

It is useful at this point to clarify notation. When dealing with classical fields, the complex conjugate representation is the usual complex conjugate of the field, $\psi \rightarrow \psi^{*}$. When dealing with quantum fields, on the other hand, it is conventional to write a Hermitian conjugate; i.e. $\psi \rightarrow \psi^{\dagger}$. This is because the quantum field contains creation and annihilation operators. The Hermitian conjugate here is the quantum version of complex conjugation.

This is the same reason why Lagrangians are often written $\mathcal{L}=$ term + h.c.. In classical physics, the Lagrangian is a scalar quantity so one could just 'c.c.' (complex conjugate) rather than 'h.c.' (Hermitian conjugate). In QFT, however, the fields in the Lagrangian are operators that must be Hermitian conjugated. When taking a first general relativity course, some students develop a dangerous habit: they think of lower-index objects as row vectors and upper-index objects as column vectors, so that

$$
V_{\mu} W^{\mu}=\left(\begin{array}{llll}
V_{0} & V_{1} & V_{2} & V_{3}
\end{array}\right) \cdot\left(\begin{array}{c}
W^{0}  \tag{7.189}\\
W^{1} \\
W^{2} \\
W^{3}
\end{array}\right) .
$$

Thus they think of the covariant vector as somehow a 'transpose' of contravariant vectors. This is is a bad, bad, bad habit and those students must pay their penance when they work with spinors. In addition to confusion generated from the antisymmetry of the metric and the anticommutation relations of the spinors, such students become confused when reading an expression like $\psi_{a}^{\dagger}$ because they interpret this as

$$
\begin{equation*}
\psi_{a}^{\dagger} \stackrel{?}{=}\left(\psi_{a}^{*}\right)^{T}=\left(\bar{\psi}_{\dot{\alpha}}\right)^{T} \stackrel{?}{=} \bar{\psi}^{\dot{a}} \tag{7.190}
\end{equation*}
$$

Wrong! Fail! Go directly to jail, do not pass go! The dagger $\left(^{\dagger}\right.$ ) on the $\psi$ acts only on the quantum operators in the field $\psi$, it doesn't know and doesn't care about the Lorentz index. Said once again, with emphasis: There is no 'transpose' in the quantum Hermitian conjugate!

To be safe, one could always write the Hermitian conjugate since this is 'technically' always correct. The meaning, however, is not always clear. Hermitian conjugation is always defined with respect to an inner product. Anyone who shows you a Hermitian conjugate without an accompanying inner product might as well be selling you a used car with no engine.

In [matrix] quantum theory the inner product comes with the appropriate Hilbert space. This is what is usually assumed when you see a dagger in QFT. In quantum wave mechanics, on the other hand, the Hermitian conjugate is defined with respect to the $L^{2}$ inner product,

$$
\begin{equation*}
\langle f, g\rangle=\int d x f^{*}(x) g(x) \tag{7.191}
\end{equation*}
$$

so that its action on fields is just complex conjugation. The structure of the inner product still manifests itself, though. Due to integration by parts, the Hermitian conjugate of the derivative is non-trivial,

$$
\begin{equation*}
\left(\frac{\partial}{\partial x}\right)^{\dagger}=-\frac{\partial}{\partial x} . \tag{7.192}
\end{equation*}
$$

As you know very well we're really just looking at different aspects of the same physics. The main point is that the inner product on the infinite dimensional function space is different from the inner product that we are used to from finite dimensional representations.

It is worth making one further note about notation. Sometimes authors will write

$$
\begin{equation*}
\bar{\psi}_{\dot{\alpha}}=\psi_{a}^{\dagger} . \tag{7.193}
\end{equation*}
$$

This is technically correct, but it can be a bit misleading since one shouldn't get into the habit of thinking of the bar as some kind of operator. The bar and its dotted index are notation to distinguish the right-handed representation from the left-handed representation. The content of the above equation is the statement that the conjugate of a left-handed spinor transforms as a right-handed spinor.

## 7.A. 6 Tensor representations

Now that we've said a few things about raised/lowered and dotted/undotted indices, it's worth repeating the mantra of tensor representations of Lie groups: symmetrize, antisymmetrize, and trace.Recall the familiar $\operatorname{SU}(N)$ case. We can write down tensor representations by just writing out the appropriate indices, e.g. if $\psi^{a} \rightarrow U^{a}{ }_{b} \psi^{b}$ and $\psi_{a} \rightarrow U^{\dagger}{ }_{a}^{b} \psi_{b}$, then we can write an $(n, m)$-tensor $\Psi$ and its transformation as

$$
\begin{equation*}
\Psi^{i_{1} \cdots i_{m}}{ }_{j_{1} \cdots j_{m}} \rightarrow U_{i_{1}^{\prime}}^{i_{1}^{\prime}} \cdots U_{i_{n}^{\prime}}^{i_{n}^{\prime}} U_{j_{1}}^{\dagger} j_{1}^{\prime} \cdots U_{j_{m}}^{\dagger} \Psi_{j_{m}^{\prime}}^{\prime} \Psi^{i_{1}^{\prime} \cdots i_{m}^{\prime}}{ }_{j_{1}^{\prime} \cdots j_{m}^{\prime}} \tag{7.194}
\end{equation*}
$$

This, however, is not generally an irreducible representation. In order to find the irreps, we can make use of the fact that tensors of symmetrized/antisymmetrized indices don't mix under the matrix symmetry group. For $U(N)$,

$$
\begin{align*}
\Psi^{i j} \rightarrow \Psi^{\prime i j} & =U_{k}^{i} U_{\ell}^{j} \Psi^{k \ell}  \tag{7.195}\\
\Psi^{j i} \rightarrow \Psi^{\prime j i} & =U_{\ell}^{j} U_{k}^{i} \Psi^{\ell k}  \tag{7.196}\\
& =U_{k}^{i} U_{\ell}^{j} \Psi^{\ell k}, \tag{7.197}
\end{align*}
$$

thus if $P(i, j)$ is the operator that swaps the indices $i \leftrightarrow j$, then $\Psi^{i j}$ and $\Psi^{j i}=P(i, j) \Psi^{i j}$ transform in the same way. $P(i, j)$ commutes with the matrices of $U(N)$, and hence we may construct simultaneous eigenstates of each. This means that the eigenstates of $P(i, j)$, i.e. symmetric and antisymmetric tensors, form invariant subspaces under $U(N)$. This argument is straightforwardly generalized to any matrix group and arbitrarily complicated index structures. Thus we may commit to memory an important lesson: we ought to symmetrize and antisymmetrize our tensor representations.

There are two more tricks we can invoke to further reduce our tensor reps. The first is taking the trace. For $U(N)$ we know from basic linear algebra that the trace is invariant under unitary rotations; it is properly a scalar quantity. What this amounts to for a general tensor is taking the contraction of an upper index $i$ and lower index $j$ with the Kronecker delta, $\delta_{i}^{j}$. This is guaranteed to commute with the symmetry group because $\delta_{i}^{j}$ is invariant under $U(N)$. This is analogous to $\varepsilon_{\alpha \beta}$ being an invariant tensor of $S L(2, \mathbb{C})$.

A second trick applicable to $S U(N)$ (but not $U(N)$ ) comes from the invariant tensor $\varepsilon_{i_{1} \cdots i_{N}}$. This is invariant under $S U(N)$ since

$$
\begin{equation*}
U_{i_{1}}^{i_{1}^{\prime}} \cdots U_{i_{N}}^{i_{N}^{\prime}} \varepsilon_{i_{1}^{\prime} \cdots i_{N}^{\prime}}^{\prime}=\operatorname{det} U \varepsilon_{i_{1} \cdots i_{N}}=\varepsilon_{i_{1} \cdots i_{N}} . \tag{7.198}
\end{equation*}
$$

Thus any time one has $N$ antisymmetric indices of an $U(N)$ tensor, one can go ahead and drop them. Note that this is totally different from the $\varepsilon_{\alpha \beta}$ of $S L(2, \mathbb{C})$.

For $S L(2, \mathbb{C})$ the irreducible two-index $\varepsilon$ tensor tells us that we can always reduce any tensorial representation into direct sums irreducible tensors which are symmetric in their dotted and (separately) undotted indices,

$$
\begin{equation*}
\Psi_{a_{1} \cdots a_{2 n} \dot{\alpha}_{1} \cdots \dot{a}_{2 m}}=\Psi_{\left(a_{1} \cdots a_{2 n}\right)\left(\dot{\alpha}_{1} \cdots \dot{a}_{2 m}\right)} . \tag{7.199}
\end{equation*}
$$

We label such an irreducible tensor-of-spinor-indices with the $\operatorname{SO}(3,1)$ notation $(n, m)$. In this notation the fundamental left$(\psi)$ and right-handed $(\bar{\chi})$ spinors transform as $\left(\frac{1}{2}, \mathrm{o}\right)$ and ( $\mathrm{o}, \frac{1}{2}$ ) respectively. One might now want to consider how to reduce Poincaré tensor products following the analogous procedure that textbooks present for $S U(2)$. Recall that $S O(3) \cong S U(2) / \mathbb{Z}_{2}$ so that such an analogy amounts to 'promoting' $S O(3)$ to $S O(3,1)$.

## 7.A. 7 Lorentz-Invariant Spinor Products

Armed with a metric to raise and lower indices, we can define the inner product of spinors as the contraction of upper and lower indices. Note that in order for inner products to be Lorentz-invariant, one cannot contract dotted and undotted indices.

There is a very nice short-hand that is commonly used in supersymmetry that allows us to drop contracted indices. Since it's important to distinguish between left- and right-handed Weyl spinors, we have to be careful that dropping indices doesn't introduce an ambiguity. This is why right-handed spinors are barred in addition to having dotted indices. Let us now define
the contractions

$$
\begin{equation*}
\psi \chi \equiv \psi^{\alpha} \chi_{\alpha} \quad \bar{\psi} \bar{\chi} \equiv \bar{\psi} \overline{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} \tag{7.200}
\end{equation*}
$$

Note that the contractions are different for the left- and right-handed spinors. This is a choice of convention such that

$$
\begin{equation*}
(\psi \chi)^{\dagger} \equiv\left(\psi^{\alpha} \chi_{\alpha}\right)^{\dagger}=\bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} \equiv \bar{\chi} \bar{\psi}=\bar{\psi} \bar{\chi} . \tag{7.201}
\end{equation*}
$$

The second equality is worth explaining. Why is it that $\left(\psi^{\alpha} \chi_{\alpha}\right)^{\dagger}=\bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}$ ? Recall from that the Hermitian conjugation acts on the creation and annihilation operators in the quantum fields $\psi$ and $\chi$. The Hermitian conjugate of the product of two Hermitian operators $A B$ is given by $B^{\dagger} A^{\dagger}$. The coefficients of these operators in the quantum fields are just $c$-numbers ('commuting' numbers), so the conjugate of $\psi^{a} \chi_{\alpha}$ is $\left(\chi^{\dagger}\right)_{\dot{\alpha}}\left(\psi^{\dagger}\right)^{\dot{\alpha}}$.

Now let's get back to our contraction convention. Recall that quantum spinor fields are Grassmann, i.e. they anticommute. Thus we show that with our contraction convention, the order of the contracted fields don't matter:

$$
\begin{gather*}
\psi \chi=\psi^{a} \chi_{a}=-\psi_{a} \chi^{a}=\chi^{a} \psi_{a}=\chi \psi  \tag{7.202}\\
\bar{\psi} \bar{\chi}=\bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{a}}=-\bar{\psi}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}}=\bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}=\bar{\chi} \bar{\psi} . \tag{7.203}
\end{gather*}
$$

## 7.A. 8 Vector-Like Spinor Products

Notice that the Pauli matrices give a natural way to go between $\operatorname{SO}(3,1)$ and $S L(2, \mathbb{C})$ indices. Using (7.169) ,

$$
\begin{align*}
\left(x_{\mu} \sigma^{\mu}\right)_{\alpha \dot{\alpha}} & \rightarrow N_{\alpha}^{\beta}\left(x_{v} \sigma^{v}\right)_{\beta \dot{\gamma}} N_{\dot{\alpha}}^{*} \dot{\dot{r}}  \tag{7.204}\\
& =\left(\Lambda_{\mu}{ }^{v} x_{v}\right) \sigma_{a \dot{\alpha}}^{\mu} . \tag{7.205}
\end{align*}
$$

Then we have

$$
\begin{equation*}
\left(\sigma^{\mu}\right)_{a \dot{\alpha}}=N_{a}^{\beta}\left(\sigma^{\nu}\right)_{\beta \dot{\gamma}}\left(\Lambda^{-1}\right)^{\mu}{ }_{v} N_{\dot{\alpha}}^{*} \dot{\gamma} \tag{7.206}
\end{equation*}
$$

One could, for example, swap between the vector and spinor indices by writing

$$
\begin{equation*}
V_{\mu} \rightarrow V_{a \dot{\beta}} \equiv V_{\mu}\left(\sigma^{\mu}\right)_{a \dot{\beta}} \tag{7.207}
\end{equation*}
$$

We can define a 'raised index' $\sigma$ matrix,

$$
\begin{align*}
\left(\bar{\sigma}^{\mu}\right)^{\dot{\alpha} \alpha} & \equiv \varepsilon^{\alpha \beta} \varepsilon^{\dot{\beta}}\left(\sigma^{\mu}\right)_{\beta \dot{\beta}}  \tag{7.208}\\
& =(\mathbb{1},-\boldsymbol{\sigma}) \tag{7.209}
\end{align*}
$$

Note the bar and the reversed order of the dotted and undotted indices. The bar is just notation to indicate the index structure, similarly to the bars on the right-handed spinors.

Now an important question: How do we understand the indices? Why do we know that the un-barred Pauli matrices have lower indices $\alpha \dot{\alpha}$ while the barred Pauli matrices have upper indices $\dot{\alpha} \alpha$ ? Clearly this allows us to maintain our convention about how indices contract, but some further checks might help clarify the matter. Let us go back to the matrix form of the Pauli matrices (7.163) and the upper-indices epsilon tensor (7.175). One may use $\varepsilon=i \sigma^{2}$ and to directly verify that

$$
\begin{equation*}
\varepsilon \bar{\sigma}_{\mu}=\sigma_{\mu}^{T} \varepsilon \tag{7.210}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\bar{\sigma}_{\mu}=\varepsilon \sigma_{\mu}^{T} \varepsilon^{T} \tag{7.211}
\end{equation*}
$$

Now let us write these in terms of dot-less indices-i.e. write all indices without dots, whether or not they ought to have dots-then we can restore the indices later to see how they turn out. To avoid confusion we'll write dot-less indices with
lowercase Roman letters $\alpha, \beta, \gamma, \delta \rightarrow a, b, c, d$.

$$
\begin{align*}
\left(\bar{\sigma}^{\mu}\right)^{a d} & =\varepsilon^{a b}\left(\sigma^{\mu T}\right)_{b c}\left(\varepsilon^{T}\right)^{c d}  \tag{7.212}\\
& =\varepsilon^{a b}\left(\sigma^{\mu}\right)_{c b} \varepsilon^{d c}  \tag{7.213}\\
& =\varepsilon^{a b} \varepsilon^{d c}\left(\sigma^{\mu}\right)_{c b} . \tag{7.214}
\end{align*}
$$

We already know what the dot structure of the $\sigma^{\mu}$ is, so we may go ahead and convert to the dotted/undotted lowercase Greek indices. Thus $c, b \rightarrow \gamma, \dot{\beta}$. Further, we know that the $\varepsilon$ s carry only one type of index, so that $a, d \rightarrow \dot{\alpha}, \delta$. Thus we see that the $\bar{\sigma}^{\mu}$ have a dotted-then-undotted index structure. A further consistency check comes from looking at the structure of the $\gamma$ matrices as applied to the Dirac spinors formed using Weyl spinors with our index convention.

## 7.A. 9 Generators of $\operatorname{SL}(2, \mathbb{C})$

We now show how Lorentz transformations act on Weyl spinors. The objects that obey the Lorentz algebra, (7.5), and generate the desired transformations are given by the matrices,

$$
\left(\sigma^{\mu \nu}\right)_{a}^{\beta}=\frac{i}{4}\left(\sigma^{\mu} \bar{\sigma}^{\nu}-\sigma^{\nu} \bar{\sigma}^{\mu}\right)_{a}^{\beta}
$$

$$
\left(\bar{\sigma}^{\mu v}\right)_{\dot{\beta}}^{\dot{\alpha}}=\frac{i}{4}\left(\bar{\sigma}^{\mu} \sigma^{v}-\bar{\sigma}^{v} \sigma^{\mu}\right)_{\dot{\beta}}^{\dot{\alpha}}
$$

It is important to note that these matrices are Hermitian. The assignment of dotted and undotted indices are deliberate: they tell us which generator corresponds to the fundamental versus the conjugate representation. The choice of which one is fundamental versus conjugate, of course, is arbitrary. Thus the left and right-handed Weyl spinors transform as

$$
\begin{equation*}
\psi_{a} \rightarrow\left(e^{-\frac{i}{2} \omega_{\mu \nu} \sigma^{\mu \nu}}\right)_{a}^{\beta} \psi_{\beta} \tag{7.216}
\end{equation*}
$$

$$
\bar{\chi}^{\dot{\alpha}} \rightarrow\left(e^{-\frac{i}{2} \omega_{k} \overline{v^{\mu}}}\right)_{\dot{\beta}}^{\dot{\alpha}} \bar{\chi}^{\dot{\beta}} .
$$

We can invoke the $S U(2) \times S U(2)$ representation (and we use that word very loosely) of the Lorentz group from (7.157) to write the left-handed $\sigma^{\mu \nu}$ generators as

$$
\begin{equation*}
J_{i}=\frac{1}{2} \varepsilon_{i j k} \sigma_{j k}=\frac{1}{2} \sigma_{i} \quad K_{i}=\sigma_{o i}=-\frac{i}{2} \sigma_{i} \tag{7.217}
\end{equation*}
$$

One then finds

$$
\begin{equation*}
A_{i}=\frac{1}{2}\left(J_{i}+i K_{i}\right)=\frac{1}{2} \sigma_{i} \quad B_{i}=\frac{1}{2}\left(J_{i}-i K_{i}\right)=0 . \tag{7.218}
\end{equation*}
$$

The left-handed Weyl spinors $\psi_{\alpha}$ are $\left(\frac{1}{2}, \circ\right)$ spinor representations Similarly, one finds that the right-handed Weyl spinors $\bar{\chi}^{\dot{a}}$ are ( $0, \frac{1}{2}$ ) spinor representations. Alternately, we could have guessed the generators of the Lorentz group acting on Weyl spinors from the algebra of rotations and boosts in (7.158). One could have made the ansatz that the $\mathbf{J}$ and $\mathbf{K}$ are represented on Weyl spinors via(7.217). From this one could exponentiate to derive a finite Lorentz transformation,

$$
\begin{equation*}
\exp \left(\frac{i}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\theta} \pm \frac{1}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\varphi}\right)=\exp \left(\underset{2}{\left.i \frac{1}{2} \cdot(\boldsymbol{\theta} \mp i \boldsymbol{\varphi})\right), ~}\right. \tag{7.219}
\end{equation*}
$$

where the upper sign refers to left-handed spinors while the lower sign refers to right-handed spinors. $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}$ are the parameters of rotations and boosts, respectively. One can then calculate the values of $\sigma^{\circ i}$ and $\sigma^{i j}$ to confirm that they indeed match the above expression.

## 7.A. 10 Chirality

Let us return to a point of nomenclature. Why do we call them left- and right-handed spinors? The Dirac equation tells us

$$
\begin{align*}
p_{\mu} \sigma^{\mu} \psi & =m \psi  \tag{7.220}\\
p_{\mu} \bar{\sigma}^{\mu} \bar{\chi} & =m \bar{\chi} \tag{7.221}
\end{align*}
$$

Equation (7.221) follows from (7.220) via Hermitian conjugation, as appropriate for the conjugate representation. In the massless limit, then, $p^{\circ} \rightarrow|\mathbf{p}|$ and hence

$$
\begin{equation*}
\left(\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{|\mathbf{p}|} \psi\right)=\psi \quad\left(\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{|\mathbf{p}|} \bar{\chi}\right)=-\bar{\chi} \tag{7.222}
\end{equation*}
$$

We recognize the quantity in parenthesis as the helicity operator, so that $\psi$ has helicity +1 (left-handed) and $\bar{\chi}$ has helicity -1 (right-handed). Non-zero masses complicate things, of course, but even though $\psi$ and $\bar{\chi}$ would no longer be helicity eigenstates, they are chirality eigenstates:.

$$
\begin{equation*}
\gamma_{5}\binom{\psi}{0}=-\binom{\psi}{0} \quad \gamma_{5}\binom{0}{\bar{\chi}}=\binom{0}{\bar{\chi}} \tag{7.223}
\end{equation*}
$$

where we've put the Weyl spinors into four-component Dirac spinors in the usual way so that we may apply the chirality operator, $\gamma_{5}$.

## 7.A. 11 Connection to Dirac Spinors

We would now explicitly connect the machinery of two-component Weyl spinors to the four-component Dirac spinors.Let us define

$$
\gamma^{\mu} \equiv\left(\begin{array}{cc}
\circ & \sigma^{\mu}  \tag{7.224}\\
\bar{\sigma}^{\mu} & \circ
\end{array}\right)
$$

This, one can check, gives us the Clifford algebra

$$
\left\{\gamma^{\mu}, \gamma^{\nu}\right\}=2 \eta^{\mu \nu} \cdot \mathbb{1}
$$

$$
(7.225)
$$

We can further define the fifth $\gamma$-matrix, the four-dimensional chirality operator,

$$
\gamma^{5}=i \gamma^{\circ} \gamma^{1} \gamma^{2} \gamma^{3}=\left(\begin{array}{cc}
-\mathbb{1} & 0  \tag{7.226}\\
\circ & \mathbb{1}
\end{array}\right)
$$

A Dirac spinor is defined, as mentioned above, as the direct sum of left- and right-handed Weyl spinors, $\Psi_{D}=\psi \oplus \bar{\chi}$, or

$$
\begin{equation*}
\Psi_{D}=\binom{\psi_{a}^{a}}{\bar{\chi}^{\dot{\alpha}}} \tag{7.227}
\end{equation*}
$$

The choice of having a lower undotted index and an upper dotted index is convention and comes from how we defined our spinor contractions. The generator of Lorentz transformations takes the form

$$
\Sigma^{\mu \nu}=\left(\begin{array}{cc}
\sigma^{\mu \nu} & 0  \tag{7.228}\\
0 & \bar{\sigma}^{\mu \nu}
\end{array}\right)
$$

with spinors transforming as

$$
\begin{equation*}
\Psi_{D} \rightarrow e^{-\frac{i}{2} \omega_{\mu \nu} \Sigma^{\mu \nu}} \Psi_{D} \tag{7.229}
\end{equation*}
$$

In our representation the action of the chirality operator is given by $\gamma_{s}$,

$$
\begin{equation*}
\gamma^{5} \Psi_{D}=\binom{-\psi_{\tilde{a}}}{\bar{\chi}^{a}} . \tag{7.230}
\end{equation*}
$$

We can then define left- and right-handed projection operators,

$$
\begin{equation*}
P_{L, R}=\frac{1}{2}\left(\mathbb{1} \mp \gamma^{s}\right) . \tag{7.231}
\end{equation*}
$$

Using the standard notation, we shall define a barred Dirac spinor as $\bar{\Psi}_{D} \equiv \Psi^{\dagger} \gamma^{\circ}$. Note that this bar has nothing to do with the bar on a Weyl spinor. We can then define a charge conjugation matrix $C$ via $C^{-1} \gamma^{\mu} \mathrm{C}=-\left(\gamma^{\mu}\right)^{T}$ and the Dirac conjugate spinor $\Psi_{D}{ }^{c}=C \bar{\Psi}_{D}{ }^{T}$, or explicitly in our representation,

$$
\begin{equation*}
\Psi_{D}{ }^{c}=\left(\frac{\chi_{\alpha}}{\bar{\psi}^{\alpha}}\right) . \tag{7.232}
\end{equation*}
$$

A Majorana spinor is defined to be a Dirac spinor that is its own conjugate, $\Psi_{M}=\Psi_{M}^{c}$. We can thus write a Majorana spinor in terms of a Weyl spinor,

$$
\begin{equation*}
\Psi_{M}=\left(\frac{\psi_{a}^{a}}{\psi^{\dot{a}}}\right) . \tag{7.233}
\end{equation*}
$$

Here our notation is that $\bar{\psi}=\psi^{\dagger}$, i.e. we treat the bar as an operation acting on the Weyl spinor (a terrible idea, but we'll do it just for now). One can choose a basis of the $\gamma$ matrices such that the Majorana spinors are manifestly real. Thus this is sometimes called the 'real representation' of a Weyl spinor. Note that a Majorana spinor contains exactly the same amount of information as a Weyl spinor. Some textbooks thus package the Weyl SUSY generators into Majorana Dirac spinors, eschewing the dotted and undotted indices.

It's worth emphasizing once more that the dots and bars are just book-keeping tools. Essentially they are a result of not having enough alphabets available to write different kinds of objects. The bars on Weyl spinors should not be associated with barred Dirac spinors, $\bar{\Psi}=\Psi^{\dagger} \gamma_{0}$. Do not make this mistake. Weyl and Dirac spinors are different objects. The bar on a Weyl spinor has nothing to do with the bar on a Dirac spinor. We see this explicitly when we construct Dirac spinors out of Weyl spinors (namely $\Psi=\psi \oplus \bar{\chi}$ ), but it's worth repeating because the notation can be misleading.

In principle $\psi$ and $\bar{\psi}$ are totally different spinors in the same way that $\alpha$ and $\dot{\alpha}$ are totally different indices. Sometimes-as we have done above-we may also use the bar as an operation that converts an unbarred Weyl spinor into a barred Weyl spinor. That is to say that for a left-handed spinor $\psi$, we may define $\bar{\psi}=\psi^{\dagger}$. To avoid ambiguity it is customary-as we have also done-to write $\psi$ for left-handed Weyl spinors, $\bar{\chi}$ for right-handed Weyl spinors, and $\bar{\psi}$ to for the right-handed Weyl spinor formed by taking the Hermitian conjugate of the left-handed spinor $\psi$.

## 7.B The Poincaré Group

In this appendix we push forward and describe the representations of the Poincaré group relevant to quantum field theory.

## 7.B. 1 Casimir Operators of the Poincaré Group

We have thus constructed the representations of the Lorentz group. In order to generalize to the Poincaré group, let us remind ourselves of the representations of the Rotation group.

The algebra is given by

$$
\begin{equation*}
\left[J_{i}, J_{j}\right]=i \varepsilon_{i j k} J_{k} \tag{7.234}
\end{equation*}
$$

$S O(3)$ has one Casimir operator, an operator built out of the generators that commutes with all of the generators. For $\mathrm{SO}(3)$ this is

$$
\begin{equation*}
\mathbf{J}^{2}=\sum J_{i}^{2} . \tag{7.235}
\end{equation*}
$$

Each irreducible representation (irrep) takes a single value of the Casimir operator. For example, the eigenvalues of $\mathbf{J}^{2}$ are $j(j+1)$ where $j=\frac{1}{2}, 1, \cdots$. Thus each irrep is labelled by $j$. To label each element of the irrep, we pick eigenvalues of $J_{3}$ from the set $j_{3}=-j, \cdots, j$. Thus each state is labelled as $\left|j ; j_{3}\right\rangle$, identifying individual states with respect to their transformation properties under the symmetry.

Let's do the analogous analysis for the Poincaré group. It is reasonable to expect that there should be a Casimir each for the Lorentz part of the Poincaré group and the translation part. The former should be some covariant generalization of the $\mathbf{J}^{2}$ of $S O(3)$ that incorporates boosts, while the latter should have something to do with translations in space-time. Thinking about conserved quantities, one would expect these to roughly correspond to spin (angular momentum) and mass (energy).

Define the Pauli-Lubanski vector,

$$
\begin{equation*}
W_{\mu}=\frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} P^{v} M^{\rho \sigma} . \tag{7.236}
\end{equation*}
$$

This unusual vector is a Lorentz-covariant generalization of angular momentum. How might we have thought to write down something like this to represent 'angular momentum' in $S O(3,1)$ ? We have three covariant or invariant tensors to use: $M^{\mu v}, P^{\mu}$, and $\varepsilon_{\mu \nu \rho \sigma \sigma}$. Taking some motivation from $\mathbf{J}=\mathbf{r} \times \mathbf{p}$, it is reasonable to expect a 'cross product' of $P^{\mu}$ and $M^{\mu \nu}$ (Up to a choice of normalization, (7.236) is the natural result.

We can now define two Lorentz- and translation-invariant Casimir operators,

$$
C_{1} \equiv P^{\mu} P_{\mu} C_{2} \quad \equiv W^{\mu} W_{\mu}
$$

That these are really Casimirs can be checked explicitly. The eigenvalue of $C_{1}$ is the particle mass as usual. We interpret $C_{2}$ below, but one can already guess that it has something to do with spin. From these two we will be able to label Poincaré irreps $|m, \omega\rangle$ by their mass, $m$, and the 'spin' eigenvalue of $C_{2}$, which we call $\omega$.

The connection between $C_{2}$ and spin will come about when we look at particular representations, so our strategy is to first show how $C_{1}$ decomposes the space of Poincaré representations and then to see what happens when we look at $C_{2}$ on these subspaces. The entire analysis that follows can be done without explicit knowledge of $C_{2}$, and in fact one can 'reverse-engineer' the expression for $C_{2}$ after we do some heavy-lifting.

## 7.B. 2 States in a Poincare Irreducible Representation

To label the particular particle states within an irreducible representation of the Poincaré group, we need to pick eigenvalues for a set of commuting generators. A good place to start is the the momentum operator $\hat{P}^{\mu}$,

$$
\begin{equation*}
\hat{p}^{\mu}\left|m, \omega ; p^{\mu}\right\rangle=p^{\mu}\left|m, \omega ; p^{\mu}\right\rangle . \tag{7.238}
\end{equation*}
$$

We shall adopt the non-standard notation of writing a hat to mean the appropriate representation (operator) of a Poincar'e group element. A more typical notation is to write the representation of the abstract group element $P$ by $d(P)$ or even $d_{R}(P)$ for a representation $R$. This gets ridiculously cumbersome and obfuscates the meaning of equations. Thus we shall write a hat to denote a representation. The particular representation shall be left implicit unless there is an ambiguity.

Are there more generators that commute with $P_{\mu}$, i.e. further labels for elements within a Poincar'e irrep? Yes, and as one might guess they have something to do with the Pauli-Lubanski vector. However, $\left[W^{\mu}, P^{\nu}\right] \neq \circ$ in general. We are led to consider the cases of massive and massless representations separately. This is not so surprising since we know that massless and massive particles are fundamentally different from the point of view of the Lorentz group: you cannot boost into the rest frame of a massless particle.

We label a unitary representation of the Poincaré group by its momentum $p_{\mu}$ and any other quantum numbers $\alpha$. For example, $a$ would include spin and any gauge quantum numbers. We shall write a state under this representation as $|p, a\rangle$ with
the property that

$$
\begin{equation*}
\hat{P}_{\mu}|p, a\rangle=p_{\mu}|p, a\rangle . \tag{7.239}
\end{equation*}
$$

To be complete, we really should have written $|m, \omega ; p, a\rangle$ to label the irrep with the values of the Casimir operators. In the current analysis, however, we'll only work within an irrep so we shall suppress these labels. Describe the space of all states of a given momentum $p$ by

$$
\begin{equation*}
\mathcal{H}_{k}=\{|k, a\rangle\}, \tag{7.240}
\end{equation*}
$$

this set is labelled by the quantum numbers $\alpha$. For example, the space $\mathcal{H}_{k}$ for an on-shell massless $S U(N)$ gauge boson is the set of states for which the gauge boson has momentum $k_{\mu}$. In that case $\alpha$ is short hand for the $(N-1)$ gauge quantum states and the two spin polarizations.

What happens when we perform a finitetranslation on the state $|p, a\rangle$ by acting with $\hat{U}(\mathbb{1}, a)$ ? We can write such a translation as the exponentiation of the translation generator, the momentum operator,

$$
\begin{equation*}
\hat{U}(\mathbb{1}, a)|p, a\rangle=e^{i a \cdot \hat{p}}|p, a\rangle=e^{i a \cdot p}|p, a\rangle . \tag{7.241}
\end{equation*}
$$

To understand how Lorentz-transformed states behave under translations, let us consider the state

$$
\begin{equation*}
|p, a\rangle_{\Lambda} \equiv \hat{U}(\Lambda, o)|p, a\rangle \tag{7.242}
\end{equation*}
$$

The operator $\hat{U}(\Lambda, \circ)$ generically has some non-trivial $\alpha$ index structure acting on the spin quantum numbers, so one would rather expect a definition more like

$$
\begin{equation*}
|p, a\rangle_{\Lambda}=\hat{U}(\Lambda, o)_{a \beta}|p, \beta\rangle . \tag{7.243}
\end{equation*}
$$

We know, however, that we are free to choose a basis of spin states where $\hat{U}(\Lambda, o)_{\alpha \beta}$ is diagonal, hence the definition (7.242) is sensible. Let us act on the state $p, a\rangle_{\Lambda}$ with $\hat{U}(\mathbb{1}, a)$ :

$$
\begin{aligned}
\hat{U}(\mathbb{1}, a)|p, a\rangle_{\Lambda} & =\hat{U}(\mathbb{1}, a) \hat{U}(\Lambda, o)|p, a\rangle \\
& =\hat{U}(\Lambda, o) \hat{U}\left(\mathbb{1}, \Lambda^{-1} a\right)|p, a\rangle \\
& =e^{i \Lambda^{-1} a \cdot p} U(\Lambda, o)|p, a\rangle \\
& =e^{i a \cdot \Lambda p}|p, a\rangle_{\Lambda}
\end{aligned}
$$

In the penultimate line we have just used the fact that $e^{i \Lambda^{-1} a \cdot p}$ is a scalar with no group structure and in the final line we've used $\left(\Lambda_{v}{ }^{\mu} a^{\nu}\right) p_{\mu}=a^{\nu}\left(\Lambda_{v}{ }^{\mu} p_{\mu}\right)$. Because of (7.241), we know that (7.247) implies that the state $|p, a\rangle_{\Lambda}=\hat{U}(\Lambda, o)|p, a\rangle$ is part of the space $\mathcal{H}_{\Lambda p}$, i.e. $|p, a\rangle_{\Lambda} \in \mathcal{H}_{\Lambda p}$. In other words, if we act on a state of momentum $p$ with a Lorentz transformation $\Lambda$, then the resulting state has momentum $\Lambda p$. In fact, since we work in a basis where $\hat{U}(\Lambda, \circ)$ is diagonal in spin indices, we may identify

$$
\begin{equation*}
|p, a\rangle_{\Lambda}=|\Lambda p, a\rangle . \tag{7.248}
\end{equation*}
$$

## 7.B. 3 Tracks and the little group

In addition to $\mathcal{H}_{k}$, let us define larger subspaces $\mathcal{H}_{\{\Lambda k\}}$ that include all states with a momentum $q$ that can be related to the momentum $k$ by a Lorentz transformation,

$$
\begin{equation*}
\mathcal{H}_{\{\Lambda k\}}=\{|q, a\rangle \mid \exists \Lambda \text { such that } q=\Lambda\} . \tag{7.249}
\end{equation*}
$$

Heuristically one could write $\mathcal{H}_{\{\Lambda k\}}=\sum_{i} \mathcal{H}_{\Lambda_{i} k}$. Since we now know with excessive formality that Lorentz transformations $\hat{U}(\Lambda, o)$ take states $|p, a\rangle$ to $|p, a\rangle_{\Lambda}=|\Lambda p, a\rangle$ and that translations do not induce any non-trivial transformation of states'
quantum numbers. (Note that this requirement can be relaxed compact extra dimensions in which the field can 'twist.') The space $\mathcal{H}_{\{\Lambda k\}}$ is invariant under the Poincaré group. Each of these spaces can be represented by a track or mass-shell surface, a hyperboloid in 4 -momentum space, as shown in Figure 7.B.1. These tracks are, by definition, invariant under the action of the Poincaré group and will be the 'elements' of the space of Poincaré irreps.


Figure 7.B.1: Hyperboloids representing tracks of different characteristic momenta (e.g. $\hat{p}$ ), i.e. different irreducible representations of the Poincaré group. The full hyperboloids are given by rotating these cross sections about the $p^{\circ}$ axis.

It is clear that each track can be completely described by one arbitrarily chosen characteristic momentum, $\hat{p}$. Canonical choices for characteristic momenta are:

| Track | Characteristic $\hat{p}$ |
| :--- | :--- |
| $p^{2}=m^{2}>0, \quad p^{\circ}>0$ | $(m, o)$ |
| $p^{2}=m^{2}>0, \quad p^{\circ}<0$ | $(-m, \circ)$ |
| $p^{2}=0, \quad p^{\circ}>0$ | $(1, \circ, \circ, \mathbf{1})$ |
| $p^{2}=0, \quad p^{\circ}<0$ | $(-\mathbf{1}, \circ, \circ, \mathbf{1})$ |
| $p^{\mu}=0$ | $(0, \circ)$ |
| $p^{2}=-m^{2}<0$ | $(0, \circ, \circ, m)$ |

We separate out the positive and negative energy tracks $\pm p^{\circ}>0$ since we know that physical states have non-negative energy. A fancy way of saying this is that the sign of $p^{\circ}$ is a sort of Casimir operator. For any non-characteristic momentum $p$ on a given track with characteristic momentum $\hat{p}$, define $\hat{L}(p)$ to be a Lorentz transformation that takes $\hat{p}$ to $p$,

$$
\begin{equation*}
L(p) \hat{p}=p \tag{7.250}
\end{equation*}
$$

The power of defining such characteristic momenta is that we may now use (7.242) to write down any state $|p, \alpha\rangle$ in terms of a state with the characteristic momentum of the track associated with $p$,

$$
\begin{equation*}
|p, a\rangle \equiv \hat{U}(L(p))|\hat{p}, a\rangle=|\hat{p}, a\rangle_{L(p)} . \tag{7.251}
\end{equation*}
$$

For simplicity we've dropped the second argument of $\hat{U}$ when it is zero: $\hat{U}(\Lambda, o) \equiv \hat{U}(\Lambda)$.

Define the little group of $p, H_{p}$ to be the subgroup of Lorentz transformations that leave the four-vector $p$ unchanged,

$$
H_{p}=\{\Lambda \mid \Lambda p=p\}
$$

Sometimes this is called the stability group or invariance group of $p$. For example, in $S O(3)$ the little group of a vector $\mathbf{v}$ is the $S O(2)$ subgroup of rotations about $\mathbf{v}$. Don't confuse this with $\mathcal{H}_{k}$ or $\mathcal{H}_{\{\Lambda k\}}$, which are spaces of states rather than transformations.

We are particularly interested in transformations that leave the characteristic momentum $\hat{p}$ unchanged, i.e. the little group $H_{\hat{p}}$. These particular transformations are useful because we write out all of our Poincaré states $|p, a\rangle$ as a Lorentz transform $L(p)$ of a representative state $|\hat{p}, a\rangle$ with the track's characteristic momentum $\hat{p}$. Distinct elements of an irreducible representation will correspond to different states in the representation of the characteristic momentum's little group. One obtains nontrivial elements of the little group if there exist more than one Lorentz transformation, e.g. $L_{1}(p), L_{2}(p)$, that takes $\hat{p}$ to $p$ :

$$
\begin{equation*}
L_{1}^{-1}(p) L_{2}(p) \hat{p}=\hat{p} \tag{7.253}
\end{equation*}
$$

and so $L_{1}^{-1}(p) L_{2}(p) \in H_{\hat{p}}$.

## 7.B. 4 The little group and the Pauli-Lubanski vector

The action of a finite Lorentz transformation can be written as the exponential of a linear combination of the generators of the Lorentz algebra,

$$
\begin{equation*}
\hat{U}(\Lambda)=e^{\frac{i}{2} \omega_{\mu \nu} M^{\mu \nu}} \tag{7.254}
\end{equation*}
$$

For a Lorentz transformation $\Lambda$, may write the action of $\hat{U}(\Lambda)$ on a state $|p, a\rangle$ as

$$
\begin{equation*}
\hat{U}(\Lambda)|p, a\rangle=e^{\frac{i}{2} \omega_{\mu \nu} M^{\mu \nu}}|p, a\rangle=\hat{C}_{\alpha \beta}\left|\left(e^{\omega}\right)^{\mu}{ }_{\nu} p^{\nu}, \beta\right\rangle \tag{7.255}
\end{equation*}
$$

where $\hat{C}_{\alpha \beta}$ is some matrix acting on the spin quantum numbers only; it's particular form is of no interest to us at the moment. If we restrict to the case where $\Lambda \in H_{p}$, then the transformed momentum must equal the original momentum,

$$
\begin{equation*}
\left(e^{\omega}\right)^{\mu}{ }_{\nu} p^{\nu}=p^{\mu} \tag{7.256}
\end{equation*}
$$

and hence $\omega^{\mu}{ }_{\nu} p^{\nu}=0$. This has a general solution given by

$$
\begin{equation*}
\omega_{\mu \nu}=\varepsilon_{\mu v \rho \sigma} p^{\rho} q^{\sigma} \tag{7.257}
\end{equation*}
$$

where $q$ is any arbitrary four-vector. Is this starting to look familiar? If we restrict the action to elements of the space $\mathcal{H}_{p}$, we may write down general elements of the little group $\Lambda_{p}(b, n)$ as

$$
\begin{equation*}
\Lambda_{p}(b, q)=\exp \left(-i b_{\mu} P^{\mu}+\frac{i}{2} \varepsilon_{\mu \nu \rho \sigma} p^{\mu} q^{\nu} M_{\rho \sigma}\right)=\exp \left(-i b_{\mu} P^{\mu}+i q_{\mu} W^{\mu}\right) \tag{7.258}
\end{equation*}
$$

We welcome back our friend, the Pauli-Lubanski vector $W_{\mu}$, which appears to describe the Lorentz action of an element of the little group $H_{p}$ on a state in $\mathcal{H}_{p}$. To be absolutely clear: $W_{\mu}$ describes the action of the little group the momentum four-vector $p$ acting on states with precisely that momentum $p$.

By definition elements of $\mathcal{H}_{p}$ have the same momentum. The only quantum number affected by the Poincaré group is spin, and so we indeed expect that the Pauli-Lubanski vector indeed transforms states' spin polarizations.

## 7.B. 5 Wigner Decomposition

Next we require the fact that acting on characteristic momentum state $|\hat{p}, a\rangle$ with an element from the little group $\Lambda_{\hat{p}} \in H_{\hat{p}}$ only acts on the $\alpha$ quantum numbers and leaves the state's momentum unchanged. Written out formally, this statement is

$$
\begin{equation*}
\hat{U}\left(\Lambda_{p}, o\right)|\hat{p}, a\rangle=\hat{D}_{a a^{\prime}}\left(\Lambda_{p}\right)\left|\hat{p}, a^{\prime}\right\rangle \tag{7.259}
\end{equation*}
$$

where $\hat{D}_{a a^{\prime}}\left(\Lambda_{\hat{p}}\right)$ is the representation of the element of the little group $\tilde{\Lambda}$. There is an implied sum over $\alpha^{\prime}$ which is shorthand for 'matrix' multiplication on the non-momentum quantum numbers. The point is that $\hat{D}_{\alpha \alpha^{\prime}}$ is a scalar with respect to the Lorentz group.

For a general Lorentz transformation $\Lambda$ acting on a general state $|p, \alpha\rangle$, we can perform some sleight of hand:

$$
\begin{align*}
\hat{U}(\Lambda, o)|p, a\rangle & =\hat{U}(\Lambda) \hat{U}(L(p))|\hat{p}, a\rangle  \tag{7.260}\\
& =\hat{U}(L(\Lambda p)) \hat{U}\left(L^{-1}(\Lambda p)\right) \hat{U}(\Lambda) \hat{U}(L(p))|\hat{p}, a\rangle  \tag{7.261}\\
& =\hat{U}(L(\Lambda p)) \hat{U}(\underbrace{L^{-1}(\Lambda p) \cdot \Lambda \cdot L(p)}_{\equiv \tilde{\Lambda}_{p} \in H_{\hat{p}}})|\hat{p}, a\rangle  \tag{7.262}\\
& =\hat{U}(L(\Lambda p)) \hat{D}_{\alpha a^{\prime}}\left(\tilde{\Lambda}_{p}\right)\left|\hat{p}, a^{\prime}\right\rangle  \tag{7.263}\\
& =\hat{D}_{\alpha a^{\prime}}\left(\tilde{\Lambda}_{p}\right) \hat{U}(L(\Lambda p))\left|\hat{p}, a^{\prime}\right\rangle  \tag{7.264}\\
& =\hat{D}_{\alpha a^{\prime}}\left(\tilde{\Lambda}_{p}\right)\left|\hat{p}, a^{\prime}\right\rangle_{\Lambda} . \tag{7.265}
\end{align*}
$$

We used the 'trick' of inserting $\mathbb{1}=\hat{U}(L(\Lambda p)) \hat{U}\left(L^{-1}(\Lambda p)\right)$ so that we could identify one of the products of operators as a representation of an element of the little group, $L^{-1}(\Lambda p) \cdot \Lambda \cdot L(p)$ in $(7.262)$. We then use the definition of the action of the little group from (7.259) and use the fact that each $\hat{D}_{\alpha a^{\prime}}$ is just a scalar (with an implied sum over $a^{\prime}$ ) push $\hat{U}(L(\Lambda p)$ ) past it and act on the state. Finally, in the last line we invoke our (slightly unusual) definition for $|p, a\rangle_{\Lambda}$ in (7.242), with the reminder that $\hat{U}$ is diagonal in $\alpha$ indices.

The action of the little group on any state on a track $\mathcal{H}_{\{\Lambda \hat{p}\}}$ has the same structure as its action on the state with the track's characteristic momentum $\hat{p}$. Thus, in order to determine the action of a general Lorentz transformation on a representation of the Poincaré group, one only needs to understand action of the little group on a state with the track's [arbitrarily chosen] characteristic momentum. This is an example of an induced representation whereby the representation of a group is given by a subgroup (in this case, the little group). This was first elucidated in Wigner's classic 1939 paper [325]; thus we now call $\hat{D}$ a Wigner rotation and the expression (7.265) a Wigner decomposition.

Now we arrive at the punchline: an irreducible representation of the Poincaré group corresponds to a characteristic momentum (i.e. a track) and a representation of the little group for that momentum. Let's update our list of tracks, then, with the little group for each characteristic momentum; this is done in Table 7.1.

Physical states correspond to those with non-negative energy and mass-squared, so we can restrict ourselves to such states when constructing representations of the Poincaré group.

## 7.B. 6 The vacuum state

Let's start with the simplest track, $\hat{p}=(\mathrm{o}, \mathrm{o})$. This corresponds to the track of the vacuum. The little group corresponds to the orthochronous Lorentz group, which we already know is universally covered by $S L(2, \mathbb{C})$. The unitary irreps of this group are infinite dimensional. This isn't a problem for the vacuum, however, since it lives in the trivial representation of the Lorentz group,

$$
\begin{equation*}
\Lambda \rightarrow \mathbb{1} \tag{7.266}
\end{equation*}
$$

| Track |  | Characteristic $\hat{p}$ | $H_{\hat{p}}$ | Physical? |
| :--- | :--- | :--- | :---: | :---: |
| $p^{2}=m^{2}>0$, | $p^{\circ}>0$ | $(m, o)$ | $S O(3)$ | Yes |
| $p^{2}=m^{2}>0$, | $p^{\circ}<0$ | $(-m, o)$ | $S O(3)$ | No |
| $p^{2}=0$, | $p^{\circ}>0$ | $(1, o, o, 1)$ | $E_{2}$ | Yes |
| $p^{2}=0$, | $p^{\circ}<0$ | $(-1,0,0,1)$ | $E_{2}$ | No |
| $p^{\mu}=0$ |  | $(0, o)$ | $\operatorname{SL}(2, \mathbb{C})$ | Yes |
| $p^{2}=-m^{2}<0$ |  | $(0, o, o, m)$ | $S U(1,1)$ | No |

Table 7.1: Irreducible representations of the Poincaré group. $E_{2}$ refers to the Euclidean group of the 2-plane. $S L(2, \mathbb{C})$, as previously established, is the universal cover of the Lorentz group. $S U(1,1)$ is the group that leaves $\left|z_{1}\right|^{2}-\left|z_{2}\right|^{2}$ invariant.

Thus we have a single vacuum $|0\rangle$ with the property that

$$
\begin{equation*}
\hat{U}(a, \Lambda)|0\rangle=|0\rangle . \tag{7.267}
\end{equation*}
$$

## 7.B. 7 Massive Representations

For the case of massive particles one can always boost into a frame where

$$
\begin{equation*}
p^{\mu}=(m, \mathrm{o}, \mathrm{o}, \mathrm{o}) . \tag{7.268}
\end{equation*}
$$

We search for generators that leave $p^{\mu}=(m, o, o, o)$ invariant. This is given by the generators of the rotation group, $\mathrm{SO}(3)$. We say that $S O(3)$ is the stability group or the little group. This implies that we may use labels $j$ and $j^{3}$ as we did before. This sheds a little light on the nature of $W_{\mu}$. We notice that $W_{\circ}=o$ and $W_{i}=m J_{i}$. In the massive representation the Pauli-Lubanski vector does not contain any new information; $\omega$ is the same as, for example, $j_{3}$. We may label elements within an irrep as $\left|m, j ; p^{\mu}, j_{3}\right\rangle$. This is what we mean by a one-particle state, it is the definition of a elementary particle.

## 7.B. 8 Massless Representations

For massless particles we are unable to boost into a rest frame. The best we can do is boost into a frame where

$$
\begin{equation*}
p^{\mu}=(E, \mathrm{o}, \mathrm{o},-E) . \tag{7.269}
\end{equation*}
$$

Looking at this, we expect that the stability group is $S O(2)$. This is indeed correct, though a proper analysis is a lot trickier. Writing out each element of the Pauli-Lubanski vector, one finds

$$
W_{\circ}=E J_{3} \quad W_{1}=E\left(-J_{1}+K_{2}\right) \quad W_{2}=E\left(J_{2}-K_{1}\right) \quad W_{3}=E J_{3}, \quad \text { (7.270) }
$$

from which one can write down the commutation relations

$$
\left[W_{1}, W_{2}\right]=0 \quad\left[W_{3}, W_{1}\right]=-i E W_{2} \quad\left[W_{3}, W_{2}\right]=i E W_{1}
$$

This is the algebra for the two dimensional Euclidean group. Evidently the little group is more than just the $S O(2)$ group we originally expected. This group has infinite-dimensional representations and hence we find a continuum label for each of our massless states. Since we don't know of any massless particles with a continuum of states, we restrict to finite dimensional
representations by imposing

$$
W_{1}=W_{2}=0
$$

(7.272)

See [326-328] for recent explorations on the mathematical consistency of 'continuum spin representations.' The $W_{3}$ generates $O(2)$, as we wanted. Then

$$
W^{\mu}=\lambda P^{\mu}
$$

with $\lambda$ defining the helicity of the particle. Recalling that the algebra requires $e^{4 \pi i \lambda}|\lambda\rangle=|\lambda\rangle$, we know that $\lambda \in \pm \frac{1}{2}, 1 \cdots$. In fact, for a field theory with massless fields in the representation $(A, B)$, the helicity is given by $\lambda=B-A$. Massless particle states can thus be labelled as

$$
\begin{equation*}
\left|o, j ; p^{\mu}, \lambda\right\rangle \tag{7.274}
\end{equation*}
$$

## Dark Matter and Direct Detection

DARK MATTER IS ONE OF THE CHIEF MOTIVATIONS FOR NEW PHYSICS. It is nearly five times more abundant in the universe than the ordinary matter with which we are familiar. Despite having very weak interactions with ordinary matter, dark matter plays a key role in the early history of the universe. For example, clumps of dark matter seed the gravitational wells that became the galaxies. In this chapter we review the calculation of the dark matter abundance and describe how dark matter interactions with matter are constrained by direct detection experiments.

### 8.1 WIMP Relic Density

We assume that dark matter is a thermal WIMP, i.e. a species that was in thermal equilibrium before freezing out and leaving a relic density [329]. This means that freeze-out occurs when the wIMP species are nearly at rest. For an 'improved analysis' of the abundance of a stable particle that does not depend on the low relative velocities, see [330]. See also [331] for notable counter-examples. Recently others have begun to explore the possibilities for non-thermal relics through the 'freeze-in' of hidden sector species [143-146]; these are beyond the scope of this document.

The material in this section follows the textbooks [332,333]. For a summary of more recent developments see [331] and references therein.

### 8.1.1 The Boltzmann Equation

The Boltzmann equation connects the particle physics of dark matter to the cosmology of dark matter. We present need-to-know details of the Friedmann-Robertson-Walker cosmology in Appendix 8.A. For a general derivation of the Boltzmann equation, see Appendix 8.B. The main idea is that in the early universe, particles were in thermal equilibrium. This means that the production rate of particles from the thermal bath is equivalent to the annihilation rate, $\Gamma$. If we adiabatically lowered the temperature of a static universe below the DM mass, then the DM abundance would freeze out to a value that is thermally suppressed by $\exp (-m / T)$. However, we know that the universe is expanding at a rate given by the Hubble parameter $H$. Because of this, freeze-out occurs when the expansion rate overtakes the annihilation rate, $H \gg \Gamma$.

The Boltzmann equation quantifies this picture and can be written as

$$
\begin{equation*}
a^{-3} \frac{d\left(n a^{3}\right)}{d t}=\langle\sigma v\rangle\left[\left(n_{\mathrm{EQ}}\right)^{2}-n^{2}\right], \tag{8.1}
\end{equation*}
$$

where $a$ is the scale factor, $n$ is the dark matter number density, $n_{\mathrm{EQ}}$ is the equilibrium number density,

$$
n_{\mathrm{EQ}, i} \equiv g_{i} \int \frac{d^{3} p}{(2 \pi)^{3}} e^{-E_{i} / T}\left\{\begin{array}{l}
g_{i}\left(\frac{m_{i} T}{2 \pi}\right)^{3 / 2} e^{-m_{i} / T}  \tag{8.2}\\
g_{i} \frac{T^{3}}{\pi^{2}}
\end{array}\right.
$$

where $g_{i}$ is the number of degrees of freedom for the field. The general Boltzmann equation is derived in Appenxix 8.B; see also $[332,333]$. To simplify this, use the fact that $(a T)$ is independent of $t$ so that one can write $n a^{3}=n a^{3} T^{3} / T^{3}$ and pull a factor of $(a T)^{3}$ out of the time derivative. It is convenient to write these quantities in terms of dimensionless quantities

$$
\begin{equation*}
Y \equiv \frac{n}{T^{3}} \sim \frac{n}{s} \quad x=\frac{m}{T} . \tag{8.3}
\end{equation*}
$$

These quantities are useful not only because they're dimensionless, but because of their scaling properties. For example, the cubed temperature scales like $R^{-3}$ so that $\dot{s}+{ }_{3} \mathrm{Hs}=\mathrm{o}$. Compare this to the Boltzmann equation, which can be written as $\dot{n}+{ }_{3} H n=\langle\sigma v\rangle\left[\left(n_{\mathrm{EQ}}\right)^{2}-n^{2}\right]$. Using the variable $Y$ cancels the ${ }_{3} H$ term.

Let's now rewrite the Boltzmann equation in a few steps,

$$
\begin{equation*}
\frac{d Y}{d t}=T^{3}\langle\sigma v\rangle\left(Y_{\mathrm{EQ}}^{2}-Y^{2}\right), \tag{8.4}
\end{equation*}
$$

where $Y_{\mathrm{EQ}}=n_{\mathrm{EQ}} / T^{3}$. See (8.113) for the non-relativistic expression of $n_{\mathrm{EQ}}$. We can write the Boltzmann equation in different ways depending on how we define $Y$ [329]. For example, for $Y=n / s, n / s_{\gamma}$, or $n / n_{\gamma}$, we have

$$
\dot{Y}=\langle\sigma \nu\rangle\left(\begin{array}{c}
s  \tag{8.5}\\
s_{\gamma} \\
n_{\gamma}
\end{array}\right)\left(Y^{2}-Y_{\mathrm{EQ}}^{2}\right) .
$$

Note that there are some prefactors that come along with whether one chooses $Y=n / T^{3}$ or $Y=n / s$, the two most common conventions. The relevant conversion is

$$
\begin{equation*}
s=\frac{2 \pi^{2}}{45} g_{* s} T^{3} . \tag{8.6}
\end{equation*}
$$

For a rough derivation see the discussion before (8.128). Be sure to remember this conversion factor when comparing to other conventions, such as [333].

We now change variables from $t$ to $x$, for which we need $d x / d t=H x$. In particular, since dark matter production typically occurs in the radiation era where energy density scales like $T^{4}$, the Hubble parameter is $H=H(m) / x^{2}$ so that

$$
\begin{equation*}
\frac{d Y}{d x}=-\frac{\lambda}{x^{2}}\left(Y^{2}-Y_{\mathrm{EQ}}^{2}\right), \tag{8.7}
\end{equation*}
$$

where the parameter $\lambda$ relates the annihilation rate to the expansion rate of the universe,

$$
\begin{equation*}
\lambda=\frac{m^{3}\langle\sigma v\rangle}{H(m)} . \tag{8.8}
\end{equation*}
$$

For $s$-wave processes $\lambda$ is constant, but in principle one can have some temperature dependence in $\langle\sigma v\rangle$. In general, we should write $\langle\sigma v\rangle(x)$ and $\lambda(x)$.

For reference, recall the cosmological formulae

$$
\begin{array}{ll}
H(T)^{2}=\frac{8 \pi}{3} G \rho(T) \\
\rho_{R}(T)=\frac{\pi^{2}}{30} g_{*} T^{4}, \quad \text { (radiation dominated) } \tag{8.10}
\end{array}
$$

where $H=(\dot{a} / a)$ and $\rho_{R}$ is the energy density of relativistic species; see (8.100) and (8.118). Note that $8 \pi G=1 / M_{\mathrm{Pl}}^{2}$.
Before proceeding, let us discuss the qualitative solution to (8.7). While the annihilation rate $\Gamma \sim\langle\sigma v\rangle T^{3}$ is much greater than the expansion rate $H$, the 'number density' $Y$ remains in thermal equilibrium and tracks $Y_{\mathrm{EL}}$. This is because $\lambda$ is large and $Y$ wants to change to match $Y_{\mathrm{EQ}}$. However, $\lambda$ is decreasing. Eventually $\Gamma \approx H$ at some 'time' $x_{f}$. From that point on, $d Y / d x$ becomes small and $Y$ doesn't want to change. We're left with $Y(x) \approx Y\left(x_{f}\right)$ so that the number of particles per comoving volume has frozen out. For neutrinos this occurs while the species are still relativistic. For wimps, this occurs when the particles are already non-relativistic.

### 8.1.2 Solving the Boltzmann equation: $s$-wave

Unfortunately, (8.7) is a type of Riccati equation with no analytic solution. Despite not being exactly solvable, we can still see this through by invoking some physics intuition. We know that most of the action happens at $x \sim 1$. In this region, we can see that the left-hand side of $(8.7)$ is $\mathcal{O}(Y)$ while the right-hand side is $\mathcal{O}\left(\lambda Y^{2}\right)$. We will see shortly that $\lambda \gg 1$, so the right-hand side must have a cancellation in the $Y^{2}-Y_{\mathrm{EQ}^{2}}$ term.

After freeze out, $Y_{\mathrm{EQ}}$ will continue to decrease according to the thermal suppression $\exp (-m / T)$ so that $Y \gg Y_{\mathrm{EQ}}$. This happens at late times $x \gg 1$ where the Boltzmann equation reduces to

$$
\begin{equation*}
\frac{d Y}{d x} \approx-\frac{\lambda(x)}{x^{2}} Y^{2} . \tag{8.11}
\end{equation*}
$$

This is not yet solvable due to the $x$-dependence of $\lambda$ coming from the temperature dependence of $\langle\sigma v\rangle$.
Since we assume freeze-out occurs when wIMPs are non-relativistic, we may expand $\langle\sigma v\rangle=a+b v^{2}+\cdots$, where $a$ corresponds to an $s$-wave piece, $b$ corresponds to a $p$-wave (and some $s$-wave) piece, and so forth. For now, let us assume that the process is $s$-wave so that we may drop all powers of $\nu^{2}$ from the thermally-averaged cross section. In this case $\lambda(x)=\lambda$.

This is now a tractable differential equation which we can solve. The trick will be to match the solution in the asymptotic future to a good approximation at $x \sim 1$; i.e. we go from an intractable ODE (8.7) to a solvable ODE (8.11) at the cost of determining a boundary condition. The solution of $(8.11)$ is

$$
\begin{equation*}
\frac{1}{Y_{\infty}}-\frac{1}{Y_{f}}=\frac{\lambda}{x_{f}} \tag{8.12}
\end{equation*}
$$

where $Y_{\infty}$ is the asymptotic dimensionless number density and $Y_{f}$ is the value at the freeze out boundary condition $x_{f}$. Typically $Y_{f} \gg Y_{\infty}$ so that we may approximate this solution as

$$
\begin{equation*}
Y_{\infty} \approx \frac{x_{f}}{\lambda} . \tag{8.13}
\end{equation*}
$$

A simple order of magnitude estimate for this solution is $x_{f} \sim 10$; more precise values are on the order of $x_{f} \approx 20$ or 25 . At this level plugging in this value is a kludge. A more honest approximation comes from solving

$$
\begin{equation*}
n_{\mathrm{EQ}}\left(x_{f}\right)\langle\sigma v\rangle=H\left(x_{f}\right) . \tag{8.14}
\end{equation*}
$$

We give an even more explicit expression below. The plot for the dark matter relic density is well-known ${ }^{1}$. The qualitative features are as follows:

[^11]- $Y$ tracks its equilibrium value $Y_{\mathrm{EQ}}$ until $x \sim 10$, and then levels off to a frozen-out constant.
- As one increases the annihilation cross section, the freeze out time is later.
- The distinction between Bose and Fermi statistics is negligible by the time the dark matter species freezes out. (The use of Boltzmann statistics was assumed in when we wrote the Boltzmann equation.)


### 8.1.3 Solving the Boltzmann equation: general

Let's consider to a more general solution to the Boltzmann equation that extends our $s$-wave analysis above. The general conclusions are the same, so we'll focus on some technical details. We will follow Scherrer and Turner [329]. Useful note: that paper uses $Y=n / s$, which differs from our definition of $Y$ by the overall conversion factor in (8.6).

Suppose that in a velocity expansion, the leading order term in the thermally averaged cross section goes like the $p$-th power of $v$,

$$
\begin{equation*}
\langle\sigma v\rangle \propto v^{p} . \tag{8.15}
\end{equation*}
$$

For $s$-wave $p=0$, while for $p$-wave $p=2$, and so forth. From the Boltzmann velocity distribution, we know that $\langle v\rangle \sim \sqrt{T}$ so that we may write

$$
\begin{equation*}
\langle\sigma v\rangle \propto x^{-n} \tag{8.16}
\end{equation*}
$$

where $n=p / 2$. If we write $\langle\sigma v\rangle=\langle\sigma v\rangle_{0} x^{-n}$, then we may define

$$
\begin{equation*}
\lambda_{\circ}=\frac{m^{3}\langle\sigma v\rangle_{\circ}}{H(m)}=\lambda x^{n} . \tag{8.17}
\end{equation*}
$$

In this way $\lambda_{0}$ is independent of $x$. In this way we may pull out the $x$-dependence from $\lambda$ in (8.7),

$$
\begin{equation*}
\frac{d Y}{d x}=-\frac{\lambda_{0}}{x^{2+n}}\left(Y^{2}-Y_{\mathrm{EQ}}^{2}\right) \tag{8.18}
\end{equation*}
$$

We will rewrite this in terms of $\Delta \equiv Y-Y_{\mathrm{EQ}}$ :

$$
\begin{equation*}
\frac{d \Delta}{d x}=-\frac{d Y_{\mathrm{EQ}}}{d x}-\frac{\lambda_{0}}{x^{2+n}} \Delta\left(2 Y_{\mathrm{EQ}}+\Delta\right) . \tag{8.19}
\end{equation*}
$$

Here we've just used $Y^{2}-Y_{\mathrm{EQ}}^{2}=\left(Y+Y_{\mathrm{EQ}}\right)\left(Y-Y_{\mathrm{EQ}}\right)$.
First consider the case where $x$ is small; say $1<x \ll x_{f}$. We'll give a more precise definition of $x_{f}$ below. In this limit, we know that $Y$ is very close to $Y_{\mathrm{EQ}}$ so that $\Delta \ll Y_{\mathrm{EQ}}$ and $\left|\Delta^{\prime}\right| \ll-Y_{\mathrm{EQ}}^{\prime}$, where we've written a prime to mean $d / d x$. In this regime we can algebraically solve (8.19):

$$
\begin{equation*}
\Delta=-\frac{d Y_{\mathrm{EQ}}}{d x} \frac{x^{2+n}}{\lambda_{0}\left(2 Y_{\mathrm{EQ}}+\Delta\right)}=\left(1-\frac{3}{2 x}\right) \frac{x^{2+n}}{\lambda_{0}\left(2+\Delta / Y_{\mathrm{EQ}}\right)} \approx \frac{x^{2+n}}{2 \lambda_{0}} . \tag{8.20}
\end{equation*}
$$

Here we have used $Y_{\mathrm{EQ}}=n_{\mathrm{EQ}} / T^{3}$ and (8.113), i.e.

$$
\begin{equation*}
Y_{\mathrm{EQ}}=\frac{g}{(2 \pi)^{3 / 2}} x^{3 / 2} e^{-x} \equiv a x^{3 / 2} e^{-x} . \tag{8.21}
\end{equation*}
$$

Now consider what happens when $x \gg x_{f}$. In this regime we know that $Y_{\mathrm{EQ}}$ is exponentially small compared to $\Delta \approx Y \gg Y_{\mathrm{EQ}}$. We can thus drop $Y_{\mathrm{EQ}}$ and $Y_{\mathrm{EQ}}^{\prime}$ in (8.19) to obtain

$$
\begin{equation*}
\frac{d \Delta}{d x}=-\frac{\lambda}{x^{2+n}} \Delta^{2} . \tag{8.22}
\end{equation*}
$$

Physically, particle creation has practically halted while annihilations are still somewhat important, leading to a slight
reduction to $Y$ compared to the value of $Y_{\mathrm{EQ}}$ at $\Gamma=H$ (the natural back-of-the-envelope rough estimate the for the relic abundance). Integrating this approximation from $x_{f}$ —which we nebulously take to be the lower-limit of the valid range for this approximation-to $x=\infty$ gives

$$
\begin{equation*}
Y_{\infty}=\frac{(n+1)}{\lambda_{0}} x_{f}^{n+1} . \tag{8.23}
\end{equation*}
$$

The importance of this quantity is that $Y$ (today) $\approx Y_{\infty}$, i.e. this is what we plug into $\rho_{\chi}$ and $\Omega_{\chi}$ to check if we've obtained the correct (observed) dark matter relic density. As mentioned in the simplified $s$-wave case, we've obtained this by resorting to an approximation in the $x \gg x_{f}$ case. The cost is that we've introduced a boundary condition at $x_{f}$, the freeze out 'time,' where we must match our approximation.

Now let's precisely define $x_{f}$. We are interested in the regime where $\Delta \approx Y_{\mathrm{EQ}}$. We define freeze-out precisely by the condition

$$
\begin{equation*}
\Delta\left(x_{f}\right)=c Y_{\mathrm{EQ}}\left(x_{f}\right), \tag{8.24}
\end{equation*}
$$

where $c=\mathcal{O}(1)$ and is determined empirically. We will plug into (8.19). We shall take two limits: first we will assume that $d \Delta / d x \ll 1$ and further that the particle is non-relativistic at freeze-out, in particular $x \gg 3 / 2$. The $3 / 2$ comes from $d Y_{\mathrm{E}} / d x=a(3 / 2-x) x^{1 / 2} e^{-x}$. Plugging in and solving gives,

$$
\begin{align*}
e^{x_{f}} & \approx \frac{a \lambda_{0} c(2+c)}{x_{f}^{n+1 / 2}}  \tag{8.25}\\
x_{f} & \approx \ln \left[a \lambda_{0} c(2+c)\right]-(n+1 / 2) \ln \ln \left[(2+c) \lambda_{0} a c\right] \tag{8.26}
\end{align*}
$$

Here we've further used the limit $x_{f} \gg 3 / 2$, as appropriate for a particle which is non-relativistic at freeze out. In (8.26) we now have a detailed expression for $x_{f}$ which we may take as a definition.

One must still pick a value for $c$. It turns out that the best fit to numerical results sets

$$
\begin{equation*}
c(c+2)=n+1 \tag{8.27}
\end{equation*}
$$

which is better than $5 \%$ for any $x_{f} \gtrsim 3$. Plugging in (8.17), (8.10), (8.9), and $M_{\mathrm{Pl}}^{2}=8 \pi G$, we obtain:

$$
\begin{equation*}
a \lambda_{\circ}=\frac{g}{(2 \pi)^{3 / 2}} \cdot \frac{m^{3}}{H(m)}\langle\sigma v\rangle_{\circ}=\frac{g}{(2 \pi)^{3 / 2}} \cdot m^{3} \frac{1}{m^{2}} \frac{1}{\sqrt{8 \pi G}} \sqrt{\frac{90}{\pi^{2} g_{*}}}\langle\sigma v\rangle_{0}=\underbrace{\sqrt{\frac{45}{4 \pi^{5}}}}_{\approx 0.19} \frac{g}{\sqrt{g_{*}}} \frac{m}{\sqrt{8 \pi G}}\langle\sigma v\rangle_{0} . \tag{8.28}
\end{equation*}
$$

Putting it all together,

$$
\begin{equation*}
x_{f} \approx \ln \left[\sqrt{\frac{45}{4 \pi^{5}}} \frac{g}{\sqrt{g_{*}}} \frac{m}{\sqrt{8 \pi G}}\langle\sigma v\rangle_{0}\right]-\left(n+\frac{1}{2}\right) \ln ^{2}[\cdots], \tag{8.29}
\end{equation*}
$$

where the second bracket contains the same junk as the first bracket. Note that the corrections to $x_{f} \approx 20$ (for $s$-wave) are only logarithmic. Note that we can write $m / \sqrt{8 \pi G}=m M_{\mathrm{Pl}}$ where $M_{\mathrm{Pl}}$ is the reduced Planck mass, $M_{\mathrm{Pl}}=2.44 \times 10^{18} \mathrm{Gev}$.

### 8.1.4 Abundance

Once a particle has frozen out, its number density falls off according to the scale factor, $a^{-3}$. Thus the [mass] density today is $m\left(a_{1} / a_{\circ}\right)^{3} n$, where $a_{1}$ is assumed to be at a sufficiently late time that $Y \approx Y_{\infty}$. Recall that the number density at this late time is $n=Y_{\infty} T_{1}^{3}$. Thus the mass density today is

$$
\begin{equation*}
\rho=m Y_{\infty} T_{\circ}^{3}\left(\frac{a_{1} T_{1}}{a_{\circ} T_{\circ}}\right)^{3} \approx \frac{m Y_{\infty} T_{\circ}^{3}}{30} \tag{8.30}
\end{equation*}
$$

This last equality is exercise 11 of Dodelson's text (the solution is in the back); the point is that $a T$ is not constant due to the reheating of photons from the annihilation of particles between 1 Mev and 100 Gev . Note that we've gone back to our normalization $Y=n / T^{3}$.

The relevant number to match is the fraction of the present-day critical density coming from $\chi$, using (8.8):

$$
\begin{equation*}
\Omega_{\chi}=\frac{x_{f}}{\lambda} \frac{m T_{o}^{3}}{30 \rho_{\text {crit }}}=\frac{H(m) x_{f} T_{o}^{3}}{30 m^{2}\langle\sigma v\rangle \rho_{\text {crit }}} . \tag{8.31}
\end{equation*}
$$

Recall that $\rho_{\text {crit }}={ }_{3} H_{0}^{2} / 8 \pi G$. Using (8.9) and (8.10), the Hubble rate at $T=m$, which we assume to be during the radiation era, is

$$
\begin{equation*}
H(T)=T^{2} \sqrt{\frac{4 \pi^{3} G g_{*}(T)}{45}}, \tag{8.32}
\end{equation*}
$$

where $g_{*}(T)$ is the effective number of degrees of freedom at temperature $T$. Plugging $H(m)$ into the expression for $\Omega_{\chi}$ shows that the latter quantity does not depend on the dark matter mass $m$ except through the implicit dependence in $x_{f}$ and $g_{*}$. This provides an important lesson: the relic abundance is primarily controlled by the cross section, $\langle\sigma v\rangle$.

The final expression is

$$
\begin{equation*}
\Omega_{\chi}=\sqrt{\frac{4 \pi G g_{*}(m) \pi^{3}}{45}} \frac{x_{f} T_{\circ}^{3}}{30\langle\sigma v\rangle \rho_{\text {cr }}}=0.3 h^{-2}\left(\frac{x_{f}}{10}\right)\left(\frac{g_{*}(m)}{100}\right)^{1 / 2} \frac{10^{-39} \mathrm{~cm}^{2}}{\langle\sigma v\rangle} . \tag{8.33}
\end{equation*}
$$

Assuming that $\chi$ makes up all of the dark matter, the correct density requires $\Omega_{\chi}=0.3$. The $10^{-39} \mathrm{~cm}^{2}$ cross section, which is right around what one would expect from a weakly interacting 100 -ish Gev particle, is the "WIMP miracle."

### 8.1.5 Polemics: wimp agnosticism

The wIMP miracle is often presented as strong evidence for new terascale physics connected to electroweak symmetry breaking. However, this should be taken with a grain of salt. First the statement of the wimp miracle is valid only at the "within a few orders of magnitude" level. Note that a typical weak cross section is $\langle\sigma v\rangle \sim \mathrm{pb}=10^{-36} \mathrm{~cm}^{2}$, so that some amount of tuning is required in the wimp coupling.

A more sobering restriction comes from a tension between the correct relic abundance and recent direct detection bounds. As of the writing of this paragraph, XENON 100 has set an upper limit on the spin-independent elastic wIMP-nucleon cross section on the order of $\sigma_{\mathrm{SI}}=7.0 \times 10^{-45} \mathrm{~cm}^{2}=7.0 \times 10^{-9} \mathrm{pb}$ for a 50 GeV WIMP at $90 \%$ confidence. A very naive assumption is that the annihilation cross section should be roughly of the same order as the direct detection cross section, and so there appears to be significant tuning required to generate a difference on the order of several orders of magnitude between the two processes.

As a case study, consider the plight of the MSSM. The prototypical MSSM wIMP is a neutralino (the LSP) whose abundance is protected by $R$-parity. A standard approach is to consider parameters in which the direct detection bounds are satisfied and then attempt to boost the relic density using handy tricks (i.e. tuning). For example, for a pure bino LSP one could set up coannihilations due to an accidental slepton degeneracy or resonant annihilations (e.g. a Higgs resonance). Alternately, one may note that Higgsinos and winos have annihilation cross sections that are typically too large allows one to tune the LSP to be a specific combination of bino, Higgsino, and wino to generate the correct abundance. The parameter space for the latter 'well-tempered neutralino' scenario, however, is now strongly constrained by XENON 100 .

There remain ways to generate honest-to-goodness wIMPs in models of new physics, but these appear to be rather special cases in extended models rather than generic phenomena.

For completeness, we offer a counterpoint: even though there appears to be a $10^{\text {few }}$ tuning required, one may argue that there is still a 'miracle' because of the orders of magnitude that have to cancel. People point out the ( $\left.T_{\circ} / M_{P 1}\right)^{3}$ factor in the explicit formulae above. Of course, the point is that the smallness of $\left(T_{\mathrm{o}} / M_{\mathrm{PI}}\right)^{3}$ is balanced by the smallness (weak scale cross section) of $\langle\sigma v\rangle$. In this sense it's a coincidence between the Weak scale, the Planck scale, and the cmb scale. And note, very importantly, that it is independent of the wIMP mass up to logarithmic corrections. Is this a miracle?

### 8.1.6 THERMALLY AVERAGED CROSS SECTION \& IDENTICAL PARTICLES

Note that $v$ is the relative velocity, so that each of the initial state $\chi$ particles in the annihilation process has velocity $v / 2 .\langle\sigma v\rangle$ is defined by

$$
\begin{equation*}
\langle\sigma v\rangle=\frac{1}{\overline{n_{1} \overline{n_{2}}}}\left(\prod_{i=1}^{4} \int \frac{d^{3} p_{i}}{2 E_{i}}\right) e^{-\left(E_{1}+E_{2}\right) / T}(2 \pi)^{4} \delta^{(4)}\left(p_{1}+p_{2}-p_{3}-p_{4}\right)|\mathcal{M}|^{2} . \tag{8.34}
\end{equation*}
$$

Compared to the usual definition of $\sigma$ in Peskin \& Schröeder, (??), the thermal average includes an integral over the initial state momenta weighted by the Maxwell-Boltzmann contribution.

Practically, we don't need to do the thermal average over and over again for each cross section. Instead, we expand in powers of $v^{2}$ and insert the moments of the Maxwell-Bolztman velocity distribution. Typically one only needs the first or second term to get the relevant behavior. Thus we would like to find

$$
\begin{equation*}
\sigma v=a+b v^{2}+\cdots . \tag{8.35}
\end{equation*}
$$

The thermal average gives $\left\langle v^{2}\right\rangle=6 / x_{f}$, for example. Note that the overall prefactor $1 /\left|v_{a}-v_{b}\right|=1 / v$ in the expression for $d \sigma$ cancels in $\sigma v$.

The annihilation cross section is given by

$$
\begin{equation*}
d \sigma=\frac{1}{2 E_{a} 2 E_{b}\left|v_{a}-v_{b}\right|}\left(\prod_{f} \frac{d^{3} p_{f}}{(2 \pi)^{3}} \frac{1}{2 E_{f}}\right)(2 \pi)^{4} \delta^{(4)}\left(p_{a}^{\mu}+p_{b}^{\mu}-\sum_{f} p_{f}^{\mu}\right)|\mathcal{M}|_{\text {s.a. }}^{2}, \tag{8.36}
\end{equation*}
$$

where $|\mathcal{M}|^{2}$ should be understood to mean the spin averaged squared amplitude. The two-body phase space is,

$$
\begin{equation*}
d \mathrm{PS}_{2}\left(p_{1}, p_{2}\right)=\left(\prod_{f} \frac{d^{3} p_{f}}{(2 \pi)^{3}} \frac{1}{2 E_{f}}\right)(2 \pi)^{4} \delta^{(4)}\left(p_{a}^{\mu}+p_{b}^{\mu}-\sum_{f} p_{f}^{\mu}\right)=\frac{d \Omega_{\mathrm{CM}}}{4 \pi} \frac{1}{8 \pi}\left(\frac{2\left|\mathbf{p}_{\mathbf{p}}\right|}{E_{\mathrm{CM}}}\right) \tag{8.37}
\end{equation*}
$$

Here 1 and 2 label final state particles.
At this stage there are model-dependent factors of two which become important. Focusing on the case of $2 \rightarrow 2$ annihilations, we are concerned about symmetry factors which pop up for identical initial states (e.g. Majorana fermion dark matter) and identical final states.

First consider the initial states. Suppose the two initial state dark matter particles are identical. There is no additional factor of two coming from identical initial states. Here's a paragraph from Dreiner, Haber, and Martin [24]:

Recall the standard procedure for the calculation of decay rates and cross-sections in field theory-average over unobserved degrees of freedom of the initial state and sum over the unobserved degrees of freedom of the final state. This mantra is well-known for dealing with spin and color degrees of freedom, but it is also applicable to degrees of freedom associated with global internal symmetries. Thus, the cross-section for the annihilation of a Dirac fermion pair into a neutral scalar boson can be obtained by computing the average of the cross-sections for $\xi_{1}\left(\mathbf{p}_{1}, s_{1}\right) \xi_{2}\left(\mathbf{p}_{2}, s_{2}\right) \rightarrow \varphi$ and $\xi_{2}\left(\mathbf{p}_{1}, s_{1}\right) \xi_{2}\left(\mathbf{p}_{2}, s_{2}\right) \rightarrow \varphi$. [Here $\xi$ is an uncharged, massive, $(1 / 2,0)$ fermion.] Since the annihilation cross-sections for $\xi_{1} \xi_{1}$ and $\xi_{2} \xi_{2}$ are equal, we confirm the resulting annihilation cross-section for the Dirac fermion pair obtained above in the $\chi-\eta$ basis. [Here $\left.\Psi_{D}=\left(\chi, \eta^{\dagger}\right)^{T}\right]$.

Thus there are no additional factors in the thermally averaged annihilation cross section $\langle\sigma v\rangle$ coming from having identical Majorana dark matter particles. It is trivial that the above argument carries over to the case of where the particles have arbitrary spin.

Now consider the final state particles. If there are $k$ identical final state particles, then we expect an additional factor of $1 / k!$, which can be understood precisely as above: the phase space integral over-counts final state configurations. For $2 \rightarrow 2$ processes this is a factor of $1 / 2$ ! which we will write out as $1 / k$ ! in the remainder of this section as a reminder.

Putting this all together, we have:

$$
\begin{align*}
d \sigma & =\frac{1}{k!} \frac{1}{4 E^{2} v}\left(\prod_{f} \frac{d^{3} p_{f}}{(2 \pi)^{3}} \frac{1}{2 E_{f}}\right)(2 \pi)^{4} \delta^{(4)}\left(p_{a}^{\mu}+p_{b}^{\mu}-\sum_{f} p_{f}^{\mu}\right) \frac{1}{4} \sum_{\text {spins }}|\mathcal{M}|^{2}  \tag{8.38}\\
& =\frac{1}{k!} \frac{1}{4 E^{2} v} \frac{d \Omega_{\mathrm{CM}}}{4 \pi} \frac{1}{8 \pi}\left(\frac{2\left|\mathbf{p}_{1}\right|}{E_{\mathrm{CM}}}\right) \frac{1}{4} \sum_{\text {spins }}|\mathcal{M}|^{2} . \tag{8.39}
\end{align*}
$$

We have written $2 E_{a} 2 E_{b}\left|v_{a}-v_{b}\right|=4 E^{2} v$. Now note that

$$
\begin{equation*}
\frac{2\left|\mathbf{p}_{1}\right|}{E_{\mathrm{CM}}}=\frac{\left|\mathbf{p}_{1}\right|}{E}=v_{1} \tag{8.40}
\end{equation*}
$$

where $v_{1}$ is the velocity of one of the final state axions. This is not integrated over (we've already done the final state phase space integrals) and must be converted into the initial state relative velocity $v$ using conservation of $E_{i}^{2}=m_{i}^{2}+p_{i}^{2}$ and $v_{i}=p_{i} / E$,

$$
\begin{equation*}
m_{\chi}^{2}+v_{\chi}^{2} E^{2}=m_{a}^{2}+v_{1}^{2} E^{2} \tag{8.41}
\end{equation*}
$$

Recalling that $v=2 v_{\chi}$, we find

$$
\begin{equation*}
v_{1}^{2}=\frac{v^{2}}{4}+\frac{m_{\chi}^{2}-m_{a}^{2}}{E^{2}} \tag{8.42}
\end{equation*}
$$

Plugging this back in to $d \sigma$,

$$
\begin{align*}
d \sigma & =\frac{1}{k!} \frac{1}{4 E^{2} v} \frac{d \Omega_{\mathrm{CM}}}{4 \pi} \frac{1}{8 \pi} \sqrt{\frac{v^{2}}{4}+\frac{m_{\chi}^{2}-m_{a}^{2}}{E^{2}}} \frac{1}{4} \sum_{\text {spins }}|\mathcal{M}|^{2}  \tag{8.43}\\
v d \sigma & =\frac{1}{k!} \frac{d \cos \theta}{(2 E)^{2}} \frac{1}{16 \pi} \frac{1}{4} \sum_{\text {spins }}|\mathcal{M}|^{2}(1+\cdots), \tag{8.44}
\end{align*}
$$

where the expansion of the square root drops terms of order $\mathcal{O}(v)$ and $\mathcal{O}\left(m_{a} / m_{\chi}\right)$ since $E \approx m_{\chi}$. To be precise, $E=\gamma m_{\chi}$ where $\gamma$ is the Lorentz factor $\left(1-v_{\chi}^{2}\right)^{-1 / 2}$. The expansion of the square root seems to give a higher order correction proportional to $v$. So it looks a term in $\sigma v$ that goes like $v^{3}$, i.e. the expansion in relative velocity includes odd powers of $v$. What is important to note is that this term comes from the factor of $v_{1}$ in (8.40). In the case when $v_{1} \approx 1$ is not a good approximation-that is, when the final state mass is appreciable-one has to treat this carefully since this Taylor expansion of $v_{1}$ in $v$ (or $v^{2}$ if you prefer even powers) breaks down and gives artificial divergences. This is an important factor to carry when considering annihilation modes via thermally-accessed channels that are otherwise kinematically forbidden. See [331].

Simplifying a bit more, we have a leading order contribution of

$$
\begin{equation*}
v d \sigma=\frac{1}{k!} \frac{d \cos \theta}{64 \pi} \frac{1}{s} \sum_{\text {spins }}|\mathcal{M}|^{2} \tag{8.45}
\end{equation*}
$$

Recall that $k$ ! encodes the symmetry of the final states: $k=1$ for non-identical final states, and $k=2$ for two identical final state particles. One can perform the $d \cos \theta$ integral and expand in powers of $v$ to obtain the coefficients in (8.35). From taking the first moment of the Boltzmann distribution, we can plug in those coefficients to obtain

$$
\begin{equation*}
\left\langle\sigma_{\mathrm{ann} .} v\right\rangle=a+6 \frac{b}{x_{f}}+\cdots \tag{8.46}
\end{equation*}
$$

### 8.1.7 A REMARK ON a factor of $1 / 2$

As a remark, the factor of $1 / 2$ discussed above regarding identical particles should not be confused with the factor of $1 / 2$ which appears when calculating indirect detection rates, which comes from the number densities in the flux,

$$
\begin{equation*}
\frac{d^{2} N}{d A d T} \sim \int d \ell n_{1} n_{2}\langle\sigma v\rangle . \tag{8.47}
\end{equation*}
$$

This is explained by Dreiner et al. as follows,
We assume that the number density of Dirac fermions and antifermions and the corresponding number density of Majorana fermions are all the same (and denoted by $n$ ). Above, we showed that $\sigma$ is the same for the annihilation of a single species of Majorana and Dirac fermions. For the Dirac case, $n_{1} n_{2}=n^{2}$. For the Majorana case, because the Majorana fermions are identical particles, given $N$ initial state fermions in a volume $V$, there are $N(N-1) / 2$ possible scatterings. In the thermodynamic limit where $N, V \rightarrow \infty$ at fixed $n \equiv N / V$, we conclude that $n_{1} n_{2}=n^{2} / 2$ for a single species of annihilating Majorana fermions. Hence the event rate of a Dirac fermion-antifermion pair is double that of a single species of Majorana fermions.
The factor of $1 / 2$ is explained in [334] and is consistent with the interpretation of a Dirac fermion as a pair of mass-degenerate Majorana fermions. Alternately,

The extra factor of $1 / 2$ can also be understood by noting that in the case of annihilating dark matter particles, all possible scattering axes occur and are implicitly integrated over. But, integrating over $4 \pi$ steradians double counts the annihilation of identical particles, hence one must include a factor of $1 / 2$ by replacing $n_{1} n_{2}=n^{2}$ by $n^{2} / 2$.

This interpretation for the factor of $1 / 2$ in indirect detection (which is not relevant for the relic abundance calculation with which we are presently concerned) carries over to the degeneracy of the final states in the annihilation cross section.

### 8.2 DIRECT DETECTION

After the above long-winded historical introduction, we now discuss general features of direct dark matter detection. Direct detection first demonstrated by Goodman and Witten (yes, that Witten) at around the time when the author was born [335]. As explained in the introduction, we study the scattering of halo dark matter particles off of highly-shielded targets to determine information about their interactions (cross sections) and kinematics (mass). Because dark matter is so weakly interacting with the Standard Model such experiments require large detector volumes, as is the case with neutrino experiments. Unlike neutrino experiments, however, dark matter is heavy and the detection methods are rather different. While neutrinos may zip through a liquid detector relativistically and leave easy-to-detect Čerenkov radiation, wimps lumber along like giant elephants that will absent-mindedly bump into target nuclei. One can intuitively appreciate that the two scenarios very different kinematics that require separate detection techniques.

The canonical review of the calculation of dark matter direct detection constraints is reviewed exceptionally well by Lewin and Smith [336]. We shall review these results following the pedagogical discussion in [337]. Additional comments and applications to the CDMS detector are presented in chapter 2 of [338]. The key result will be to understand the structure of dark matter exclusion plots. We will also briefly survey and classify the experimental techniques used in the range of direct detection experiments to help place our specific study of XENON 100 into proper context.

### 8.2.1 General strategy

A garden-variety neutralino-like wIMP interacts with a target material primarily through elastic collisions with the target nuclei. Experiments can then use complementary detection techniques to detect and distinguish such interactions from background events to compare to theoretical predictions. These theoretical predictions can be parameterized by the dark matter mass and a single effective coupling for typical wIMPs or up to four effective couplings for more general dark matter
models depending on, e.g., spin coupling. The primary quantity to connect experimental data to theoretical models is the elastic nuclear recoil spectrum, $d R / d E_{R}$, where $R$ is the recoil event rate and $E_{R}$ is the energy of the recoiling nucleus.

We will start by assembling some pieces required to construct the recoil spectrum: the astrophysical input data about the wIMP velocity distribution and the effective ('phenomenological') cross section. Since we will see that most events occur with low recoil energy, it will be advantageous to further parameterize the cross section in terms of a zero momentum transfer part and a form factor that encodes the momentum and target dependence. In doing so we will uncover important general features that feed into the design of direct detection experiments.

### 8.2.2 Astrophysical input

Our primary astrophysical assumption is that the dark matter in the halo has a 'sufficiently' Maxwellian velocity distribution. The Maxwell-Boltzmann distribution describes the velocities of particles which move freely up to short collisions and is derived in one's favorite statistical physics textbook. Here one assumes that the wIMPs are isothermal and isotropically distributed in phase space (i.e. gravitationally relaxed). It is important to remark that this is not actually fully accurate and thus that wIMPs cannot have an exactly Maxwellian distribution even though such an approximation should be sufficient (i.e. with uncertainties smaller than those coming from the wimp-nucleus cross section) for garden-variety wimp models. For a recent discussion of the implications of the expected departures from the Maxwell distribution at the large velocity tail and the kinds of models that would be affected by this, see [339].

The complete phase space distribution for such a halo for a dark matter species of mass $m_{\chi}$, gravitational potential $\Phi(\mathbf{x})$, and velocity in the galaxy frame $\mathbf{v}_{\text {gal }}$ is

$$
\begin{equation*}
f(\mathbf{x}, \mathbf{v}) d^{3} x d^{3} v \propto \exp \left(-\frac{m_{\chi}\left[v^{2} / 2+\Phi(\mathbf{x})\right]}{k_{B} T}\right) . \tag{8.48}
\end{equation*}
$$

The Earth is effectively at a fixed point in the gravitational potential so that the position dependence is is also fixed and can be absorbed into the overall normalization. We may thus write

$$
\begin{equation*}
f\left(v_{\mathrm{gal}}\right)=\frac{1}{k_{\mathrm{o}}} e^{v_{\mathrm{gal}}^{2} / v_{\mathrm{o}}^{2}} \tag{8.49}
\end{equation*}
$$

where $k$ is a factor to normalize the distribution

$$
\begin{equation*}
k_{0}=\int d^{3} \mathbf{v}_{\mathrm{gal}}{ }^{v_{\mathrm{gal}}^{2} / v_{0}^{2}}=\left(\pi v_{0}^{2}\right)^{3 / 2} \tag{8.50}
\end{equation*}
$$

and $v_{0}$ is the most probable wIMP speed and is given by the characteristic kinetic energy:

$$
\begin{equation*}
\frac{1}{2} m_{\chi} v_{\circ}^{2}=k_{B} T \quad v_{0} \approx 220 \mathrm{~km} / \mathrm{s} \approx 0.75 \cdot 10^{-3} c . \tag{8.51}
\end{equation*}
$$

Note that in (8.50) we have not defined the region of integration in velocity space, we will discuss this shortly. For now one can assume that we are integrating over the entire space. It is typically to write the $\mathbf{v}_{\text {gal }}$ explicitly in terms of the velocity in the Earth (lab) frame, $\mathbf{v}$, and the velocity of this frame relative to the dark matter halo, $\mathbf{v}_{E}$,

$$
\begin{equation*}
\mathbf{v}_{\mathrm{gal}}=\mathbf{v}+\mathbf{v}_{\mathrm{E}} . \tag{8.52}
\end{equation*}
$$

The orbit of the Earth about the sun in the galactic halo frame provides the input for an annual modulation:

$$
\begin{equation*}
v_{E}=232+15 \cos \left(2 \pi \frac{t-152.5 \text { days }}{365.25 \text { days }}\right) \mathrm{km} \mathrm{~s}^{-1} . \tag{8.53}
\end{equation*}
$$

All astrophysical data in this section come from [338]. Further discussion this data can be found in, e.g., [340, 341].
A key observation on the right-hand side of $(8.51)$ is that the dark matter particle is very non-relativistic (we include an explicitly factor of $c=1$ ). This will have important implications on our wimp-nucleon cross section.

Let us remark once again that for the remainder of this document (except for isolated remarks), we will assume this
astrophysical input. While we have mentioned in Section ?? that there are many new phenomenological dark matter models that can deviate from these assumptions, we will not consider them in our primary analysis.

### 8.2.3 Phenomenological cross section

Given a matrix element $\mathcal{M}(q)$ for the scattering of wIMPs of lab frame velocity $\mathbf{v}$ against target nuclei with characteristic momentum transfer $q$, we may use Fermi's Golden Rule to determine the differential wimp-nucleus cross section,

$$
\begin{equation*}
\frac{d \sigma_{N}(q)}{d q^{2}}=\frac{1}{\pi v^{2}}|\mathcal{M}|^{2}=\hat{\sigma}_{N} \cdot \frac{F^{2}(q)}{4 m_{r}^{2} v^{2}} \tag{8.54}
\end{equation*}
$$

The $\left(\pi v^{2}\right)^{-1}$ factor comes from the density of final states and the usual $2 \pi$ in the Golden Rule formula. In the last equality we've written the cross section in terms of a $q$-independent factor $\hat{\sigma}_{N}=\sigma_{N}(q=0)$ and fit all of the momentum dependence into the remaining form factor, $F(q)$. We have written $m_{r}$ for the reduced mass of the wimp-nucleus system,

$$
\begin{equation*}
m_{r}=\frac{m_{\chi} m_{N}}{m_{\chi}+m_{N}} . \tag{8.55}
\end{equation*}
$$

For a general interaction Lagrangian between wIMPs and nucleons, one can show that the $q=o$ cross section can be parameterized by four effective couplings $f_{p, n}$ and $a_{p, n}$ (subscripts refer to proton and neutron couplings) according to

$$
\begin{equation*}
\hat{\sigma}_{N}=\frac{4 m_{r}^{2}}{\pi}\left[Z f_{p}+(A-Z) f_{n}\right]^{2}+\frac{32 G_{F}^{2} m_{r}^{2}}{\pi} \frac{(J+1)}{J}\left[a_{p}\left\langle S_{p}\right\rangle+a_{n}\left\langle S_{n}\right\rangle\right] \tag{8.56}
\end{equation*}
$$

where $J$ is the nuclear spin, $Z(A)$ is the atomic (mass) number, and $S_{p, n}$ are the spin content of the proton and neutron [342]. There is an implied sum over nucleons, $p$ and $n$. We have separated the zero momentum transfer cross section into spin independent (SI) and spin-dependent (SD) pieces. The relevant point is that this is still a general formula for the effective, zero momentum transfer cross section.

Now one must consider the coherence effect coming from summing over nucleons. Nuclear physicists knew all about coherence effects in atomic interactions... but they're all old and wrinkly now. In this day and age, we have to invoke highfalutin ideas like decoupling: as good effective field theorists, we know that the nuclear scale is 'macroscopic' relative to the dark matter scale. We thus have to ask if it it is appropriate to sum the quantum mechanically over the amplitudes coming from each target nucleon. This is a question of energy dependence since higher energies probe smaller scales. We already know from our discussion of the wIMP velocity distribution that wIMPs are very non-relativistic in the lab frame so that they have a large de Broglie wavelength that indeed probes the entire target nucleus.

We harp upon this because this already provides a dramatic simplification. It is not surprising that an electrically neutral dark matter particle should couple in (roughly) the same way to the proton and neutron since these are related by isospin. Thus we may take $f_{p}=f_{n} \equiv f$ and note that the first term in (8.56) takes the form

$$
\begin{equation*}
\left.\hat{\sigma}_{N}\right|_{\mathrm{SI}} \approx \frac{4 m_{r}^{2}}{\pi} f^{2} A^{2} \tag{8.57}
\end{equation*}
$$

i.e. the spin-independent cross section is enhanced by a factor of $A^{2}$ due to coherence. Further, since spins form anti-parallel pairs in ground state nuclei, most of the spin-dependent cross section cancels and only leaves a leftover coupling to an odd number of protons or neutrons in the nucleus. Thus for our garden-variety wIMP interacting with a garden-variety (e.g. Ge) target with low spin, we can completely neglect the spin-dependence,

$$
\begin{equation*}
\left.\hat{\sigma}_{N} \approx \hat{\sigma}_{N}\right|_{\mathrm{SI}} \tag{8.58}
\end{equation*}
$$

We remark that this simplification (assumed in standard direct detection exclusion plots) provides a place for the DAMA results to hide since DAMA's NaI target is much more sensitive to spin-dependent coupling than other direct detection experiments of comparable volume. In case this is being read by LHC physicists, the detector volume $\sim$ [instantaneous] luminosity.

### 8.2.4 DIFFERENTIAL RECOIL RATE, A FIRST PASS

Let us now turn to the kinematics of the process. We assume elastic scattering since this dominates for point-like dark matter interacting with nuclei. This assumption provides another place to hide DAMA results, c.f. inelastic dark matter [343]. In the center of mass frame,


The kinematics of this scattering process are worked out thoroughly in first-year mechanics,

$$
\begin{equation*}
E_{R}=E_{i} r \frac{1-\cos \theta}{2} \tag{8.59}
\end{equation*}
$$

where $r$ is a kinematic factor built out of the particle masses

$$
\begin{equation*}
r=\frac{4 m_{r}}{m_{\chi} m_{N}}=\frac{4 m_{\chi} m_{N}}{\left(m_{\chi}+m_{N}\right)^{2}} \tag{8.60}
\end{equation*}
$$

The key feature is that $0<r \leq 1$ with the upper bound saturated for $m_{\chi}=m_{N}$. In other words, recoil energy is maximized when the masses of the wIMP and target nuclei are matched. The conventional cartoon to understand this is to consider the scattering of ping pong balls and bowling balls.

Now let us proceed to calculate the differential recoil rate for the case of zero momentum transfer $q=0$ where we've already parameterized the relevant cross section. We will later correct for the $q$-dependence in the form factor. In the center of mass frame the scattering is isotropic so that $E_{R}$ is uniform in $\cos \theta$ over the range

$$
\begin{equation*}
\circ<E_{R} \leq E_{i} r=E_{R}^{\max } \tag{8.61}
\end{equation*}
$$

This gives us a relatively boring plot of differential recoil rate for an incident energy


Nondescriptness notwithstanding, it is important to understand what is being plotted here. The vertical axis gives the rate of nuclear recoils for a sliver of recoil energies between $E_{R}$ and $E_{R}+d E_{R}$ and a sliver of incident energies between $E_{i}$ and $E_{i}+d E_{i}$. This is the differential of the recoil energy spectrum for the distribution of input wimp velocities (i.e. $E_{i}$ ). The area of the shaded box represents the contribution to this differential rate coming from integrating over $E_{R}$ for a given $E_{i}$. As promised this distribution is flat due to isotropy. The length of the box is given by $E_{R}^{\max }\left(E_{i}\right)$. The height of the box is a function of our zero momentum transfer cross section $\hat{\sigma}_{N}$ and $E_{i}$ through the dependence of the rate on the wIMP velocity distribution. Thus we may write

$$
\begin{equation*}
\frac{d}{d E_{i}} \frac{d R}{d E_{R}}=\frac{\text { area }}{\text { length }}=\frac{d R}{E_{i} r} . \tag{8.62}
\end{equation*}
$$

We would have a boring rectangular plot like this for each incident velocity (i.e. each $E_{i}$ ). The length of each rectangle is $E_{i} r$ and the height will be a more complicated function (given below) of the velocity distribution. In order to get the recoil spectrum, $d R / d E_{R}$, we can imagine stacking all of these boring rectangular plots on top of each other:


Now we can imagine summing together the contribution from each box to get the recoil spectrum, i.e. we can integrate (8.62)

$$
\begin{equation*}
\frac{d R}{d E_{R}}=\int_{E_{i}^{\min }}^{E_{i}^{\max }} \frac{d R\left(E_{i}\right)}{E_{i} r} \longrightarrow \int_{\mathbf{v}} \frac{d R\left(\mathbf{v}+\mathbf{v}_{E}\right)}{E_{i} r} \tag{8.63}
\end{equation*}
$$

where on the right we convert to an integral over wimp velocity, i.e. $E_{i}=E_{i}\left(\mathbf{v}+\mathbf{v}_{E}\right)$. As we noted above when normalizing the Maxwellian velocity distribution, we have been glib about the limits of integration. To simplify our first pass, will take $E_{i}^{\max } \rightarrow \infty$ and $E_{i}^{\min }=E_{R} / r$ from the second inequality in (8.61). We will later address the effect of a finite $E_{i}^{\max }$ coming from the characteristic escape velocity $v_{\text {esc }}$ of wIMPs in the dark matter halo.

To perform this integral we need an explicit form of the differential rate $d R\left(E_{i}\right)$ of scattering from an incident energy $E_{i}$ to a recoil energy $E_{R}$. (We have only explicitly written the argument that is integrated over.) $d R\left(E_{i}\right)$ tells us how many such recoil events occur per kilogram-day of a target material of atomic mass $A$. Heuristically this is written as

$$
\begin{equation*}
d R=\frac{\# \text { nuclei }}{\mathrm{kg}} \cdot \frac{\text { rate }}{\text { nucleus }} \tag{8.64}
\end{equation*}
$$

i.e. the number of nuclei per unit mass multiplied by the rate per nuclei. To determine this latter quantity we can imagine each target nucleus traveling through space at velocity $\mathbf{v}_{g a l}=\mathbf{v}+\mathbf{v}_{E}$ in the wimp rest frame with a cross section $\hat{\sigma}_{N}$.


The nucleus effectively carves out an interaction volume $\hat{\sigma}_{N} v d t$ across a space with wimp number density $n_{0} f\left(\mathbf{v}+\mathbf{v}_{E}\right) d^{3} \mathbf{v}$. Thus the number of events is

$$
\begin{equation*}
\frac{\text { rate }}{\text { nucleus }} d t=\hat{\sigma}_{N} v_{\text {gal }} n_{\mathrm{o}} f\left(\mathbf{v}+\mathbf{v}_{E}\right) d^{3} \mathbf{v} d t \tag{8.65}
\end{equation*}
$$

and the rate per nucleus is given by dropping the $d t$.
Plugging everything into (8.64), including the Maxwellian velocity distribution (8.49),

$$
\begin{equation*}
d R=\frac{N_{0}}{A} \cdot \hat{\sigma}_{N} v_{\text {gal }} n_{0} \frac{1}{k} e^{\left(\mathbf{v}+\mathbf{v}_{E}\right)^{2} / v_{0}^{2}} d^{3} \mathbf{v} \tag{8.66}
\end{equation*}
$$

where $N_{0}$ is Avogadro's number. Let us now perform the integral (8.63) in a very simplified 'toy' case which we will gradually make more sophisticated. In addition to setting $v_{\text {esc }} \rightarrow \infty$, let us turn off the annual modulation from the Earth's motion in the galaxy, $\mathbf{v}_{E}=\circ$ (this also sets $v_{\text {gal }}=v$ ). The resulting integral is then

$$
\begin{equation*}
\frac{d R}{d E_{R}}=\int_{v_{\text {min }}}^{\infty} \frac{1}{\left(\frac{1}{2} m_{\chi} v^{2}\right) r} \frac{R_{0}}{2 \pi v_{0}^{4}} v e^{-\nu^{2} / v_{o}^{2}} 4 \pi v^{2} d v . \tag{8.67}
\end{equation*}
$$

The first term is just $\left(E_{i} r\right)^{-1}$, the second term defines $R_{\circ}$ to absorb constants in a way that will be convenient later, and the remainder contains the $v$ dependence of $d R$. The minimum velocity is given by

$$
\begin{equation*}
E_{i}^{\min }=\frac{E_{R}}{r}=\frac{1}{2} m_{\chi} v_{\min }^{2} . \tag{8.68}
\end{equation*}
$$

Proceeding to simplify and perform the integral,

$$
\begin{equation*}
\frac{d R}{d E_{R}}=\frac{R_{\circ}}{r\left(\frac{1}{2} m_{\chi} v_{0}^{2}\right)} \int_{v_{\min }}^{\infty} \frac{2}{v_{0}^{2}} e^{-v^{2} / v_{o}^{2}} v d v=\frac{R_{\circ}}{E_{\circ} r} e^{-E_{R} / E_{o} r}, \tag{8.69}
\end{equation*}
$$

where we have defined $E_{\circ}=\frac{1}{2} m_{\chi} v_{0}^{2}$ to be the most probable incident wimp energy and $R_{\circ}$ can now be simply interpreted as the total rate for isotropic nuclear recoil from a non-relativistic point-like particle moving through the galaxy. Explicitly writing in all of the factors that went into this constant, we find

$$
\begin{equation*}
R_{\circ}=\frac{2}{\sqrt{\pi}} \frac{N_{\circ}}{A} n_{0} \hat{\sigma}_{N} v_{0} \approx \frac{500 \mathrm{Gev}}{A m_{\chi}} \cdot \frac{\hat{\sigma}_{N}}{1 \mathrm{pb}} \cdot \frac{\rho_{\mathrm{DM}}}{0.4 \mathrm{Gev} / \mathrm{cm}^{3}} \cdot \frac{\text { events }}{\mathrm{kg} \mathrm{day}} . \tag{8.70}
\end{equation*}
$$

Sometimes people define silly units like tru ('total rate unit') $=$ event $^{\mathrm{kg}^{-1}}$ day $^{-1}$ for this rate or the dru ('differential rate unit') for event $\mathrm{kg}^{-1}$ day ${ }^{-1} \mathrm{keV}^{-1}$ [336]. However, the last thing particle physics needs is more units so we will not use these.

It is useful to pause for a moment to admire this toy result since it already gives a very rough estimate for what one might expect in the real world. Given a 100 kg detector made up of $\mathrm{Xe}(A \approx 100)$ and a 100 Gev wimp with typical weak-scale nuclear cross section $\hat{\sigma}_{N} \sim 1 \mathrm{pb}$, one ends up with about 5 events per day. This scales linearly with cross section, wimp density (astrophysics), and inversely with the wimp mass. Now suppose the target nucleus happens to have the same mass, $m_{N}=m_{\chi}=100 \mathrm{Gev}$ (this is the right ballpark for Xe ) so that $r=1$, then we can calculate the mean recoil energy,

$$
\begin{equation*}
\left\langle E_{R}\right\rangle=E_{0} r=\frac{1}{2} m_{\chi} v_{0}^{2}=\frac{1}{2} 50 \mathrm{Gev}\left(.75 \cdot 10^{-3}\right) \approx 30 \mathrm{kev} . \tag{8.71}
\end{equation*}
$$

This number is remarkably small, even though we're in the 'best case' scenario where the wimp and target masses are matched. To compare to experiments that collider physicists (especially those at Fermilab) might appreciate a bit better, neutrino beam experiments typically detect events of mev-scale energies. Dark matter experiments have to be significantly better than this.

### 8.2.5 COMPARING APPLES TO APPLES

Before moving on to make our toy model more realistic, let us pause to make an important point about meaningful ways to convey the information from a direct detection experiment. Assuming we have run such an experiment for some time and have detected no signal, we can make an exclusion plot to convey what our experiment has learned. We present such a plot in Figure 8.2.1. The plot assumes that there are no events detected within the energy threshold; effectively one assumes that there was a maximal number of events of energy less than the threshold that would still be consistent with no observed events above threshold. Integrating (8.69) gives such a value for $R$ for which one can plot $R_{\mathrm{o}} / r \sim \hat{\sigma}_{N}$ over $m_{\chi}$. One can qualitatively understand the features of this graph: at the minimum the kinetic factor $r$ is maximized for $m_{\chi} \approx m_{N}$. Below this value there's not enough kinetic energy transferred (ping pong balls don't transfer much energy to bowling balls) and above this value the density of dark matter decreases $\left(n \sim \rho / m_{\chi}\right)$ so that the bounds away from $m_{\chi} \approx m_{N}$ become weaker.


Figure 8.2.1: Model log-log exclusion plots from (8.69) in arbitrary units. Each line excludes points above it. Solid lines indicate increasing energy threshold (worse sensitivity) following the solid arrow while the dashed lines indicate increasing target atomic mass $A$.

Such a plot can be generated for each direct detection experiment with null results. The key question is how one ought to combine the results of different experiments. Since we know that different experiments use different target material (and this is good since this provides sensitivity for a broad range of wimp masses), we are particularly concerned about the dependence of the exclusion plot on the target. This can be summarized by fact that we are setting bounds on the [zero momentum transfer] wimp-nucleus cross section $\hat{\sigma}_{N}$ for various wimp masses. This clearly is not a useful quantity when comparing experiments with different target nuclei. Fortunately, there is a trivial fix: rescale everything so that we provide bounds on the wIMP-nucleon cross section $\hat{\sigma}_{n}$ which is thus independent of the particular nucleus. Note that we use the convention that lowercase $n$ refers to nucleon (or 'neutron') while capital $N$ refers to the entire nucleus. The conversion is straightforward,

$$
\begin{equation*}
\hat{\sigma}_{N}=\frac{m_{r}^{2}}{m_{r n}^{2}} A^{2} \hat{\sigma}_{n} \tag{8.72}
\end{equation*}
$$

where $m_{r n}$ is the reduced mass for the wimp-nucleon system. Note that we pick up an additional factor of $A^{2}$ which, combined with (8.57), gives us a total coherence enhancement of $A^{4}$ in the wimp-nucleon rate (the rate which is sensible to compare
between experiments). Let us remind ourselves that we are restricting ourselves to the case of dominant spin-independent interactions, the case where spin-dependent scattering is appreciable requires more caution.

Plugging this back into our very rough (back of a very small envelope) estimate (8.70) and using $m_{r}^{2} / m_{r n}^{2} \sim A^{2}$, we find that for our 100 kg Xe detector and 100 Gev wimp, we get five events per day for a zero momentum transfer wimp-nucleon cross section of $\hat{\sigma}_{n} \sim 10^{-8} \mathrm{pb}$.

### 8.2.6 MORE REALISTIC VELOCITIES

The differential recoil rate in Section 8.2.4 is a handy estimate for what one would expect for an experiment, but it is a dramatic simplification. Let us make our toy expression slightly more sophisticated by taking into account the effect of a finite escape velocity and replace the effect of the Earth's annually modulated velocity relative to the dark matter halo. To make it clear which spectrum we are referring to, let us write

$$
\begin{equation*}
\frac{d R}{d E_{R}} \longrightarrow \frac{d R\left(v_{E}, v_{\mathrm{esc}}\right)}{d E_{R}} \tag{8.73}
\end{equation*}
$$

where we explicitly write the dependence on the Earth's velocity and the escape velocity. The toy-model spectrum we derived above then $d R(\mathrm{o}, \infty) / d E_{R}$.

Because the dark matter halo is gravitationally bound, there is a characteristic escape velocity at which the Maxwell distribution necessarily breaks down since any particles with such energies would escape the halo. Thus our integration over WIMP velocity (or, equivalently, incident energy) should have some upper limit. Technically, the gravitational potential modifies the Maxwell distribution at its tail, but it is typically sufficient to impose a hard cutoff. Typically $v_{\text {esc }} \approx 600 \mathrm{~km} \mathrm{~s}^{-1}$ should be used as the upper limit for the integration in (8.69). Note that this also requires a modification of the overall normalization of the Maxwell distribution. We define the finite $v_{\text {esc }}$ normalization by

$$
\begin{equation*}
k_{\mathrm{esc}}=k_{\mathrm{o}}\left[\operatorname{erf}\left(\frac{v_{\mathrm{esc}}}{v_{0}}\right)-\frac{2}{\sqrt{\pi}} \frac{v_{\mathrm{esc}}}{v_{0}} e^{-v_{\mathrm{esc}}^{2} / v_{\mathrm{o}}^{2}}\right] \tag{8.74}
\end{equation*}
$$

where the error function is a convenient shorthand for the integral over the finite velocity domain. The modified recoil spectrum can be written in terms of the $v_{\text {esc }} \rightarrow \infty$ spectrum as

$$
\begin{equation*}
\frac{d R\left(\mathrm{o}, v_{\mathrm{esc}}\right)}{d E_{R}}=\frac{k_{\circ}}{k_{\mathrm{esc}}}\left[\frac{d R(\mathrm{o}, \infty)}{d E_{R}}-\frac{R_{\circ}}{E_{\mathrm{o}} r} e^{-v_{\mathrm{esc}}^{2} / v_{\mathrm{o}}^{2}}\right], \tag{8.75}
\end{equation*}
$$

where we see the effect of the rescaled normalization and an additional term which vanishes in the $v_{\text {esc }} \rightarrow \infty$ limit. Let us remark that typically these large velocity effects are negligible relative to our toy model since our garden-variety wimps tend to be rather heavy and don't carry much kinetic energy. This allowed us, for example, to simply truncate the distribution above the escape velocity. However, light wIMP candidates can populate more of the tail of the velocity distribution and proper treatment of this region is important [339].

Now let us account for the modulated velocity of the Earth relative to the dark matter halo, which we wrote above as:

$$
\begin{equation*}
v_{E}=232+15 \cos \left(2 \pi \frac{t-152.5 \text { days }}{365.25 \text { days }}\right) \mathrm{km} \mathrm{~s}^{-1} . \tag{8.76}
\end{equation*}
$$

Due to the unfortunate placement of our solar system in the Milky Way galaxy, the average velocity ( $232 \mathrm{~km} / \mathrm{s}$ ) is not very well known, though the amplitude of the modulation ( $15 \mathrm{~km} / \mathrm{s}$ ) is well measured. We should further remark that there are small errors since the modulation isn't exactly sinusoidal. This modulation clearly does not affect the finite $v_{\text {esc }}$ term in (8.75) since the large $v_{\text {esc }}$ dominates over $v_{E}$. However, this does affect the $d R(o, \infty) / d E_{R}$ term. Going through the same analysis as Section 8.2.4 with $\nu^{2} \rightarrow\left(\mathbf{v}+\mathbf{v}_{E}\right)^{2}$, we find

$$
\begin{equation*}
\frac{d R\left(v_{E}, \infty\right)}{d E_{R}}=\frac{R_{\circ}}{E_{\mathrm{o}} r} \frac{\sqrt{\pi}}{4} \frac{v_{\circ}}{v_{E}}\left[\operatorname{erf}\left(\frac{v_{\min }+v_{E}}{v_{\circ}}\right)-\operatorname{erf}\left(\frac{v_{\min }-v_{E}}{v_{\mathrm{o}}}\right)\right] . \tag{8.77}
\end{equation*}
$$

Combining this with (8.75) finally gives us

$$
\begin{equation*}
\frac{d R\left(v_{E}, v_{\mathrm{esc}}\right)}{d E_{R}}=\frac{k_{0}}{k_{\mathrm{esc}}}\left[\frac{d R\left(v_{E}, \infty\right)}{d E_{R}}-\frac{R_{0}}{E_{0} r} e^{-v_{\mathrm{esc}}^{2} / v_{0}^{2}}\right] . \tag{8.78}
\end{equation*}
$$

This certainly brings us closer to a realistic expression (though we still have not included $q$-dependence), but (8.77) and (8.78) leaves much to be desired in terms of having something tractable to interpret. Fortunately, it turns out that (8.77) can be approximated very well by a simpler form,

$$
\begin{equation*}
\frac{d R\left(v_{E}, \infty\right)}{d E_{R}}=c_{1} \frac{R_{\circ}}{E_{0} r} e^{-c_{2} E_{R} / E_{\circ} r}, \tag{8.79}
\end{equation*}
$$

for some fitting 'constants' $c_{1}$ and $c_{2}$ which vary slightly with the time of year

$$
\begin{equation*}
.73 \leq c_{1} \leq .77 \quad .53 \leq c_{2} \leq .59 \tag{8.80}
\end{equation*}
$$

A detailed time-dependence can be found in Appendix C of [336], but for most cases it is sufficient to set them to their average values $\left\langle c_{1}\right\rangle=0.75$ and $\left\langle c_{2}\right\rangle=0.56$. Note that these are not independent, since integration of the above equation forces $c_{1} / c_{2}=R\left(v_{E}, \infty\right) / R_{0}$. In this simplified form we can see that the that the effects of the Earth's motion can increase rate and make the spectrum slightly harder (from $c_{2}$ ).

Finally, let's remark that integrating the spectrum (8.77) to get a total rate and differentiating with respect to the Earth's velocity gives

$$
\begin{equation*}
\frac{d}{d v_{E}}\left(\frac{R}{R_{\circ}}\right)=\frac{1}{v_{E}}\left[\frac{R}{R_{\circ}}-\frac{\sqrt{\pi} v_{o}}{2 v_{E}} \operatorname{erf}\left(\frac{v_{E}}{v_{\circ}}\right)\right] \approx \frac{1}{2 v_{E}} \frac{R}{R_{\circ}} \tag{1}
\end{equation*}
$$

where our final approximation assumes $v_{E} \approx v_{0}$. From this we can see that the $6 \%$ modulation in $v_{E}$ causes a $3 \%$ modulation in the rate.

### 8.2.7 FORM FACTOR SUPPRESSION: COHERENCE LOST

Perhaps the most obvious omission in our toy model thus far has been the approximation of zero momentum transfer, $q=0$. This came from our ansatz all the way back in (8.54) that we could reliably treat the $q$-dependence as a correction to the $q=0$ cross section which we parameterized as a form factor, $F(q)$. Now we should justify this parameterization and determine the form of $F(q)$. See [344] for a discussion.

Momentum transfer from the wimp-nucleus collision is

$$
\begin{equation*}
q=\sqrt{2 m_{N} E_{R}} \tag{8.82}
\end{equation*}
$$

For large enough values of $q$ we expect coherence to break down as the de Broglie wavelength becomes smaller than the scale of the nucleus. A simple way to develop an intuition for the form factor is to work in the first Born approximation (i.e. plane wave approximation):

$$
\begin{equation*}
\mathcal{M}(q)=f_{n} A \int d^{3} \mathbf{x} \rho(\mathbf{x}) e^{i \mathbf{q} \cdot \mathbf{x}} \tag{8.83}
\end{equation*}
$$

where $\rho$ is the density distribution of scattering sites. The form factor is precisely the this Fourier transform over the scattering lattice,

$$
\begin{equation*}
F(q)=\int d^{3} \mathbf{x} \rho(x) e^{i \mathbf{q} \cdot \mathbf{x}}=\frac{4 \pi}{q} \int_{0}^{\infty} r \sin (q r) \rho(r) d r . \tag{8.84}
\end{equation*}
$$

For spin independent interactions, a simple model of the nucleus as a solid sphere turns out to be a very good approximation.

In this case the form factor takes the form

$$
\begin{equation*}
F\left(q r_{N}\right)=\frac{j_{1}\left(q r_{N}\right)}{q r_{N}}=3 \frac{\sin \left(q r_{N}\right)-q r_{N} \cos \left(q r_{N}\right)}{\left(q r_{N}\right)^{3}} \tag{8.85}
\end{equation*}
$$

where we've written the momentum dependence in terms of a dimensionless quantity $q r_{N}$ where $r_{N} \sim A^{1 / 3}$ is a characteristic nuclear radius. Recall that $q \sim \sqrt{A E_{R}}$ where the $A$-dependence comes from $m_{N} \sim A$. Thus the leading $A$ and $E_{R}$ dependence of $q r_{N}$ goes like

$$
\begin{equation*}
q r_{N} \sim A^{5 / 2} E_{R}^{1 / 2} . \tag{8.86}
\end{equation*}
$$

A more accurate parameterization from [336] is

$$
\begin{equation*}
q r_{N}=6.92 \cdot 10^{-3} A^{1 / 2}\left(\frac{E_{R}}{\mathrm{keV}}\right)^{1 / 2}\left(a_{N} A^{1 / 3}+b_{N}\right), \tag{8.87}
\end{equation*}
$$

where $a_{N}$ and $b_{N}$ are 'fudge factors' to give the correct nuclear radius $r_{N}$ from its $A$ dependence. We will simply take $a_{N}=1$ and $b_{N}=\circ$ (to this precision $6.92 \rightarrow 7$ ) so that a reasonable-to-detect 100 keV recoil of a $\mathrm{Xe}(A \approx 100)$ nucleus gives $q r_{N} \approx 3.2$. From our argument about length scales one might worry that this is the regime where coherence breaks down. Indeed, plugging into our solid sphere nuclear model, we get an $F^{2}\left(q r_{N}\right)$ suppression.

For light target nuclei, the form factor doesn't make much difference. For heavy nuclei, on the other hand, we can resolve the structure of the Bessel function (the Fourier transform of our solid sphere nuclear model) and we find ourselves hitting the zeroes of $j_{1}$ and brushing up against its exponential suppression.

This is a very important plot to take into account when designing a direct detection experiment. We saw in (8.57) that the spin-independent nuclear cross section scales as $A^{2}$. This is enhanced to $A^{4}$ when considering the more useful nucleon cross section. While we know that having too large an $A$ (so that $m_{N} \gg m_{\chi}$ ) leads to penalty in the kinetic factor $r$, we know from (8.60) that this is only $A^{-1}$. Thus it would still seem advantageous to build detectors with the heaviest target materials available to maximize the interaction cross section. As we've now seen (and could have expected), this breaks down when the WIMP is no longer able to scatter coherently off the entire nucleus. One must then balance the coherence from having heavy nuclei with the form factor suppressing coming from decoherence.

As we consider larger nuclei (large $A$ ), the region around $q=0$ where $F^{2}\left(q r_{N}\right)$ is not prohibitive becomes smaller. The trade off when designing an experiment then depends crucially on how low one can push the energy threshold: what is the smallest nuclear recoil that one can measure? If you can efficiently detect arbitrarily low threshold recoils, then you can go ahead and use the heaviest nuclei you can find for your detector. However, real experiments only have a finite energy threshold (partially a function of the target material). For this minimum recoil energy, one must consider to what extent the form factor suppression from one's target material will suppress one's signal.

Thus in Figure ??, the $A \sim 20$ detector takes a big hit in the interaction cross section because of its low $A$ value. However, we see that one is free to use a detector technology with a less prohibitive energy threshold since $F^{2}$ doesn't decrease very quickly. The $A \sim 120$ detector, on the other hand, gives a nice enhancement from coherence, but only for sufficiently low energy recoils so that one must be very sensitive to low energy signals. As a rule of thumb, targets lighter than Ge start start to lose a lot from $A^{2}$ suppression; i.e. current detector technology does not require $A$ any lower than this to ensure reasonable efficiency.

This is an important lesson to put the CDMS and XENON experiments in context. While Xe is appreciably heavier than Ge , form factor suppression (decoherence) in Xe leads to the two being roughly the same in their ability to detect wimps. For a discussion of spin-dependent scattering, see [344-347].

### 8.2.8 FURTHER REFINEMENT

In addition to proper inclusion of spin-dependence and refinements of the models used above (e.g. the halo, Born approximation with a hard sphere), further refinements of this model are discussed pedagogically in [336] and [338]. These are typically of the following form:

- Detection efficiency. Nuclear recoils and electron recoils are very different interactions. Given an electron and a nuclear interaction with the same recoil energy, a given detector technology will measure different values for such events due to the nature of the detection technique. This means that instead of the spectrum with respect to the recoil energy $d R / d E_{R}$, one should calculate the spectrum with respect to the visible energy $d R / d E_{v}$ where $E_{v}=f_{n} E_{R}$ so that

$$
\begin{equation*}
\frac{d R}{d E_{R}} \approx f_{n}\left(1+\frac{E_{R}}{f_{n}} \frac{d f_{n}}{d E_{R}}\right) \frac{d R}{d E_{v}} . \tag{8.88}
\end{equation*}
$$

A related issue that is important to discuss is quenching, see [348] for a nice discussion. Because detectors respond differently to nuclear recoils than to electron recoils, we need useful units to measure our visible energy. The difference between the visible energy coming from electron and nuclear events of the same recoil energy is parameterized by a quenching factor, $Q$. This leads to some silly notation: $\mathrm{kev}_{\text {ee }}$ for the "electron equivalent" energy (i.e. observed energy had the event come from an electron) and $\mathrm{kev}_{\mathrm{r}}$ for the energy signature from a "nuclear recoil."

$$
\begin{equation*}
E_{e}\left(\operatorname{kev}_{\mathrm{ee}}\right)=Q \times E_{r}\left(\operatorname{kev}_{\mathrm{r}}\right) \tag{8.89}
\end{equation*}
$$

- Energy resolution. The next effect to consider is the finite resolution for any real detector. This means that if there were exactly $N$ signal recoils each of a single energy $E_{v}=E_{v}^{\prime}$, then our real detector would observe a spread of energies smeared out in an approximately Gaussian manner with some energy-dependent width $\Delta E$,

$$
\begin{equation*}
\frac{d N}{d E_{v}}=\frac{N}{\sqrt{2 \pi} \Delta E} e^{\left(E_{v}-E_{v}^{\prime}\right)^{2}} 2 \Delta E^{2} . \tag{8.90}
\end{equation*}
$$

Thus the actual spectrum that we measure should be transformed to

$$
\begin{equation*}
\frac{d R}{d E_{v}}=\frac{1}{\sqrt{2 \pi}} \int d E_{v}^{\prime} \frac{1}{\Delta E} \frac{d R}{d E_{v}^{\prime}} e^{\left(E_{v}-E_{v}\right)^{2} / 2 \Delta E^{2}} \tag{8.91}
\end{equation*}
$$

where $\Delta E\left(E_{v}^{\prime}\right) \sim \sqrt{E_{v}^{\prime}}$. Real experimentalists should also 'fold in' the other terms in $\Delta E$ relevant to a given detector technology. For low energy events one should also worry that the Gaussian statistics above might lead to erroneous loss of counts due to negative energies. This can be solved by using a Poisson distribution, but leads to issues regarding the energy threshold.

- Energy threshold. As discussed above, the most favorable rates come from low energy events where the de Broglie wavelength of the wIMP is large enough to permit coherent scattering against an entire target nucleus. However, detectors (e.g. photomultiplier tubes) can only resolve events above a given threshold energy. Noise reduction also sets a threshold dependent on nearby radioactive sources (e.g. impurities in the target material) and shielding. These cutoffs must be taken into account for each experiment when constructing exclusion plots.
- Target mass fractions. Let us comment in passing that in detectors with compound targets (e.g. NaI for DAMA) one must calculate the rate limit separately for each target. To summarize, let use write out the recoil spectrum with respect to measured energy as a handy mnemonic:

$$
\begin{equation*}
\frac{d R}{d E_{v}}=R_{\circ} \sum_{A} f_{A} S_{A} F_{A}^{2} I_{A} \tag{8.92}
\end{equation*}
$$

where $R_{0}$ is the total rate, $A$ runs over the relevant atomic mass numbers, $f_{A}$ gives the detection efficiency for nuclear recoil, $S_{A}$ is the spectral function, $F^{2}$ is the form factor suppression, and $I_{A}$ is a reminder about which sort of interaction (spin-independent or spin-dependent) we are considering. $S_{A}$ is essentially the spectrum in (8.69) modified by all of the above velocity and detector effects.

## 8.A Cosmology basics

Here quickly review relevant background topics in cosmology at a very low level. These are based primarily on [85] and [332,333]. For further details, see the brief review in the PDG [349] or any more advanced cosmology review.

## 8.A. 1 Friedmann equation

For a spatially homogeneous and isotropic (Friedmann-Robertson-Walker or FRW) universe, the non-trivial part of the metric reduces to an overall scale factor $a(t)$ such that

$$
\begin{equation*}
d s^{2}=d t^{2}-a^{2}(t) d \mathbf{x}^{2} . \tag{8.93}
\end{equation*}
$$

One may now turn the crank of general relativity to derive the Friedmann equation. For our purposes, a Newtonian example is more physically intuitive and almost gives the exact correct answer, see e.g. [85]. Consider a comoving sphere of radius $R(t)$ containing the total mass of the universe, $M$. In such a comoving volume the number density of 'static' objects do not change with the expansion of the universe. A test mass $m$ on the surface of the sphere experiences a Newtonian gravitational force

$$
\begin{equation*}
F=-\frac{G M m}{R(t)^{2}} . \tag{8.94}
\end{equation*}
$$

This means that the gravitational acceleration on the test mass is

$$
\begin{equation*}
\ddot{R}(t)=-\frac{G M}{R(t)^{2}} . \tag{8.95}
\end{equation*}
$$

We can convert this into familiar energies by multiplying by $\dot{R}$ and integrating to give

$$
\begin{equation*}
\frac{1}{2} \dot{R}^{2}=\frac{G M}{R}+U, \tag{8.96}
\end{equation*}
$$

for a constant of integration $U$. We identify the left-hand side as the kinetic energy per unit mass and the right-hand side as (minus) a potential energy per unit mass. We see that kinetic plus potential energy is constant.

Now let's massage things into more common quantities. The radius of the sphere can be written in terms of a reference radius times the scale factor,

$$
\begin{equation*}
R(t)=a(t) r . \tag{8.97}
\end{equation*}
$$

Next, we can write the total mass within the sphere in terms of the density, $M=(4 \pi / 3) \rho(t) R(t)^{3}$, from which (8.96) takes the form

$$
\begin{equation*}
\frac{1}{2} r^{2} \dot{a}^{2}=\frac{4 \pi}{3} G r^{2} \rho(t) a(t)^{2}+U \tag{8.98}
\end{equation*}
$$

Finally, we can divide by $r^{2} a^{2} / 2$ to obtain the Newtonian Friedmann equation,

$$
\begin{equation*}
\left(\frac{\dot{a}}{a}\right)^{2}=\frac{8 \pi G}{3} \rho(t)+\frac{2 U}{r^{2}} \frac{1}{a^{2}} . \tag{8.99}
\end{equation*}
$$

One can see that if you assume an expanding universe, $\dot{a}>0$, the fate of the universe is controlled by the value of $U$. While the above argument gives a correct heuristic picture of what's going on, one must honestly solve Einstein's equations to obtain the relativistic Friedmann equation,

$$
\begin{equation*}
\left(\frac{\dot{a}}{a}\right)^{2}=\frac{8 \pi G}{3} \rho(t)-\frac{\kappa}{R_{0}^{2}} \frac{1}{a^{2}} . \tag{8.100}
\end{equation*}
$$

This is derived from the o-o component of the Einstein equation. We implicitly promoted the Newtonian mass density $\rho$ to
the relativistic energy density. Further, we have associated the potential $U$ to the curvature $\kappa$ via

$$
\begin{equation*}
\frac{2 U}{r^{2}}=-\frac{\kappa}{R_{\circ}^{2}} \tag{8.101}
\end{equation*}
$$

where $R_{\circ}$ is related to the radius of curvature of the universe, $R(t)=a(t) R_{0}$. The different fates of the universe thus correspond to different values of the curvature. Note that it is typical to write the Friedmann equation in terms of the Hubble parameter, $H(t) \equiv \dot{a} / a$.

## 8.A. 2 Density of the universe

Define the critical density, $\rho_{\mathrm{c}}(t)$, to be the energy density for which the universe is flat, $\kappa=0$ :

$$
\begin{equation*}
\rho_{\mathrm{c}}(t) \equiv \frac{3}{8 \pi G} H(t)^{2} \tag{8.102}
\end{equation*}
$$

This gives a natural way to define dimensionless density parameters,

$$
\begin{equation*}
\Omega \equiv \frac{\rho}{\rho_{c}} \tag{8.103}
\end{equation*}
$$

so that the Friedmann equation may be written

$$
\begin{equation*}
1-\Omega(t)=\frac{-\kappa}{R^{2} H^{2}} \tag{8.104}
\end{equation*}
$$

The right-hand side does not change sign so that the universe cannot change the sign of its curvature. For $\Omega>1$ we have $\kappa=+1$ and a closed universe. Conversely, for $\Omega<1$ we have $\kappa=-1$ and an open universe. The intermediate case $\Omega=1$ and $\kappa=\mathrm{o}$ yields a flat universe.

## 8.A. 3 THE FLUID AND ACCELERATION EQUATIONS

The Friedmann equation is essentially a statement about comoving conservation of energy. This has another manifestation in physics, the First Law of Thermodynamics,

$$
\begin{equation*}
d Q=d E+P d V \tag{8.105}
\end{equation*}
$$

For a perfectly homogeneous universe there is no bulk heat flow so that the expansion of the universe is adiabatic, $d Q=0$. Writing $V=4 \pi R^{3} / 3$ and $E=V \rho$ and then plugging into the First Law we find the fluid equation,

$$
\begin{equation*}
\dot{\rho}+3 \frac{\dot{a}}{a}(\rho+P)=0 \tag{8.106}
\end{equation*}
$$

We mentioned above that the Friedmann equation corresponds to the o-o component of the Einstein equation given the FRW ansatz. We could also solve for the $i-i$ components, but it turns out that this is related to the Friedmann and fluid equations through the Bianchi identity. Indeed, combining the two equations gives the acceleration equation,

$$
\begin{equation*}
\frac{\ddot{a}}{a}=-\frac{4 \pi G}{3}(\rho+3 P) \tag{8.107}
\end{equation*}
$$

Ordinary stuff has a positive pressure, whereas dark energy has negative pressure $P=-\rho$.

## 8.A. 4 EqUATIONS OF STATE

An equation of state relates pressure and energy density, $P=w \rho$ for some constant $w$. The assumption that the equation of state is linear and time-independent is good for dilute gases. Requiring that the speed of sound waves $c_{s}=d P / d \rho$ is
non-tachyonic, $c_{\mathrm{s}}<1$, imposes $w \leq 1$. One way of recasting the First Law of Thermodynamics is

$$
\begin{equation*}
d\left[R^{3}(\rho+P)\right]=R^{3} d P \tag{8.108}
\end{equation*}
$$

from which we note that the evolution of a given species of energy densities goes like

$$
\begin{equation*}
\rho \propto R^{-3(1+w)} \tag{8.109}
\end{equation*}
$$

The most important examples are

- $w=o$ for non-relativistic matter. (Non-relativistic matter has zero pressure.)
- $w=1 / 3$ for a relativistic gas (e.g. of photons).
- $w=-1$ for vacuum energy.


## 8.A. 5 EQUILIBRIUM AND OUT OF EQULIBRIUM THERMODYNAMICS

For dark matter, we are primarily interested in thermodynamics out of equilibrium since this is the regime in which thermal freeze out occurs. As background, however, let us review salient aspects of equilibrium thermodynamics. First: 'temperature' is something which is species dependent. When we refer to 'the temperature' $T$, we mean the photon temperature, $T=T_{\gamma}$. Next recall the Fermi-Dirac $(+)$ and Bose-Einstein $(-)$ phase space distributions,

$$
\begin{equation*}
f(\mathbf{p})=\frac{1}{\exp ((E-\mu) / T) \pm 1} \tag{8.110}
\end{equation*}
$$

where the chemical potential (the free energy cost of adding an additional particle, e.g. due to a conserved charge) may be related to the chemical potentials of other species which are in chemical equilibrium with the particle. From this we can write the number density, energy density, and pressure of a dilute, weakly interacting as

$$
\begin{equation*}
n=g \int d^{3} p f(\mathbf{p}) \quad \rho=g \int d^{3} p E(\mathbf{p}) f(\mathbf{p}) \quad P=g \int d^{3} p \frac{|\mathbf{p}|^{2}}{3 E} f(\mathbf{p}) \tag{8.111}
\end{equation*}
$$

where $g$ is the number of internal degrees of freedom, e.g. spin. The last expression is explained in chapter 7.13 of [350].
The integrals for $n, \rho$, and $P$ may be computed to yield analytic results. In the relativistic limit $T \gg m$ with $T \gg \mu$,

$$
n=\left\{\begin{array}{lll}
\frac{\zeta(3)}{\pi^{2}} g T^{3} & (\text { Bose })  \tag{8.112}\\
\frac{3}{4} \frac{\zeta(3)}{\pi^{2}} g T^{3} & (\text { Fermi })
\end{array} \quad \rho=\left\{\begin{array}{ll}
\frac{\pi^{2}}{30} g T^{4} & (\text { Bose }) \\
\frac{7}{8} \frac{\pi^{2}}{30} g T^{3} & (\text { Fermi })
\end{array} \quad P=\frac{\rho}{3}\right.\right.
$$

Note the famous factor of $7 / 8$ in the relativistic Fermi-Dirac energy density. In the non-relativistic limit $m \gg T$,

$$
n=g\left(\frac{m T}{2 \pi}\right)^{3 / 2} e^{-(m-\mu) / T} \quad \rho=m n \quad P=n T \ll \rho
$$

A useful quantity for $\mathbf{C P}$ violation is the number excess of a fermion species over its antiparticle. Assuming that reactions like $f+\bar{f} \leftrightarrow \gamma+\gamma$ occur rapidly, then $\mu=-\bar{\mu}$ and the net fermion number density is

$$
\begin{align*}
n-\bar{n} & =\frac{g}{2 \pi^{2}} \int_{m}^{\infty} d E E \sqrt{E^{2}-m^{2}}\left(\frac{1}{1+\exp [(E-\mu) / T]}-\frac{1}{1+\exp [(E+\mu) / T]}\right)  \tag{8.114}\\
& = \begin{cases}\frac{g T^{3}}{6 \pi^{2}}\left[\pi^{2}\left(\frac{\mu}{T}\right)+\left(\frac{\mu}{T}\right)^{3}\right] & (T \gg m) \\
2 g\left(\frac{m T}{2 \pi}\right)^{3 / 2} \sinh (\mu / T) \exp (-m / T) & (T \ll m)\end{cases} \tag{8.115}
\end{align*}
$$

In the early universe, interactions between different species kept them in equilibrium with a common temperature. As the universe cooled, species decoupled from thermal equilibrium. It turns out to be handy to measure the total energy density and
pressure of all species in equilibrium in terms of the photon temperature $T$ :

$$
\begin{align*}
& \rho_{R}=T^{4} \sum_{i}\left(\frac{T_{i}}{T}\right)^{4} \frac{g_{i}}{2 \pi^{2}} \int_{x_{i}}^{\infty} \frac{\sqrt{u^{2}-x_{i}^{2}} u^{2} d u}{\exp \left(u-y_{i}\right) \pm 1}  \tag{8.116}\\
& P_{R}=T^{4} \sum_{i}\left(\frac{T_{i}}{T}\right)^{4} \frac{g_{i}}{6 \pi^{2}} \int_{x_{i}}^{\infty} \frac{\left(u^{2}-x_{i}^{2}\right)^{3 / 2} u^{2} d u}{\exp \left(u-y_{i}\right) \pm 1} \tag{8.117}
\end{align*}
$$

where $i$ runs over all species and we have defined the dimensionless variables $x_{i} \equiv m_{i} / T$ and $y_{i} \equiv \mu_{i} / T$. Further, since the energy density and pressure of non-relativistic species $(m \gg T)$ are exponentially suppressed, we may restrict the sum to only relativistic species so that the above expressions simplify,

$$
\begin{equation*}
\rho_{R}=\frac{\pi^{2}}{30} g_{*} T^{4} \quad P_{R}=\frac{\pi^{2}}{90} g_{*} T^{4}, \tag{8.118}
\end{equation*}
$$

where $g_{*}$ counts the number of effectively massless degrees of freedom,

$$
\begin{equation*}
g_{*}=\sum_{i=\text { bosons }} g_{i}\left(\frac{T_{i}}{T}\right)^{4}+\frac{7}{8} \sum_{i=\text { fermions }} g_{i}\left(\frac{T_{i}}{T}\right)^{4} \tag{8.119}
\end{equation*}
$$

The famous factor of $7 / 8$ accounts for the difference in Bose and Fermi statistics in the equilibrium distribution function. The value of $g_{*}$ is monotonically decreasing.

## 8.A. 6 Entropy

In the early universe, the interaction rate of particles in the thermal bath was much greater than the expansion rate so that local thermal equilibrium is maintained. In this case, the entropy per comoving volume is preserved and this becomes a useful fiducial quantity. Further, for most of the early universe, the chemical potential is much smaller than the temperature and the distribution functions depend only on $E / T$. This means that

$$
\begin{equation*}
\frac{\partial P}{\partial T}=g \int d^{3} p \frac{\partial f(\mathbf{p})}{\partial T} \frac{|\mathbf{p}|^{2}}{3 E}=g \int d^{3} p\left(\frac{-E}{T}\right) \frac{\partial f(\mathbf{p})}{\partial E} \frac{|\mathbf{p}|^{2}}{3 E} \tag{8.120}
\end{equation*}
$$

Integrating this by parts (dropping the surface term) yields

$$
\begin{equation*}
\frac{\partial P}{\partial T}=\frac{\rho+P}{T} \tag{8.121}
\end{equation*}
$$

This can also be derived from integrability, $\partial^{2} S / \partial T \partial V=\partial^{2} S / \partial V \partial T$. The right-hand side is identified with entropy density. To remember this, recall that the Second Law tells us that

$$
\begin{equation*}
T d S=d(\rho V)+P d V=d[(\rho+P)+V]-V d P . \tag{8.122}
\end{equation*}
$$

Making use of (8.121), we may write

$$
\begin{equation*}
d S=\frac{d[(\rho+P) V]}{T}-\frac{(\rho+P) V d T}{T^{2}}=d\left[\frac{(\rho+P) V}{T}+\text { const. }\right] . \tag{8.123}
\end{equation*}
$$

Ignoring the overall constant, the entropy per comoving volume is

$$
\begin{equation*}
S=R^{3} \frac{\rho+P}{T} \tag{8.124}
\end{equation*}
$$

so that we may identify (8.121) with the entropy density,

$$
\begin{equation*}
s \equiv \frac{S}{V}=\frac{\rho+P}{T} \tag{8.125}
\end{equation*}
$$

Now invoke the First Law (8.105) with $d Q=0$ and $E=\rho V$, which we may write as

$$
\begin{equation*}
d[(\rho+P) V]=V d P . \tag{8.126}
\end{equation*}
$$

combining this with (8.121) gives

$$
\begin{equation*}
s=d\left[\frac{(\rho+P) V}{T}\right]=\mathrm{o} \tag{8.127}
\end{equation*}
$$

so that entropy is indeed conserved.
Entropy is dominated by the contribution of relativistic particles, (8.118), so that

$$
\begin{equation*}
s=\frac{2 \pi^{2}}{45} g_{* s} T^{3} \tag{8.128}
\end{equation*}
$$

where

$$
\begin{equation*}
g_{* s}=\sum_{i=\text { bosons }} g_{i}\left(\frac{T_{i}}{T}\right)^{3}+\frac{7}{8} \sum_{i=\text { fermions }} g_{i}\left(\frac{T_{i}}{T}\right)^{3} \tag{8.129}
\end{equation*}
$$

which differs from (8.119) only in the exponent of the $\left(T_{i} / T\right)$ factors. However, since most particles had the same temperature in the early (equilibrium) universe, $g_{* s}=g_{*}$. This is depicted in Figure ??. Note that by virtue of its dependence on $T$, $s$ is proportional to the number density of relativistic particles, (8.112). We also remark that (8.128) is a useful equation when converting between the definitions $Y=n / T^{3}$ versus $Y=n / s$.

It is convenient to normalize $s$ relative to the photon density,

$$
\begin{equation*}
s=1.80 g_{* s} n_{\gamma} \tag{8.130}
\end{equation*}
$$

Since $s \sim a^{-3}$, the total number of particles in a comoving volume, $N=R^{3} n$, is equal to the number density divided by the entropy, $N=n / s$.

Why are there two $g_{*}$ values? Even though $g_{*}=g_{* s}$ when all relativistic particles share the same temperature, these quantities differ when one species decouples and has a lower temperature. Such a species would contribute less to the effective number of relativistic degrees of freedom by a factor that depends on whether we're looking at $g_{*}$ or $g_{* s}$. The reason why we need two counts of the number of degrees of freedom is that $g_{*}$ relates the temperature to energy density via (8.118), while $g_{* s}$ relates the temperature to the scale factor via $T \sim g_{* s}^{-1 / 3} a^{-1}$, c.f. (8.130).

## 8.B Kinetic Theory and the Boltzmann Equation

We present a thorough derivation of the Boltzmann equation. For details, see one's favorites statistical mechanics or chapter 4 of [332].

## 8.B. 1 Kinetic theory

Define the unconditional s-particle probability in an $N$-particle system,

$$
\begin{equation*}
\rho_{s}\left(\mathbf{p}_{1}, \mathbf{q}_{1}, \cdots, \mathbf{p}_{s}, \mathbf{q}_{s} ; t\right)=\int \prod_{i=s+1}^{N} d \boldsymbol{\mu}_{i} \rho(\mathbb{p}, \mathbb{q}, t) \tag{8.131}
\end{equation*}
$$

Here $\rho(\mathbb{p}, \mathbb{q}, t)$ is the one-particle probability in phase space. The product on the right-hand side runs over the ( $N-s$ ) particles which are not specified by the arguments of the left-hand side. From this we can define particle densities. We begin with the single particle density which is the expectation for finding any of the $N$ particles in the state ( $\mathbf{p}, \mathbf{q}$ ),

$$
\begin{equation*}
f_{1}(\mathbf{p}, \mathbf{q} ; t)=\left\langle\sum_{j=1}^{N} \delta^{(3)}\left(\mathbf{p}-\mathbf{p}_{j}\right) \delta^{(3)}\left(\mathbf{q}-\mathbf{q}_{j}\right)\right\rangle=N \int \prod_{j=2}^{N} d \boldsymbol{\mu} \rho\left(\mathbf{p}_{1}=\mathbf{p}, \mathbf{q}_{1}=\mathbf{q}, \mathbf{p}_{2}, \mathbf{q}_{2}, \cdots, \mathbf{p}_{\mathbf{N}}, \mathbf{q}_{N} ; t\right) . \tag{8.132}
\end{equation*}
$$

For simplicity, define this to be $N \rho_{1}(\mathbf{p}, \mathbf{q} ; t)$. We've written the phase space measure as $d \mu$. From here we can generalize to an $s$-particle density,

$$
\begin{equation*}
f_{s}\left(\mathbf{p}, \cdots, \mathbf{q}_{s} ; t\right)=N(N-1) \cdots(N-S+1) \rho_{s}\left(\mathbf{p}_{1}, \cdots, \mathbf{q}_{s} ; t\right)=\frac{N!}{(N-S)!} \rho_{s}\left(\mathbf{p}_{1}, \cdots, \mathbf{q}_{s} ; t\right) . \tag{8.133}
\end{equation*}
$$

We can ask how these densities evolve with time. Fortunately, we only have to look at $\rho_{1}$ :

$$
\begin{equation*}
\frac{\partial \rho_{1}}{\partial t}=\int \prod_{i=2}^{N} d \boldsymbol{\mu}_{i} \frac{\partial \rho}{\partial t}=-\int \prod_{i=2}^{N} d \boldsymbol{\mu}_{i}\{\rho, H\}, \tag{8.134}
\end{equation*}
$$

where $\rho$ is the full phase space density ( 6 N variables) and we've use Liouville's theorem. Let us organize the Hamiltonian into three pieces, $H=H_{1}+H_{N-1}+H^{\prime}$, where,

$$
\begin{align*}
H_{1} & =\frac{\mathbf{p}_{1}^{2}}{2 m}+U\left(\mathbf{q}_{1}\right)  \tag{8.135}\\
H_{N-1} & =\sum_{i=2}^{N}\left[\frac{\mathbf{p}_{1}^{2}}{2 m}+U\left(\mathbf{q}_{1}\right)\right]+\frac{1}{2} \sum_{i, j=2}^{N} V\left(\mathbf{q}_{i}-\mathbf{q}_{j}\right)  \tag{8.136}\\
H^{\prime} & =\sum_{i=2}^{N} V\left(\mathbf{q}-\mathbf{q}_{i}\right) . \tag{8.137}
\end{align*}
$$

Here $U(\mathbf{q})$ is an external potential, while $V\left(\mathbf{q}_{i}-\mathbf{q}_{j}\right)$ is an interaction potential between different particles. We can thus write

$$
\begin{equation*}
\frac{\partial \rho_{1}}{\partial t}=-\int \prod_{i=2}^{N} d \mu_{i}\left\{\rho,\left(H_{1}+H_{N-1}+H^{\prime}\right)\right\} \tag{8.138}
\end{equation*}
$$

Let us consider each term one at a time.

$$
\begin{equation*}
\int \prod_{i=2}^{N} d \boldsymbol{\mu}_{i}\left\{\rho, H_{1}\right\}=\int \prod_{i=2}^{N} d \boldsymbol{\mu}_{i}\left\{\rho, H_{1}\right\}=\left\{\rho_{1}, H_{1}\right\} . \tag{8.139}
\end{equation*}
$$

Here we've used the fact that $H_{1}$ is independent of $\boldsymbol{\mu}_{i}$ for $i \neq 1$.

$$
\begin{align*}
\int \prod_{i=2}^{N}\left\{\rho, H_{N-1}\right\} & =\int \prod_{i=2}^{N} d \boldsymbol{\mu}_{i} \sum_{j=1}^{N}\left(\frac{\partial \rho}{\partial \mathbf{p}_{j}} \frac{\partial H_{N-1}}{\partial \mathbf{q}_{j}}-\frac{\partial \rho}{\partial \mathbf{q}_{j}} \frac{\partial H_{N-1}}{\partial \mathbf{p}_{j}}\right)  \tag{8.140}\\
& =\int \prod_{i=2}^{N} d \boldsymbol{\mu}_{i} \sum_{j=1}^{N}\left[\frac{\partial \rho}{\partial \mathbf{p}_{j}}\left(\frac{\partial U}{\partial \mathbf{q}_{j}}+\frac{1}{2} \sum_{k=2}^{N} \frac{\partial V\left(\mathbf{q}_{j}-\mathbf{q}_{k}\right)}{\partial q_{j}}\right)-\frac{\partial \rho}{\partial \mathbf{q}_{j}} \frac{\mathbf{p}_{j}}{m}\right]=0 . \tag{8.141}
\end{align*}
$$

Here we've noted that the term in the parentheses is independent of $\mathbf{p}_{j}$ while the remaining term is independent of $\mathbf{q}_{j}$; thus the
entire line vanishes upon the appropriate integration by parts.

$$
\begin{align*}
\int \prod_{i=2}^{N} d \mu_{i} \sum_{j=1}^{N}\left[\frac{\partial \rho}{\partial \mathbf{p}_{j}} \frac{\partial H^{\prime}}{\partial \mathbf{q}_{j}}-\frac{\partial \rho}{\partial \mathbf{q}_{j}} \frac{\partial \mu^{\prime}}{\partial \mathbf{p}_{j}}\right] & =\int \prod_{i=2}^{N} d \boldsymbol{\mu}_{i} \sum_{j=1}^{N}\left[\frac{\partial \rho}{\partial \mathbf{p}_{1}} \sum_{j=2}^{N} \frac{\partial V\left(\mathbf{q}_{i}-\mathbf{q}_{j}\right)}{\partial \mathbf{q}_{1}}+\sum_{j=2}^{N} \frac{\partial \rho}{\partial \mathbf{p}_{j}} \frac{\partial V\left(\mathbf{q}_{i}-\mathbf{q}_{j}\right)}{\partial \mathbf{q}_{j}}\right] \\
& =(N-1) \int \prod_{i=2}^{N} d \boldsymbol{\mu}_{i} \frac{\partial \rho}{\partial \mathbf{p}_{1}} \cdot \frac{\partial V\left(\mathbf{q}_{i}-\mathbf{q}_{j}\right)}{\partial \mathbf{q}_{1}}  \tag{8.142}\\
& =(N-1) \int d \mu_{2} \frac{\partial V\left(\mathbf{q}_{i}-\mathbf{q}_{j}\right)}{\partial \mathbf{q}_{1}} \cdot \frac{\partial}{\partial \mathbf{p}_{1}}\left(\prod_{i=3}^{N} d \mu_{i} \rho\right)  \tag{8.143}\\
& =(N-1) \int d \mu_{2} \frac{\partial V\left(\mathbf{q}_{i}-\mathbf{q}_{j}\right)}{\partial \mathbf{q}_{1}} \cdot \frac{\partial \rho_{2}}{\partial \mathbf{p}_{1}} . \tag{8.144}
\end{align*}
$$

On the first line we used the independence of $H^{\prime}$ on $\mathbf{p}$ and, on the right-hand side, integration by parts. What a mess. Fortunately we can clearn this all up and then generalize. Plugging this into (8.138) yields

$$
\begin{equation*}
\frac{\partial \rho_{1}}{\partial t}-\left\{H_{1}, \rho_{1}\right\}=(N-1) \int d \mu_{2} \frac{\partial V\left(\mathbf{q}_{1}-\mathbf{q}_{2}\right)}{\partial \mathbf{q}_{1}} \cdot \frac{\partial \rho_{2}}{\partial \mathbf{p}_{1}} . \tag{8.145}
\end{equation*}
$$

Multiplying by $N$ allows us to convert this into an expression for the time evolution of $f_{1}$,

$$
\begin{equation*}
\frac{\partial f_{1}}{\partial t}-\left\{H_{1}, f_{1}\right\}=\int d \boldsymbol{\mu}_{2} \frac{\partial V\left(\mathbf{q}_{1}-\mathbf{q}_{2}\right)}{\partial \mathbf{q}_{1}} \cdot \frac{\partial f_{2}}{\partial \mathbf{p}_{1}} . \tag{8.146}
\end{equation*}
$$

The right-hand side of this equation is a collision integral that tells us about the pair-wise interactions of particles in this system. It is now straightforward to see how this generalizes for the time evolution of a general multi-particle density,

$$
\begin{equation*}
\frac{\partial f_{s}}{\partial t}-\left\{H_{s}, f_{s}\right\}=\sum_{n=1}^{s} \int d \boldsymbol{\mu}_{2} \frac{\partial V\left(\mathbf{q}_{n}-\mathbf{q}_{s+1}\right)}{\partial \mathbf{q}_{n}} \cdot \frac{\partial f_{s+1}}{\partial \mathbf{p}_{n}} \tag{8.147}
\end{equation*}
$$

The general point that one should glean from this is that the expression for $\partial f_{s} / \partial t$ requires knowledge of $f_{s+1}$. In order to find out $f_{1}$, one needs to know $f_{2}$, but to know $f_{2}$ one needs $f_{3}$, an so forth. This is sometimes referred to as the BGGKY hierarchy.

## 8.B. 2 The Boltzmann equation

The physical approximation that allows us to bypass the BGGKY hierarchy is the Boltzmann equation. The key assumption is that interactions are short range. Even with this assumption, one should take pause: mechanics was already boring and tedious for two-particle scattering. Now we will be going to $N \sim 10^{23}$-particle scattering! We will give a loose, 'plausible' presentation. You may fill in the details as you feel necessary.

Let us explicitly write out the first two equations of the hierarchy:

$$
\begin{array}{r}
{\left[\frac{\partial}{\partial t}-\frac{\partial U}{\partial \mathbf{q}_{1}} \frac{\partial}{\partial \mathbf{p}_{1}}+\frac{\mathbf{p}_{1}}{m} \frac{\partial}{\partial \mathbf{q}_{1}}\right] f_{1}=\int d \mu_{2} \frac{\partial V\left(\mathbf{q}_{1}-\mathbf{q}_{2}\right)}{\partial \mathbf{q}_{1}} \frac{\partial f_{2}}{\partial \mathbf{p}_{1}}} \\
{\left[\frac{\partial}{\partial t}-\frac{\partial U}{\partial \mathbf{q}_{1}} \frac{\partial}{\partial \mathbf{p}_{1}}-\frac{\partial U}{\partial \mathbf{q}_{2}} \frac{\partial}{\partial \mathbf{p}_{2}}+\frac{\mathbf{p}_{1}}{m} \frac{\partial}{\partial \mathbf{q}_{1}}+\frac{\mathbf{p}_{2}}{m} \frac{\partial}{\partial \mathbf{q}_{2}}-\frac{\partial V\left(\mathbf{q}_{\mathbf{1}} \mathbf{q}_{2}\right)}{\partial \mathbf{q}_{1}}\left(\frac{\partial}{\partial \mathbf{p}_{1}}-\frac{\partial}{\partial \mathbf{p}_{2}}\right)\right] f_{2}} \\
 \tag{8.149}\\
=\int d \boldsymbol{\mu}_{3}\left[\frac{\partial V\left(\mathbf{q}_{1}-\mathbf{q}_{3}\right)}{\partial \mathbf{q}_{1}} \frac{\partial}{\partial \mathbf{p}_{1}} \frac{\partial V\left(\mathbf{q}_{2}-\mathbf{q}_{3}\right)}{\partial \mathbf{q}_{2}} \frac{\partial}{\partial \mathbf{p}_{2}}\right] f_{3}
\end{array}
$$

## Time scales

Now we get to do some physics. Let us identify the (inverse) time scales that appear in the expressions above (this is just dimensional analysis). In fact, before we identify any of the terms, you should already have some intuition for the relevant scales in the problem.

- The length scale of the external potential
- The length scale of particle-particle interactions
- The length scale for free particle propagation.

These can be converted into time scales though the average particle velocity of the system. First we have the time scale of the external potential,

$$
\begin{equation*}
\frac{1}{\tau_{U}}=\frac{\partial U}{\partial \mathbf{q}} \frac{\partial}{\partial \mathbf{p}} \sim \frac{v}{L} . \tag{8.150}
\end{equation*}
$$

Recall that $\partial U / \partial \mathbf{q}$ is a force and that momentum divided by force indeed gives the time scale for momentum change. We've written $v$ for the average velocity of the particles and $L$ to be the characteristic length scale for changes in $U$. Similarly, note that $(\mathbf{p} / m) \partial / \partial \mathbf{q}$ is a velocity times gradient, or $v \cdot \nabla f$.

Next there is a time scale associated with the mean free time between particle interactions. Consider the right-hand side of (8.148), which we may write heuristically as

$$
\begin{equation*}
\left[\int d \boldsymbol{\mu}_{2} \frac{\partial V}{\partial \mathbf{q}_{1}} \frac{\partial f_{2}}{\partial \mathbf{p}_{1}} \frac{1}{f_{1}}\right] f_{1} . \tag{8.151}
\end{equation*}
$$

We've written it this way to obtain a quantity that may sensibly be compared to the left-hand side of the same equation. Indeed, this allows us to define the mean free time more generally as

$$
\begin{equation*}
\frac{1}{\tau_{X}} \sim \int d \boldsymbol{\mu} \frac{\partial V}{\partial \mathbf{q}} \frac{\partial}{\partial \mathbf{p}} \frac{f_{s+1}}{f_{s}} \sim \frac{v}{d} \cdot n d^{3}, \tag{8.152}
\end{equation*}
$$

where $d$ is a length scale characterizing the range of the interaction. $\tau_{X}$ is the timescale between particle interactions: given an interaction, when is the next interaction? The factor $f_{2} / f_{1}$ in the $s=2$ case is the conditional probability of finding a second particle given the first. This should be associated with the factor of $n d^{3}$ on the right-hand side, where $n$ is the number density (so that this is just the probability of finding another particle per unit volume). The right-hand sides of both (8.148) and (8.149) are thus terms which represent free particle propagation within the system.

Finally, we can consider the collision duration, which appears as term containing a gradient of $V$ on the left-hand side of (8.149).

$$
\begin{equation*}
\frac{1}{\tau_{c}} \sim \frac{\partial V}{\partial \mathbf{q}} \frac{\partial}{\partial \mathbf{p}} \sim \frac{v}{d} \tag{8.153}
\end{equation*}
$$

We see that (8.148) is an equation that compares $\tau_{U}$ with $\tau_{X}$, while (8.149) also introduces $\tau_{c}$. Our goal is to try to truncate the BGGKY hierarchy by taking the correct (physically motivated) limits. First we take the dilute limit, where

$$
\begin{equation*}
n d^{3} \ll 1 \quad \Longleftrightarrow \quad \frac{1}{\tau_{c}} \gg \frac{1}{\tau_{X}} . \tag{8.154}
\end{equation*}
$$

Next, we can augment this with the assumption that the external potential is not vary much on short time scales,

$$
\begin{equation*}
\frac{1}{\tau_{U}} \ll \frac{1}{\tau_{X}} \ll \frac{1}{\tau_{c}} . \tag{8.155}
\end{equation*}
$$

In fact, typically the last relation is $\tau_{X}^{-1} \lll \tau_{c}^{-1}$. Lastly, we will need to assume molecular chaos, which is the statement that the two-particle density is well approximated by the product of one-particle densities. We will quantify this shortly.

## Deriving the Boltzmann equation

First not that the limits that we have chosen do not allow us to truncate (8.148). In the regime $\tau_{U}^{-1} \ll \tau_{X}^{-1}$, we cannot drop the right-hand side of the one-particle kinetic equation and we're stuck with the full expression. We can do more with (8.149). Here the dilute limit allows us to note that

$$
\begin{equation*}
\frac{\tau_{c}}{\tau_{X}} \approx n d^{3} \ll 1 \tag{8.156}
\end{equation*}
$$

In other words, as long as we have a $\tau_{c}^{-1}$ floating around (and only when we have such a term), we are free to drop terms that go like $\tau_{X}^{-1}$. Needless to say we can also drop the $\tau_{U}$ term on the left-hand side. Further, as we are interested in long time scales, i.e. 'steady state' situations. We can thus drop the $\partial / \partial t$ on the left-hand side. Typically $\tau_{U}^{-1} \ll 1 / t \ll \tau_{c}^{-1}$. Thus means that we can simplify (8.149) quite a bit:

$$
\begin{equation*}
\left[\frac{\mathbf{p}_{1}}{m} \cdot \frac{\partial}{\partial \mathbf{q}_{1}}+\frac{\mathbf{p}_{2}}{m} \cdot \frac{\partial}{\partial \mathbf{q}_{2}}-\frac{\partial V\left(\mathbf{q}_{1}-\mathbf{q}_{2}\right)}{\partial \mathbf{q}_{1}} \cdot\left(\frac{\partial}{\partial \mathbf{p}}-\frac{\partial}{\partial \mathbf{p}_{2}}\right)\right] f_{2}=0 \tag{8.157}
\end{equation*}
$$

Our assumption regarding the slow variation of the external potential motivates us to use relative spacetime coordinates,

$$
\begin{equation*}
\mathbf{Q} \equiv \frac{1}{2}\left(\mathbf{q}_{1}+\mathbf{q}_{2}\right) \quad \mathbf{q} \equiv \mathbf{q}_{2}-\mathbf{q}_{1} \tag{8.158}
\end{equation*}
$$

where the factor of $1 / 2$ is intentionally only on $\mathbf{Q}$. We note that in these coordinates,

$$
\begin{equation*}
\frac{\partial f_{2}}{\partial \mathbf{q}_{1}} \approx-\frac{\partial f_{2}}{\partial \mathbf{q}} \approx-\frac{\partial f_{2}}{\partial \mathbf{q}_{2}} . \tag{8.159}
\end{equation*}
$$

Using (8.157), we may thus write

$$
\begin{equation*}
\frac{\partial V\left(\mathbf{q}_{1}-\mathbf{q}_{2}\right)}{\partial \mathbf{q}_{1}} \cdot\left(\frac{\partial}{\partial \mathbf{p}}-\frac{\partial}{\partial \mathbf{p}_{2}}\right) f_{2}=\left(\frac{\mathbf{p}_{1}-\mathbf{p}_{2}}{m}\right) \cdot \frac{\partial f_{2}}{\partial \mathbf{q}} \tag{8.160}
\end{equation*}
$$

We can now use this to rewrite the right-hand side of (8.148). We start by adding a term proportional to $0=\partial f_{2} / \partial \mathbf{p}_{2}$ (this vanishes upon integration by parts),

$$
\int d \boldsymbol{\mu}_{2} \frac{\partial V\left(\mathbf{q}_{1}-\mathbf{q}_{2}\right)}{\partial \mathbf{q}_{1}} \frac{\partial f_{2}}{\partial \mathbf{p}_{1}}=\int d \boldsymbol{\mu}_{2} \frac{\partial V\left(\mathbf{q}_{1}-\mathbf{q}_{2}\right)}{\partial \mathbf{q}_{1}}\left(\frac{\partial}{\partial \mathbf{p}_{1}}-\frac{\partial}{\partial \mathbf{p}_{2}}\right) f_{2}=\int d \boldsymbol{\mu}_{2}\left(\frac{\mathbf{p}_{1}-\mathbf{p}_{2}}{m}\right) \cdot \frac{\partial f_{2}}{\partial \mathbf{q}} .
$$

Now we express the collision integral in terms of the collision kinematics. We need to recall some of our favorite quantities from two-particle scattering. In particular, we introduce the impact vector, $\mathbf{b}$, which lives in the plane perpendicular to the scattering axis and quantifies how off-axis the initial particle trajectories are. We choose angular coordinates so that $\theta$ measures the particle deflection from scattering axis and $\varphi$ is the azimuthal angle. We may thus write

$$
\begin{array}{r}
\int d^{3} \mathbf{p}_{2} d^{3} \mathbf{q}_{2}\left(\frac{\mathbf{p}_{2}-\mathbf{p}_{1}}{m}\right) \frac{\partial}{\partial \mathbf{q}} f_{2}\left(\mathbf{p}_{1}, \mathbf{q}_{1}, \mathbf{p}_{2}, \mathbf{q}_{2} ; t\right) \\
=\int d^{3} \mathbf{p}_{2} d^{2} \mathbf{b}\left|\mathbf{v}_{1}-\mathbf{v}_{2}\right|\left[f_{2}\left(\mathbf{p}_{1}, \mathbf{q}_{1}, \mathbf{p}_{2}, \mathbf{b},+; t\right)-f_{2}\left(\mathbf{p}_{1}, \mathbf{q}_{1}, \mathbf{p}_{2}, \mathbf{b},-; t\right)\right] \tag{8.161}
\end{array}
$$

where we've introduced different arguments in $f_{2}: \pm$ denotes the state before $(-)$ or after $(+)$ the collision. We would like to work exclusively in terms of the 'before collision' variables (we are taking the limit of an instantaneous collision). We thus write

$$
\begin{equation*}
f_{2}\left(\mathbf{p}_{1}, \mathbf{q}_{1}, \mathbf{p}_{2}, \mathbf{b},+; t\right)=f_{2}\left(\mathbf{p}_{1}^{\prime}, \mathbf{q}_{1}^{\prime}, \mathbf{p}_{2}^{\prime}, \mathbf{b},-; t\right), \tag{8.162}
\end{equation*}
$$

where we've defined the primed phase space coordinates to denote the momenta which trace into the unprimed coordinates upon collision. In some sense this is just a slick use of time reversal; but really it's just a definition of the primed coordinates.

Finally, the most drastic approximation we shall make is that of molecular chaos. Here we assume that particles 1 and 2 are
independent before collision so that the two-particle phase space density is well-approximated by the product of single-particle densities,

$$
\begin{equation*}
f_{2}(\cdots, \mathbf{b},-; t)=f_{1}\left(\mathbf{p}_{1}, \mathbf{q}_{1} ; t\right) f_{1}\left(\mathbf{p}_{2}, \mathbf{q}_{2} ; t\right) . \tag{8.163}
\end{equation*}
$$

Taking all of this into account in (8.148), we finally obtain

$$
\begin{equation*}
\left.\frac{d f_{1}}{d t}\right|_{\text {coll }}=\int d^{3} \mathbf{p}_{2} d^{2} \mathbf{b}\left|\mathbf{v}_{1}-\mathbf{v}_{2}\right|\left[f_{1}\left(\mathbf{p}_{1}, \mathbf{q}_{1} ; t\right) f_{1}\left(\mathbf{p}_{2}, \mathbf{q}_{2} ; t\right)-f_{1}\left(\mathbf{p}_{1}^{\prime}, \mathbf{q}_{1} ; t\right) f_{1}\left(\mathbf{p}_{2}^{\prime}, \mathbf{q}_{2} ; t\right)\right] \tag{8.164}
\end{equation*}
$$

# Goldstone Fermion Dark Matter 

Even before quarks were discovered, scientists were able to write down theories for the hadrons which they bind into. The reason for this is largely due to the idea that even though quarks are strongly coupled to one another, the bound states that they form needn't be. Further, the lightest states in the spectrum are typically particles called Goldstone bosons which prefer to be massless due to the symmetry structure of the theory. In this chapter we supersymmetrize this story and show that the supersymmetric partner to certain Goldstone bosons may be natural dark matter candidates.

### 9.1 Overview

We propose that the fermionic superpartner of a weak-scale Goldstone boson can be a natural WIMP candidate. The $p$-wave annihilation of this 'Goldstone fermion' into pairs of Goldstone bosons automatically generates the correct relic abundance, whereas the XENON 100 direct detection bounds are evaded due to suppressed couplings to the Standard Model. Further, it is able to avoid indirect detection constraints because the relevant $s$-wave annihilations are small. The interactions of the Goldstone supermultiplet can induce non-standard Higgs decays and novel collider phenomenology.

### 9.2 INTRODUCTION

Cosmological observations now provide overwhelming evidence that about $20 \%$ of the energy density of the universe is some unknown form of cold dark matter [351]. The most popular candidates are weakly interacting massive particles (WIMPs) which can produce the correct relic abundance after freeze out,

$$
\begin{equation*}
\Omega_{\mathrm{DM}} h^{2} \approx 0.1 \frac{\mathrm{pb}}{\langle\sigma v\rangle} . \tag{9.1}
\end{equation*}
$$

A natural candidate for WIMP dark matter arises in extensions of the Standard Model with low-scale supersymmetry (SUSY) and $R$-parity. In such models the lightest supersymmetric particle (LSP) is automatically stable and generically has mass on the order of the weak scale [142].

The 'WIMP miracle' is the statement that a particle with a mass and annihilation cross section typical of the weak scale will automatically yield a relic abundance that is within a few orders of magnitude of the observed value. This paradigm has been challenged by recent direct detection searches for WIMPs. In particular, XENON 100 recently set the most stringent upper limit on the spin-independent elastic WIMP-nucleon scattering cross section, $\sigma_{\text {SI }}=7.0 \times 10^{-45} \mathrm{~cm}^{2}=7.0 \times 10^{-9} \mathrm{pb}$, for a 50 GeV WIMP at $90 \%$ confidence [352]. This large discrepancy between the necessary annihilation cross section and the direct detection bound is increasingly difficult to explain in the usual WIMP dark matter scenarios.

For example, within the minimal supersymmetric Standard Model (MSSM), one must typically tune parameters in order to explain this difference [353]. A standard approach is to consider parameters in which $\sigma_{\text {SI }}$ is suppressed below direct detection constraints. At generic points in the parameter space this will also imply a suppressed annihilation cross section and thus a relic abundance that is too large. In order to overcome this problem one needs to assume special relations among a priori unrelated parameters in order to boost the annihilation rate. For example, a pure bino LSP would require coannihilation (due to an accidental slepton degeneracy) or resonant annihilation to obtain the correct annihilation cross section [331]. Alternately, the observation that Higgsinos and winos have annihilation cross sections that are typically too large allows one to tune the LSP to be a specific combination of bino, Higgsino, and wino to generate the correct abundance [354]. This 'well-tempered neutralino' scenario, however, is now strongly disfavored by XENON 100 [353].

In light of this tension, it is natural to consider non-minimal SUSY models in which

- the WIMP is a weak scale LSP,
- the direct detection cross section is suppressed while maintaining the correct relic abundance without any fine tuning, and
- the experimental prospects in near future include novel collider signatures.

We therefore extend the MSSM by a new sector with an approximate global symmetry which is spontaneously broken in the supersymmetric limit. A natural WIMP candidate that satisfies the above criteria is the fermionic partner of the Goldstone boson which we refer to as the Goldstone fermion, $\chi$. This particle can naturally sit at the bottom of the spectrum because it lives in the same chiral supermultiplet as the Goldstone boson $a$ and is thus protected by Goldstone's theorem and SUSY. Even when SUSY is broken the Goldstone fermion can remain light with mass at or below $M_{\text {SUSY }}$ [355-357]. This scenario is a weak scale version of axino dark matter [358]. Similar realizations also appear in dark matter models where the LSP has a large "singlino" component [359]; such models can reproduce the mass spectrum of Goldstone fermion dark matter but do not have a limit where the global symmetry is broken while SUSY is exact. In particular the singlino dark matter effective interactions do not come from an effective low-energy Kähler potential as discussed in Section 9.3.1. Further, due to singlino-Higgsino mixing, such models typically require tuning to avoid direct detection bounds.

The SUSY non-linear sigma model is a generic low-energy theory of the Goldstone supermultiplet based only on the symmetry breaking pattern [360]. It can be organized as an expansion in inverse powers of the symmetry breaking scale, $f$. In particular the leading order contribution to dark matter annihilation is controlled by a trilinear derivative coupling $\bar{\chi} \gamma^{\mu} \gamma^{5} \chi \partial_{\mu} a / f$. If the global symmetry is anomalous with respect the SM gauge group, the Goldstone bosons will, in turn, decay to stable SM particles, $a \rightarrow g g, \gamma \gamma$. All the interactions can be perturbative and compatible with gauge coupling unification if the mediators of the anomaly come in complete GUT multiplets. If the Goldstone fermion mass $m_{\chi}$ is around the weak scale and the symmetry breaking scale $f$ is around the TeV scale, then the resulting annihilation cross section is automatically in the thermal WIMP range

$$
\begin{equation*}
\langle\sigma v\rangle \approx\left(m_{\chi}^{2} / f^{4}\right)\left(T_{f} / m_{\chi}\right) \approx 1 \mathrm{pb} . \tag{9.2}
\end{equation*}
$$

The freeze-out temperature $T_{f} / m_{\chi} \simeq 1 / 20$ is insensitive to details of the model and appears because $\chi \chi \rightarrow a a$ is a $p$-wave process.

After electroweak symmetry breaking at $v_{\mathrm{EW}}=175 \mathrm{GeV}$, the CP-even scalar component of the Goldstone chiral multiplet mixes with the Higgs boson and generates an effective $h \chi \chi$ coupling which is suppressed by $m_{\chi} v_{\mathrm{EW}} / f \sim 0.01$. While standard Higgsino-like dark matter in the MSSM gives a large direct detection cross section, Goldstone fermion scattering off nuclei lies just below the XENON 100 bound,

$$
\begin{equation*}
\sigma_{\mathrm{SI}} \approx\left(\frac{m_{\chi} v_{\mathrm{EW}}}{f^{2}}\right)^{2} \sigma_{\mathrm{SI}}^{\mathrm{MSSM}} \approx 10^{-45} \mathrm{~cm}^{2} . \tag{9.3}
\end{equation*}
$$

Note that this suppression factor in $\sigma_{\text {SI }}$ is roughly of the same order as the suppression needed in the annihilation cross section for a standard weak-scale WIMP, $\langle\sigma v\rangle$ wimp $\sim \pi \alpha_{\text {weak }}^{2} /(100 \mathrm{GeV})^{2} \sim 150 \mathrm{pb}$, to obtain the correct abundance (9.1).

Finally, Goldstone fermion dark matter has novel consequences on Higgs phenomenology at the LHC. The global symmetry requires a derivative coupling between the Goldstone boson and the Higgs boson $\sim v_{\mathrm{EW}} / f^{\mathcal{}}(\partial a)^{2} h$. If kinematically allowed, the Higgs boson decays into four light unflavored jets, $h \rightarrow 2 a \rightarrow 4 j$, with a sizeable branching ratio. This decay mode is 'buried' under the QCD background. Such non-standard Higgs decays have recently been investigated in SUSY models motivated by the little hierarchy problem [361,362]. For Goldstone fermion dark matter, the Higgs might only be 'partially buried' with a branching ratio of $30 \%$ to the Standard Model. Alternately, one can hope to discover the Goldstone boson itself by looking for the $a \rightarrow 2 g$ decay. Together with the direct detection of its fermionic superpartner, such a discovery would be strong evidence that the dark matter particle emerges because of the Goldstone mechanism and SUSY.

The chapter is organized as follows. We introduce the effective low-energy theory of a Goldstone supermultiplet in Section 9.3 and extend this by including SUSY and explicit global symmetry breaking in Sections 9.4 and 9.5. Readers who are primarily interested in dark matter phenomenology can proceed directly to Sections 9.6 and 9.7 , where we review (in)direct detection prospects and calculate the relic abundance. We discuss the LHC phenomenology in Section 9.8. In Appendix 9.A we present simple models that realize this scenario. Details of the annihilation cross section calculation are given in Appendix 9.B. Remarks on a possible Sommerfeld enhancement are presented in Appendix 9.C.

### 9.3 The Goldstone Supermultiplet

We consider a supersymmetric gauge theory with a global $\mathrm{U}(1)$ symmetry that is broken by fields $\Psi_{i}$ which obtain vacuum expectation values (vevs) $f_{i}$. In the limit of unbroken supersymmetry, the theory has a massless Goldstone chiral superfield,

$$
\begin{equation*}
A=\frac{1}{\sqrt{2}}(s+i a)+\sqrt{2} \theta \chi+\theta^{2} F \tag{9.4}
\end{equation*}
$$

which is the low-energy degree of freedom of the high-energy fields,

$$
\begin{equation*}
\Psi_{i}=f_{i} e^{q_{i} A / f} \tag{9.5}
\end{equation*}
$$

where the effective symmetry breaking scale is

$$
\begin{equation*}
f^{2}=\sum_{i} q_{i}^{2} f_{i}^{2}, \tag{9.6}
\end{equation*}
$$

and $q_{i}$ is the $\mathrm{U}(1)$ charge of $\psi_{i}$. We refer to the component fields as the Goldstone boson $a$, the sGoldstone $s$, and the Goldstone fermion $\chi$. In models where the $U(1)$ is a Peccei-Quinn symmetry, these are typically called the axion, saxion, and axino, respectively. The mass of the CP-odd scalar $a$ is directly protected by the Goldstone theorem while the $s$ and $\tilde{a}$ masses are, in turn, protected by supersymmetry.

The Goldstone boson shift symmetry acts on the chiral superfield as $A \rightarrow A+i c f$. It is thus often convenient to consider a non-linear realization of the Goldstone chiral superfield, $G=e^{A / f}$, which naturally transforms under the $\mathrm{U}(1)$ shift symmetry, $G \rightarrow e^{i c} G$. In the absence of explicit global symmetry breaking, this shift symmetry forbids any superpotential term involving A.

### 9.3.1 Effective Kähler potential

The shift symmetry restricts the dependence of the Kähler potential on the Goldstone superfield to take the form

$$
\begin{equation*}
K=K\left(A+A^{\dagger}, \Phi_{\mathrm{L}}\right) . \tag{9.7}
\end{equation*}
$$

We have written $\Phi_{\mathrm{L}}$ to denote light fields which are uncharged under the global symmetry. Note this general form includes the canonical term $A A^{\dagger}$ which is $\left(A+A^{\dagger}\right)^{2}$ up to a Kähler transformation.

We may examine the Goldstone self-interactions by expanding the canonically normalized Kähler metric in inverse powers
of the scale $f$ :

$$
\begin{equation*}
K_{(2)}=\frac{\partial^{2} K}{\partial A \partial A^{\dagger}}=1+b_{1} \frac{q}{f}\left(A+A^{\dagger}\right)+b_{2} \frac{q^{2}}{2!f^{2}}\left(A+A^{\dagger}\right)^{2}+\ldots, \tag{9.8}
\end{equation*}
$$

where $q$ is an reference $U(1)$ charge of the theory. The choice of $q$ is arbitrary and irrelevant since the combination $f / q$ is invariant under charge rescaling. For simplicity we set $q=1$ henceforth. After integrating out the auxiliary fields, the general form of the Lagrangian is

$$
\begin{align*}
\mathcal{L}= & K_{(2)}(s)\left(\frac{1}{2} \partial^{\mu} s \partial_{\mu} s+\frac{1}{2} \partial^{\mu} a \partial_{\mu} a+\frac{i}{2} \chi^{\dagger} \bar{\sigma}^{\mu} \partial_{\mu} \chi-\frac{i}{2} \partial_{\mu} \chi^{\dagger} \bar{\sigma}^{\mu} \chi\right) \\
& -\frac{1}{\sqrt{2}} K_{(3)}(s)\left(\chi^{\dagger} \bar{\sigma}^{\mu} \chi \partial_{\mu} a\right)+\frac{1}{4}\left(K_{(4)}(s)-\frac{K_{(3)}^{2}(s)}{K_{(2)}(s)}\right)(\chi \chi)\left(\chi^{\dagger} \chi^{\dagger}\right), \tag{9.9}
\end{align*}
$$

where $K_{(n)}=\partial^{n} K / \partial A^{n}$. Passing to four-component Dirac spinors and expanding the Lagrangian in inverse powers of $1 / f$ yields,

$$
\begin{align*}
\mathcal{L}= & \left(1+b_{1} \frac{\sqrt{2}}{f} s+b_{2} \frac{1}{f^{2}} s^{2}+\cdots\right)\left(\frac{1}{2} \partial^{\mu} s \partial_{\mu} s+\frac{1}{2} \partial^{\mu} a \partial_{\mu} a+\frac{i}{2} \bar{\chi} \gamma^{\mu} \partial_{\mu} \chi\right)  \tag{9.10}\\
& +\frac{1}{2 \sqrt{2}}\left(b_{1} \frac{1}{f}+b_{2} \frac{\sqrt{2}}{f^{2}} s+\cdots\right)\left(\bar{\chi} \gamma^{\mu} \gamma^{5} \chi\right) \partial_{\mu} a+\frac{1}{16 f^{2}}\left(b_{2}-b_{1}^{2}+\cdots\right)\left[(\bar{\chi} \chi)^{2}-\left(\bar{\chi} \gamma^{5} \chi\right)^{2}\right] .
\end{align*}
$$

The coefficients $b_{1,2}, \ldots$ completely characterize the self-interactions of the Goldstone multiplet in the symmetric limit. The $b_{1}$ coefficient is particularly important for the dark matter abundance since it controls the size of the $\chi \chi a$ vertex. The tree-level contribution to $b_{1}$ can be determined by comparing ( 9.8 ) to the canonical Kähler potential of the high-energy fields $\psi_{i}$,

$$
\begin{equation*}
K=\sum_{i} \Psi_{i}^{\dagger} \Psi_{i}=\sum_{i} f_{i} e^{q_{i}\left(A+A^{\dagger}\right) / f} \tag{9.11}
\end{equation*}
$$

Note that in the absence of explicit $\mathrm{U}(1)$-breaking terms, $A$ does not get a vev and $K$ is canonically normalized with respect to the Goldstone superfield. All Goldstone self-interactions are calculable from the physical Kähler metric,

$$
\begin{equation*}
K_{A}{ }^{A^{\dagger}}=\frac{1}{f^{2}} \sum_{i} f_{i}^{2} q_{i}^{2} e^{q_{i}\left(A+A^{\dagger}\right) / f}=1+\frac{\left(A+A^{\dagger}\right)}{f^{3}}\left(\sum_{i} q_{i}^{3} f_{i}\right)+\ldots . \tag{9.12}
\end{equation*}
$$

In particular, the tree-level contribution to the $b_{1}$ coefficient is given by

$$
\begin{equation*}
b_{1}=\frac{1}{f^{2}} \sum_{i} q_{i}^{3} f_{i}^{2} . \tag{9.13}
\end{equation*}
$$

Note that $b_{1}$ is invariant under overall charge scaling. In simple models with just two fields $\Psi_{ \pm}$of opposite charge, $b_{1}$ is bounded, $-_{1} \leq b_{1} \leq 1$. In general, however, there is no such restriction on $b_{1}$ in theories with more fields or with dynamical $\mathrm{U}(1)$ breaking.

### 9.3.2 Interactions and mixing With light fields

Even though the MSSM fields are uncharged under the global symmetry, they may couple to the spontaneously broken sector through higher-order terms in the Kähler potential. We will particularly be interested in the coupling of the Goldstone multiplet with the Higgs doublets $H_{u, d}$. Explicit symmetry breaking terms can generate superpotential couplings between the MSSM and the Goldstone sector; these are discussed in Section 9.5.

The Kähler potential interactions between the Higgses and the Goldstone superfield can be parameterized by expanding in

1/f,

$$
\begin{equation*}
K=\frac{1}{f}\left(A+A^{\dagger}\right)\left(c_{1} H_{u} H_{d}+\ldots+\text { h.c. }\right)+\frac{1}{2 f^{2}}\left(A+A^{\dagger}\right)^{2}\left(c_{2} H_{u} H_{d}+\ldots+\text { h.c. }\right)+\mathcal{O}\left(1 / f^{\beta}\right) . \tag{9.14}
\end{equation*}
$$

Note that the first term vanishes if there is a $\mathbb{Z}_{2}$ discrete symmetry $A \rightarrow-A$. The presence of such symmetry depends on the choice of UV completion. A mixing between the Higgs and the sGoldstone arises, for example, from the Kähler metric term

$$
\begin{equation*}
K_{H_{u}}^{A^{\dagger}}=\partial^{2} K /\left(\partial H_{u} \partial A^{\dagger}\right)=\frac{1}{f} c_{1} H_{d}+\ldots \rightarrow \frac{v_{\mathrm{EW}}}{f} c_{1} \cos \beta+\ldots \tag{9.15}
\end{equation*}
$$

The $c_{2}$ terms can also give rise to mixing if the sGoldstone also gets a VEV of order $\langle s\rangle \sim f$. After rotating the Higgs and sGoldtone fields, the coupling between $h$ and the Goldstone multiplet appears in the effective Lagrangian as

$$
\begin{equation*}
\mathcal{L}_{\mathrm{eff}}=\left[\frac{1}{2}(\partial a \partial a)+\frac{i}{2} \bar{\chi} \gamma^{\mu} \partial_{\mu} \chi\right]\left(1+b_{1} \frac{\sqrt{2}}{f} s+c_{h} \frac{v_{\mathrm{EW}}}{f^{2}} h+\ldots\right)+\ldots \tag{9.16}
\end{equation*}
$$

where $c_{h}$ is a function of the coefficients $c_{1,2}, d_{1,2}$ and the Higgs sector mixing angles. This coupling is suppressed in the large $m_{s}$ limit, $c_{h} \rightarrow\left(m_{h} / m_{s}\right)^{2}$. At this order in $q^{2} v_{\mathrm{EW}} / f^{2}$ there are additional Higgs doublet couplings of the form

$$
\begin{equation*}
-\frac{i}{4 f^{2}} c_{2} \bar{\chi} \gamma^{\mu} \gamma^{5} \chi\left(H_{u} \partial_{\mu} H_{d}+\partial_{\mu} H_{u} H_{d}-\text { h.c. }\right) \tag{9.17}
\end{equation*}
$$

which give rise to additional interactions of the heavy Higgses with the Goldstone fermion, but do not involve the light higgs $h$. We neglect these couplings and the mixing of the heavy Higgses with the sGoldstone.

Besides the scalar mixing, there is kinetic mixing between the Higgsino and Goldstone fermion of the form

$$
\begin{align*}
\mathcal{L}_{\mathrm{KM}} & =\frac{i}{2 f}\left[\left(\chi^{\dagger} \bar{\sigma}^{\mu} \partial_{\mu} \tilde{H}_{u}-\partial_{\mu} \chi^{\dagger} \bar{\sigma}^{\mu} \tilde{H}_{u}\right)\left(c_{1} H_{d}+\ldots\right)+\text { h.c. }+\left(H_{u} \leftrightarrow H_{d}\right)\right] \\
& \rightarrow i \varepsilon_{u} \chi^{\dagger} \bar{\sigma}^{\mu} \partial_{\mu} \tilde{H}_{u}^{\circ}+i \varepsilon_{d} \chi^{\dagger} \bar{\sigma}^{\mu} \partial_{\mu} \tilde{H}_{d}^{o}+\text { h.c. } \tag{9.18}
\end{align*}
$$

where $\varepsilon_{u, d} \sim v_{\mathrm{EW}} / f$. In the case where $\mu \gg m_{x}$, the Goldstone Fermion has a small Higgsino component roughly given by $\varepsilon_{u, d} m_{\chi} / \mu \sim v_{\mathrm{EW}} m_{\chi} / f \mu$.

The Kähler terms involving the other MSSM matter fields are typically more suppressed. Assuming minimal flavor violation to control flavor-changing neutral currents, these terms take the form

$$
\begin{equation*}
K=\frac{1}{f}\left(A+A^{\dagger}\right)\left(\frac{Y_{u}}{M_{u}} \bar{Q} H_{u} u+\frac{Y_{d}}{M_{d}} \bar{Q} H_{d} d+\frac{Y_{l}}{M_{L}} \bar{L} H_{d} e+\text { h.c. }\right) . \tag{9.19}
\end{equation*}
$$

The suppression scales $M_{u, d, l}$ are not necessarily related to the global symmetry breaking scale $f$, and can be much larger depending on the UV completion of the theory.

### 9.4 SUSY BREAKING

We assume that soft SUSY breaking terms which simultaneously break the $U(1)$ global symmetry are negligible. The remaining soft terms generate an explicit sGoldstone mass, but leave the Goldstone boson massless. The Goldstone fermion may only get a mass from the superpotential or from $D$-terms via mixing with gauginos. For simplicity we ignore the latter possibility so that the fermion mass matrix is the second derivative matrix of the superpotential,

$$
\begin{equation*}
\left(m_{\text {fermion }}\right)_{i j}=W_{i j} . \tag{9.20}
\end{equation*}
$$

While the superpotential terms are $U(1)$ invariant and supersymmetric, a Goldstone fermion mass can still be induced if the vacuum is shifted from its supersymmetric value due to the presence of soft breaking terms. The $U(1)$ invariance of the
superpotential implies

$$
\begin{equation*}
\sum_{j} \frac{1}{f} W_{i j} q_{j} f_{j}=-\frac{1}{f} q_{i} W_{i}=-\frac{1}{f} q_{i} F_{i} \tag{9.21}
\end{equation*}
$$

so that the Goldstone fermion $\chi=\sum_{i} q_{i} f_{i} \psi_{i} / f$ is indeed a zero mode of the fermion mass matrix when none of the $\mathrm{U}(1)$-charged $F$-terms obtain a vev [355]. The mass of the Goldstone fermion then depends on whether the $\mathrm{U}(1)$-charged fields pick up F-terms in the presence of soft breaking terms [355, 356]:

$$
\begin{equation*}
m_{\chi} \approx q_{i}\left\langle F_{i}\right\rangle / f \tag{9.22}
\end{equation*}
$$

If the superpotential has an unbroken $R$-symmetry which is left unbroken by the soft terms, then $\chi$ necessarily carries $R$-charge -1 and a Majorana mass is prohibited ${ }^{1}$. In particular, soft scalar masses always preserve $R$-symmetry and hence cannot generate a Goldstone fermion mass in the $R$-symmetric case. On the other hand, $A$-terms are holomorphic and generically break $R$-symmetries. Thus $A$-terms are expected to always contribute to the Goldstone fermion mass, while soft scalar masses may or may not contribute.

The effect of $A$-terms is equivalent to the mixing between the $F$-terms between $U(1)$-charged fields and the SUSY breaking spurion $\langle X\rangle=F \theta^{2}+\cdots$. For concreteness, we consider gravity mediation with $F / M_{\mathrm{Pl}} \sim m_{\text {soft }}$. It was recently emphasized in [363] that $F F_{i}^{\dagger}$ type mixing terms are always expected and will contribute a mass of order $m_{3 / 2}$ to the Goldstone fermion. Indeed, such mixing terms arise from higher dimensional Kähler terms of the form

$$
\begin{equation*}
K=\sum_{i} Z\left(X, X^{\dagger}\right) \Phi_{i}^{\dagger} \Phi_{i} \tag{9.23}
\end{equation*}
$$

Using the technique of analytic continuation into superspace [364], one may absorb $Z$ into a redefinition of the chiral superfields

$$
\Phi \rightarrow \Phi^{\prime} \equiv Z^{1 / 2}\left(1+\frac{\partial \ln Z}{\partial X} F \theta^{2}\right) \Phi
$$

This canonically normalizes $K$ and generates soft terms that include the $A$-terms

$$
\begin{equation*}
\Delta \mathcal{L}_{\text {soft }}=\left.\frac{\partial W}{\partial \Phi}\right|_{\Phi=\varphi} Z^{-1 / 2}\left(-\frac{\partial \ln Z}{\partial \ln X} \frac{F}{M}\right) \tag{9.25}
\end{equation*}
$$

These terms completely incorporate the mixing between $F$-terms of the form $F F_{i}^{\dagger} \Phi_{i}$. The Goldstone fermion mass is determined by the induced $F_{i}$ s obtained by minimizing the scalar potential,

$$
\begin{equation*}
V=\left|\frac{\partial W}{\partial \varphi_{i}}\right|^{2}+A_{i} \frac{\partial W}{\partial \varphi_{i}} \varphi_{i}+\text { h.c. }+m_{i}^{2}\left|\varphi_{i}\right|^{2} \tag{9.26}
\end{equation*}
$$

To summarize this section, we find that $A$-terms will always contribute to the Goldstone fermion mass. Assuming that $A_{i}, m_{i}<f_{i}$ for all $i$, the generic size of the induced $F$-terms is $\left|F_{i}\right| \approx A_{i} f_{i}$ and, consequently, the induced Goldstone fermion mass is $\sim A_{i} q_{i}$. In many situations the $A$-terms can be suppressed relative to other soft breaking terms and it is thus reasonable to expect that the Goldstone fermion remains lighter than the other superpartners. Soft scalar masses may also contribute. If they do, their contribution to the $F_{i}$ is expected to be of order $F_{i} \sim m_{i}^{2}$ so that the contribution to the Goldstone fermion mass is of order $\sim m_{i}^{2} / f_{i}$, which again can easily be suppressed.

### 9.5 SUPERPOTENTIAL TERMS FROM EXPLICIT BREAKING

The shift symmetry forbids any superpotential for the Goldstone chiral multiplet $A$. In order to generate a small Goldstone boson mass one must include terms which break the global symmetry. These can come from an anomaly in the global

[^12]symmetry or through explicit breaking terms.

### 9.5.1 Anomaly

If the global symmetry is anomalous then the triangle diagram generates a $a G \tilde{G}$ term which fits into a superpotential term

$$
\begin{equation*}
W_{\mathrm{anomaly}}=-\frac{c_{\mathrm{an}}}{f} A W^{a} W^{a} \tag{9.27}
\end{equation*}
$$

where $W^{a}=\lambda^{a}-i \sigma^{\mu \nu} \theta G_{\mu \nu}^{a}+\ldots$ is the field strength chiral superfield for the gauge group $G$ which has a $U(1) G^{2}$ anomaly. In practice we will take $G$ to be $S U(3)_{\text {color }}$ or $U(1)_{\text {QED }}$ since we will be interested in the coupling to massless gauge fields.
$W_{\text {anomaly }}$ generates non-derivative couplings in the effective Lagrangian:

$$
\begin{equation*}
\mathcal{L}_{\text {anomaly }} \supset \frac{c_{\mathrm{an}}}{f \sqrt{2}}\left(a G_{\mu \nu}^{a} \tilde{G}_{\mu \nu}^{a}+\frac{i}{2} \bar{\chi} G_{\mu \nu}^{a}\left[\gamma^{\mu}, \gamma^{\nu}\right] \gamma^{5} \lambda^{a}\right) \tag{9.28}
\end{equation*}
$$

where $\tilde{G}_{\mu \nu}^{a}=\frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} G_{\rho \sigma}^{a}$.
For the remainder of this document we assume that the global $U(1)$ is anomalous. For example, the anomalous coupling $c_{\mathrm{an}}$ is generated when the Goldstone boson $a$ couples to $N_{\psi}$ fermions $\Psi_{i}$ that transform in the fundamental of the gauged $\operatorname{SU}(N)$ and carry a global charge $q_{\Psi}$,

$$
\begin{equation*}
c_{\mathrm{an}}=\frac{a}{8 \pi} \sqrt{2} \sum_{i}^{N_{\Psi}}\left(\frac{y_{i} f}{m_{\Psi_{i}}}\right) \quad \mathcal{L}_{y}=i a \sum_{i=1}^{N_{\Psi}} y_{i} \bar{\Psi}_{i} \gamma^{5} \Psi_{i} \tag{9.29}
\end{equation*}
$$

The result for a $\mathrm{U}(1)$ gauge group is similar and is obtained by including the different $q_{i}$ charges,

$$
\begin{equation*}
c_{\mathrm{an}}^{(1)}=\frac{a_{U(1)}}{8 \pi} \sqrt{2} N_{c} \sum_{i} 2 q_{i}^{2}\left(\frac{y_{i} f}{m_{\Psi_{i}}}\right) \tag{9.30}
\end{equation*}
$$

where $N_{c}=3$ and the factor of 2 comes from the normalization of the generators in $\operatorname{SU}(N), \operatorname{Tr}\left[T^{a} T^{b}\right]=\delta^{a b} / 2$. The simplest and most common case is when all masses are degenerate, $m_{\Psi_{i}}=m_{\Psi}$, and the $y_{i}$ are equal, $y_{i}=m_{\Psi} q_{\Psi} /(f \sqrt{2})$, so that

$$
\begin{equation*}
c_{\mathrm{an}}=\frac{a}{8 \pi} q_{\Psi} N_{\Psi} \tag{9.31}
\end{equation*}
$$

Note that gauge coupling unification is preserved if the mediator fields $\Psi_{i}$ are embedded in complete GUT multiplets. For example, one may consider $N_{\Psi} \times(5 \oplus \overline{5})$ representations of $S U(5)$ which decompose into ${ }^{2}(3,1)_{1 / 3}$ and $(1,2)_{-1 / 2}$ under $\mathrm{SU}_{\mathrm{c}}(3) \times \mathrm{SU}_{\mathrm{L}}(2) \times \mathrm{U}_{Y}(1)$. In this case the mediators $\Psi_{i}$ are both colored and electrically charged; they thus allow the dominant decay to be $a \rightarrow g g$ with subdominant contributions from $a \rightarrow \gamma \gamma$ with branching ratio $\sim 10^{-3}$.

### 9.5.2 EXPLICIT BREAKING SPURIONS

Sources of explicit global symmetry breaking terms can be parametrized by spurion chiral superfields $R_{a}$ which carry charge $a$ under the global symmetry and obtain a vev $\left\langle R_{a}\right\rangle=\lambda_{a} f$, where $\lambda_{\alpha} \ll 1$. This permits a superpotential term $\Delta W=f^{2} \sum_{a} R_{-\alpha} G^{\alpha}$ where $G^{\alpha}=\exp (\alpha A / f)$. Unbroken supersymmetry requires that there are two sources of global symmetry breaking, $R_{-\alpha}$ and $R_{-\beta}$, with opposite charges, $\alpha \beta<0$. This produces a sGoldstone boson vev and an effective superpotential with a common supersymmetry preserving masses $m_{\chi}=m_{a}=m_{s}$. Explicit breaking terms may also generate new interactions which are completely determined by the Goldstone boson mass,

$$
\begin{equation*}
\mathcal{L} \supset-\frac{m_{a}^{2}}{2}\left(a^{2}+s^{2}\right)-\frac{m_{a}}{2} \bar{\chi} \chi+\frac{m_{a}}{2 \sqrt{2} f}(a+\beta)\left(i a \bar{\chi} \gamma^{5} \chi-s \bar{\chi} \chi-m_{a} s a^{2}\right)+\frac{m_{a}}{8 f^{2}}\left(a^{2}+a \beta+\beta^{2}\right) a^{2} \bar{\chi} \chi+\ldots \tag{9.32}
\end{equation*}
$$

[^13]The only model-dependent inputs are the charges $\alpha$ and $\beta$ of the explicit breaking operators. After SUSY breaking, the sGoldstone boson and the Goldstone fermion masses are lifted while the Goldstone boson remains light, $m_{s} \gg m_{\chi}>m_{a}$. Up to integration by parts, the the on-shell trilinear axial coupling $a \bar{\chi} \gamma^{5} \chi$ is equivalent to an effective $b_{1}$ coupling in ( 9.10 ). Finally, the explicit breaking can also generate additional terms in the superpotential of the form

$$
\begin{equation*}
W=c_{i} R_{-a} G^{a}\left(H_{u} H_{d}+\frac{Y_{u}}{M_{u}} \bar{Q} H_{u} u+\frac{Y_{d}}{M_{d}} \bar{Q} H_{d} d+\frac{Y_{l}}{M_{L}} \bar{L} H_{d} e+\text { h.c. }\right) . \tag{9.33}
\end{equation*}
$$

These lead to mixing with the Higgs and decays to SM fermions.

### 9.6 Relic Abundance

The Goldstone fermion $\chi$ is a natural dark matter candidate if it is the LSP and produces the observed abundance [349, 351],

$$
\begin{equation*}
\Omega_{\mathrm{DM}} h^{2}=0.112 \pm 0.0056, \tag{9.34}
\end{equation*}
$$

where $h$ is the Hubble constant. A key observation is that the effective interactions between the Goldstone fermion $\chi$ and Goldstone boson $a$ lead to an annihilation cross section $\chi \chi \rightarrow a a$ of the correct magnitude for a thermal relic with $\mathcal{O}\left({ }_{1}\right)$ couplings and mass at the SUSY breaking scale $M_{\text {susy }}$,

$$
\begin{align*}
\langle\sigma v\rangle & \approx \frac{b_{1}^{4}}{8 \pi} \frac{T_{f}}{m_{\chi}} \frac{m_{\chi}^{2}}{f^{4}} \simeq 1 \mathrm{pb}  \tag{9.35}\\
\Omega_{\mathrm{DM}} h^{2} & \approx \frac{0.1 \mathrm{pb}}{\langle\sigma v\rangle} . \tag{9.36}
\end{align*}
$$

Note that an explicit factor of the temperature appears in (9.35) because parity forbids the $s$-wave channel. Thus the Goldstone fermion is an almost ideal WIMP candidate. Due to the slight thermal suppression, the coupling $b_{1}$ has to be slightly larger than 1. Otherwise, with the natural choices of parameters, the correct annihilation cross section is obtained.

### 9.6.1 SUMMARY OF MODEL PARAMETERS

Below we provide a summary of the Goldstone fermion model parameters and the values used in our parameter space scan:

| Parameter | Description | Scan Range |
| :---: | :---: | :---: |
| $f$ | Global symmetry breaking scale | $500 \mathrm{GeV}-1.2 \mathrm{TeV}$ |
| $m_{\chi}$ | Goldstone fermion mass | $50-150 \mathrm{GeV}$ |
| $m_{a}$ | Goldstone boson mass | $8 \mathrm{GeV}-f / 10$ |
| $b_{1}$ | XXa coupling, (9.13) | [0, 2] |
| $c_{\text {an }}$ | Anomaly coefficient, (9.28) | 0.06 |
| $c_{h}$ | Higgs coupling, (9.16) | $[-1,1]$ |
| $\delta=(\beta-\alpha) / 2$ | Explicit breaking ia $\bar{\chi} \gamma^{5} \chi$ coupling, (9.32) | 3/2 |
| $\rho=\left(\alpha^{2}+\alpha \beta+\beta^{2}\right) / 8$ | Explicit breaking $a^{2} \bar{\chi} \chi \chi$ coupling, (9.32) | 13/8 |

These values represent a natural cross section of the full parameter space.

### 9.6.2 SUMMARY OF ANNIHILATION CHANNELS

In addition to
(a) $\chi \chi \rightarrow a a$ in the $t$-channel and $u$-channel via the self-interactions (9.10),
a detailed analysis shows that there may be appreciable contributions from
(b) $\chi \chi \rightarrow$ aa from explicit breaking terms (9.32);
(c) $\chi \chi \rightarrow g g$ with $a$ in the $s$-channel via the anomaly (9.28).

In fact, these can overcome the $p$-wave suppression in the annihilation into $2 a$. Note that $\chi \chi \rightarrow g g$ also gives an $s$-wave contribution which may contribute up to $\sim 1 / 3$ of the total annihilation cross section. Less significant are the decays into Higgs bosons,
(d) $\chi \chi \rightarrow a h$ with $a$ in the $s$-channel via the Higgs coupling (9.16) when $m_{h}+m_{a}<2 m_{\chi}$;
(e) $\chi \chi \rightarrow h h$ via the coupling to two Higgs bosons (9.15) when $m_{h}<m_{\chi}$.

Note that in some cases the Higgs boson may be lighter than the 115 GeV because of non-standard Higgs decays (see Section 9.8 for the relevant collider phenomenology). Other annihilations involving a virtual gluino, the sGoldstone, or the Higgs boson are typically suppressed by large masses or small Yukawa couplings. A detailed calculation of each contribution is presented in Appendix 9.B. The model generates the correct abundance for Goldstone fermion masses between $50-150 \mathrm{GeV}$ and Goldstone boson masses between $10 \%-100 \%$ of $m_{\chi}$ for couplings $b_{1} \sim \mathcal{O}(1)$. Figure 9.6.1 shows the contours for different values of $m_{a} / m_{\chi}$ subject to the correct relic abundance in the ( $m_{\chi}, b_{1}$ ) plane. It may further be possible to open up a


Figure 9.6.1: Contours for different values of the Goldstone boson mass: $m_{a} / m_{\chi}=0.1$ (black dotted), 0.5 (blue dashed), and 0.7 (red solid) for fixed relic density $\Omega h^{2}=0.11$ in the ( $m_{\chi}, b_{1}$ ) plane. Gray lines include the subleading contributions from annihilations into Higgs bosons, $\chi \chi \rightarrow a h$ and $\chi \chi \rightarrow h h$. The kink at 60 GeV comes from threshold effects. We set $f=700 \mathrm{GeV}, a=-4, \beta=1, m_{h}=116 \mathrm{GeV}, c_{\mathrm{an}}=0.06$.
different region of parameter space with lighter Goldstone boson and fermion scales through a Sommerfeld enhancement due to an attractive force between the exchange of multiple low-energy Goldstone bosons [365]. We briefly discuss this possibility in Appendix 9.C

### 9.7 Direct and Indirect Detection of GFDM

We have seen above that this model can easily produce the correct dark matter abundance. Next we estimate the generic size of direct and indirect detection bounds.

### 9.7.1 Dark Matter Effective Operators for Direct detection

In order to evaluate the cross section of Goldstone fermion scattering off nuclei in direct detection experiments, one must evaluate the nucleon matrix elements. Usually one parametrizes the light quark mass content $f_{i}^{(N)}$ of the nucleons,

$$
\begin{equation*}
m_{i}\langle N| \bar{q}_{i} q_{i}|N\rangle \equiv f_{i}^{(N)} m_{N} \quad i=u, d, s, \quad N=p, n \tag{9.37}
\end{equation*}
$$

where $m_{N}$ is the nucleon mass. The largest contribution comes from the strange quark [366], but with sizeable uncertainties [367]. We assume the default value in the micrOMEGAs code, $f_{u, d}^{(N)} \ll f_{s}^{(p)}=f_{s}^{(n)}=0.26$ [368]. For heavy quarks, the contribution $f_{h}^{(N)}$ is induced via gluon exchange and can be calculated by means of the conformal anomaly [369],

$$
\begin{equation*}
m_{h}\langle N| \bar{q}_{h} q_{h}|N\rangle \equiv f_{h}^{(N)} m_{N}=\frac{2}{27} m_{N}\left(1-\sum_{i=u, d, s} f_{i}^{(N)}\right), \quad h=c, b, t \tag{9.38}
\end{equation*}
$$

## Coupling to quarks via Higgs exchange

In Section 9.3.2 we showed that after electroweak symmetry breaking, it is natural to expect a non-vanishing coupling between $\chi$ and the lightest neutral Higgs boson $h$,

$$
\begin{equation*}
\mathcal{L}_{h}=c_{h} \frac{v_{\mathrm{EW}}}{2 f^{2}}\left(\bar{\chi} i \gamma^{\mu} \partial_{\mu} \chi\right) h \tag{9.39}
\end{equation*}
$$

where the size of the coupling $c_{h}$ depends on the specific realization. The Higgs coupling to nucleons is set by the Yukawa couplings and—in the presence of more Higgses—the mixing angles, $c_{q} m_{q} /\left(\sqrt{2} v_{\mathrm{EW}}\right) h \bar{q} q$. Integrating out the Higgs generates an effective four-Fermi interaction,

$$
\begin{equation*}
\mathcal{L}_{\chi N}^{\mathrm{eff}}=G_{\chi N} \bar{N} N \bar{\chi} \chi \quad G_{\chi N}=c_{h} \frac{\lambda_{N}}{2 \sqrt{2}}\left(\frac{m_{\chi} m_{N}}{m_{h}^{2} f^{2}}\right) \tag{9.40}
\end{equation*}
$$

where we used the equations of motion for $\chi$ and the quark content of the nucleons (9.38) to write

$$
\begin{equation*}
\lambda_{N}=\sum_{q=u, d, s} c_{q} f_{q}^{(N)}+\frac{2}{27}\left(1-\sum_{q=u, d, s} f_{q}^{(N)}\right)\left(\sum_{q^{\prime}=c, b, t} c_{q^{\prime}}\right) \tag{9.41}
\end{equation*}
$$

The resulting scattering cross section per nucleon at zero momentum transfer is ${ }^{3}$

$$
\begin{equation*}
\sigma_{\mathrm{SI}}^{\mathrm{Higgs}}=\frac{4 \mu_{\chi}^{2}}{A^{2} \pi}\left[G_{\chi p} Z+G_{\chi n}(A-Z)\right]^{2} \tag{9.42}
\end{equation*}
$$

where $\mu_{\chi}=\left(m_{\chi}^{-1}+m_{N}^{-1}\right)^{-1}$ is the reduced mass. The typical value for $\sigma_{\mathrm{SI}}^{\text {Higgs }}$ is just below the XENON 1 oo direct detection bound [352],

$$
\begin{equation*}
\sigma_{\mathrm{SI}}^{\mathrm{Higgs}} \approx 3 c_{h}^{2} \times 10^{-45} \mathrm{~cm}^{2}\left(\frac{115 \mathrm{GeV}}{m_{h}}\right)^{4}\left(\frac{700 \mathrm{GeV}}{f}\right)^{4}\left(\frac{m_{\chi}}{100 \mathrm{GeV}}\right)^{2}\left(\frac{\mu_{\chi}}{1 \mathrm{GeV}}\right)^{2}\left(\frac{\lambda_{N}}{0.5}\right)^{2} \tag{9.43}
\end{equation*}
$$

[^14]Note the $\left(m_{\chi} v / f^{2}\right)^{2}$ suppression present in this cross section (due to the Goldstone nature of $\chi$ ) relative to that of a generic Higgs exchange. For example, Higgs-mediated neutralino decay in the MSSM with coupling $\mathcal{L} \approx c g / 2 \bar{\chi} \chi h$ needs a very small coupling $c$ to avoid the XENON 100 bounds:

$$
\begin{equation*}
\sigma_{\mathrm{SI}}^{\mathrm{MSSM}} \sim \frac{c^{2} g^{2}}{2 \pi} \frac{\lambda_{\mathrm{N}}^{2} \mu^{2} m_{N}^{2}}{m_{h}^{4} \nu_{\mathrm{EW}}^{2}} \approx c^{2} \times 10^{-42} \mathrm{~cm}^{2} . \tag{9.44}
\end{equation*}
$$

Thus, Goldstone fermion dark matter offers a natural suppression of the direct detection cross section while retaining the correct WIMP annihilation cross section and abundance. Figure 9.7.1 plots typical values of the direct detection cross section for parameters with correct relic abundance.


Figure 9.7.1: Black line: XENON 100 bound; Left: scan over parameter space with $500<f<700 \mathrm{GeV}$ (red), $700<f<800 \mathrm{GeV}$ (Orange), $800<f<900 \mathrm{GeV}$ (Yellow), $900<f<1000 \mathrm{GeV}$ (Green), $1000<f<$ 1100 GeV (Purple). We scan $0<b_{1}<2,-1<c_{h, h h}<1,8 \mathrm{GeV}<m_{a}<\frac{f}{10}$, $50 \mathrm{GeV}<m_{\chi}<200 \mathrm{GeV}$. Right: Blue points have $0.5<m_{a} / m_{\chi}$ whereas red points have $m_{a} / m_{\chi}<0.5$.

## Coupling to gluons

Integrating out the massive gaugino in (9.28) generates two dimension-7 operators that couple $\chi$ to gauge bosons,

$$
\begin{equation*}
\mathcal{L}_{\text {eff }}^{(1)}=-\left(\frac{c_{\mathrm{an}}^{2}}{2 M_{\lambda} f^{2}}\right)[\bar{\chi} \chi] G_{\alpha \beta}^{a} G_{a \beta}^{a} \quad \mathcal{L}_{\mathrm{eff}}^{(2)}=-i\left(\frac{c_{\mathrm{an}}^{2}}{2 M_{\lambda} f^{2}}\right)\left[\bar{\chi} \gamma^{5} \chi\right] G_{a \beta}^{a} \tilde{G}_{a \beta}^{a}, \tag{9.45}
\end{equation*}
$$

where $M_{\lambda}$ is the gaugino mass. In the limit of zero momentum transfer, only $\mathcal{L}_{\text {eff }}^{(1)}$ contributes to direct detection since $G \tilde{G}$ is a total derivative. We therefore neglect $\mathcal{L}_{\text {eff }}^{(2)}$ hereafter. The $\langle N| G G|N\rangle$ nucleon matrix element can be extracted from the conformal anomaly ( 9.38 ) so that $\mathcal{L}_{\text {eff }}^{(1)}$ can be mapped to a standard four-Fermi operator

$$
\begin{equation*}
\mathcal{L}_{\text {eff }}^{(1)} \longrightarrow \mathcal{L}_{\text {eff }, N}^{(1)}=G_{N} \bar{\chi} \chi \bar{N} N, \quad G_{N}=\frac{4 \pi c_{\mathrm{an}}^{2}}{9 a_{s}} \frac{m_{N}}{M_{\chi} f^{2}}\left(1-\sum_{i=u, d, s} f_{i}^{(N)}\right) \tag{9.46}
\end{equation*}
$$

The corresponding cross section per nucleon at zero momentum transfer is

$$
\begin{equation*}
\sigma_{\mathrm{SI}}^{\mathrm{gg}} \approx 2 \times 10^{-48} \mathrm{~cm}^{2}\left(\frac{700 \mathrm{GeV}}{M_{\lambda}}\right)^{2}\left(\frac{700 \mathrm{GeV}}{f}\right)^{4}\left(\frac{N_{\Psi}}{5}\right)^{4}\left(\frac{q_{\Psi}}{2}\right)^{4}\left(\frac{\mu}{1 \mathrm{GeV}}\right)^{2} \tag{9.47}
\end{equation*}
$$

where $c_{\mathrm{an}}=\alpha_{s} q_{\Psi} N_{\Psi} /(8 \pi)$ has been used. This value is much smaller than both the recent upper bound by the XENON 100 collaboration [352] and the expected reach at the LHC, $\sigma_{g g}^{\mathrm{SI}}=\mathrm{few} \times 10^{-46} \mathrm{~cm}^{2}$ [370].

### 9.7.2 INDIRECT DETECTION

Many experiments are searching for indirect signals of annihilation of dark matter in dense environments such as the galactic center or the solar core. The rate of such events is set by the present-day thermally averaged annihilation cross section. Note, however, that the dominant annihilation channels at freeze-out are $p$-wave and hence are strongly velocity suppressed in the current era. Thus, the relevant annihilation channels for indirect detection are $s$-wave and were sub-dominant at freeze-out. These cross sections are relatively small and astrophysical observations do not impose severe constraints.

## Fermi-LAT: Lines and the isotropic diffuse $\gamma$-RAy spectrum

Dark annihilation in the galactic halo may produce photons either directly (e.g. $\chi \chi \rightarrow \gamma \gamma$ ) or through secondaries (bremsstrahlung off charged products or decays of neutral pions). The Fermi experiment has searched for excesses in the gamma ray spectrum both in the form of lines arising from prompt annihilation to photons and in contributions to the diffuse spectrum from secondary products of annihilation.

Fermi currently searches for $\gamma$-ray lines from $30-200 \mathrm{GeV}$ [371], with upcoming bounds that are an order of magnitude stronger in the $7-30 \mathrm{GeV}$ region [372]. The lack of a bump in the Fermi data implies an upper bound on $\langle\sigma v\rangle_{\gamma \gamma}$ between $(0.2-2.5) \cdot 10^{-27} \mathrm{~cm}^{3} / \mathrm{s}$ when using the Einasto dark matter halo profile which predicts the largest photon flux among those examined in the Fermi analysis.

In the Goldstone fermion model, prompt annihilation to photons occurs through an anomaly vertex similar to the one which mediates $a \rightarrow g g$. This rate depends on the $\mathrm{U}(1) \times \mathrm{U}(1)_{\mathrm{EM}}^{2}$ anomaly coefficient which is determined by the choice of electric charges for fields carrying global charge. The cross section for annihilation into gluons is given in (9.54). The analagous expression for annihilations into photons is given by replacing $a_{s}^{2} N_{c} \rightarrow a_{\mathrm{EM}}^{2}\left(2 \sum_{i}\left(q_{\mathrm{EM}}^{i}\right)^{2}\right)^{2}$. For the case where the $\Psi$ are taken to be in the (anti-)fundamental of an $\operatorname{SU}(5)$ unified group, we find $\langle\sigma v\rangle_{\gamma \gamma} \sim 2 \cdot 10^{-3}\langle\sigma v\rangle_{\mathrm{gg}}$. Even with the most extreme choices of the model's free parameters, this rate remains more than an order of magnitude smaller than the Fermi bounds.

Fermi has also measured the isotropic diffuse $\gamma$-ray spectrum in the range $20-100 \mathrm{GeV}$ [373]. This bounds the annihilation of dark matter into charged particles and neutral pions. For example, for a 400 GeV dark matter particle which annihilates into a $b \bar{b}$ pair, Fermi sets a bound on $\langle\sigma v\rangle_{b \bar{b}}$ which is roughly an order of magnitude above the cross section required to reproduce the right relic abundance. The Goldstone fermion model generates diffuse photons primarily through annihilation to gluons produced in the $s$-wave annihilation channel $\chi \chi \rightarrow g g$. However, this cross section is at least an order of magnitude smaller than the Fermi bound and hence the Fermi diffuse $\gamma$-ray data do not constrain this model.

Preliminary results from a Fermi analysis of 10 dark-matter-rich dwarf spheroidal galaxies also place limits on photo-production from dark matter annihilation [374]. For low-mass ( $\lesssim 60 \mathrm{GeV}$ ) dark matter annihilating into $b \bar{b}$ pairs, constraints on the annihilation rate extend slightly below the thermal relic rate of $3 \cdot 10^{-26} \mathrm{~cm}^{3} / \mathrm{s}$, with the strongest constraint of $\sim 1 \cdot 10^{-26} \mathrm{~cm}^{3} / \mathrm{s}$ at $m_{\chi}=10 \mathrm{GeV}$. In this mass window and for reasonable parameter choices, the Goldstone fermion annihilation cross section is always at least a factor of 3 lower than these limits.

Other constraints, such as those that come from $\gamma$-rays originating in clusters of galaxies, typically set weaker bounds [375].

## PAMELA: THE ANTIPROTON FLUX

PAMELA has recently published data on the absolute cosmic ray antiproton flux from $60 \mathrm{MeV}-180 \mathrm{GeV}$ [377]. This places constraints on dark matter models with a substantial annihilation rate to hadrons. For a 100 GeV WIMP, the annihilation cross section to $Z s, W s$, and $b$ quarks has an upper bound comparable to the rate required for the observed relic abundance,
$\langle\sigma v\rangle_{\text {relic }} \sim 3 \cdot 10^{-26} \mathrm{~cm}^{2} / \mathrm{s}$ [378]. For Goldstone fermion dark matter, the dominant annihilation channel in the galactic halo,


Figure 9.7.2: Antiproton flux at the Earth for $f=700 \mathrm{GeV}, Q_{\Psi}=2, \delta=3 / 2, N_{\Psi}=5$ at fixed density $\Omega h^{2} \simeq$ o.1. Red (blue) lines represent the propagation parameters MAX (MIN) used in [376] with the Einasto DM halo profile. The dots represent the PAMELA data [377]. Left: $m_{a} / m_{\chi}=0.5$ at $m_{\chi}=50 \mathrm{GeV}$ and $b_{1}=3$ (solid); $m_{\chi}=100 \mathrm{GeV}$ and $b_{1}=1.5$ (dashed); $m_{\chi}=150 \mathrm{GeV}$ and $b_{1}=1$ (dotted). Right: $m_{a} / m_{\chi}=0.8$ at $m_{\chi}=50 \mathrm{GeV}$ and $b_{1}=2.5$ (solid); $m_{\chi}=100 \mathrm{GeV}$ and $b_{1}=1.2$ (dashed); $m_{\chi}=150 \mathrm{GeV}$ and $b_{1}=0.5$ (dotted).
$\chi \chi \rightarrow g g$, is $s$-wave. This has a typical cross section of $\langle\sigma v\rangle \sim 10^{-27} \mathrm{~cm}^{3} / \mathrm{s}$ and can be pushed up as high as $10^{-26} \mathrm{~cm}^{3} / \mathrm{s}$. Using recent numerical recipes [376], one may estimate the anti-proton flux as a function of the thermally averaged annihilation cross section and the Goldstone fermion mass. This is depicted in Figure 9.7.2 for different model parameters that yield the correct relic abundance. The anti-proton flux varies considerably as a function of the galactic propagation parameters and the halo profile. The solid, dashed, and dotted curves each correspond to different underlying Goldstone fermion model parameters. Choosing different halo profiles and propagation parameters leads to a spread in the predicted $\bar{p}$ flux such that for each choice of Goldstone fermion model parameters, the actual flux from dark matter annihilation is expected to lie between the two solid, dashed, or dotted curves respectively.

For each choice of model parameters, there is a sizeable region where the predicted flux from dark matter annihilation lies well below the measured anti-proton flux. Thus the PAMELA data do not place significant constraints on the Goldstone fermion dark matter model.

### 9.8 Collider Phenomenology

In addition to (in)direct detection, Goldstone fermion models lend themselves to novel collider signatures coming from the Goldstone supermultiplet. As discussed in Section 9.4, the sGoldstone $s$ is typically heavy with small couplings to the SM sector so we may neglect its collider signatures.

### 9.8.1 COLLIDER SIGNALS OF DARK MATTER

The most direct way of testing the dark matter annihilation mechanism is through dark matter pair production coming, for example, from the $\bar{\chi} \chi G G$ operator in ( 9.45 ). One signature of this process at colliders is a monojet coming from hard initial state QCD radiation. For a range of masses up to the TeV scale, the LHC will set the most stringent bound on this operator with a sensitivity of $\sigma_{\mathrm{SI}}^{N} \sim 10^{-46}-10^{-45} \mathrm{~cm}^{2}$ for a $5 \sigma$ discovery with $100 \mathrm{fb}^{-1}$ [370]. The effective scale that suppresses the dimension-7 operator ( 9.45 ) is roughly an order of magnitude larger than the LHC reach. However, the process $g g \rightarrow a^{*} \rightarrow \chi \chi$ via an off-shell Goldstone boson gives a larger contribution and results in a naive effective scale $M^{*} \sim\left(m_{x} f^{\prime} / c_{\text {an }}\right)^{1 / 3} \sim 1 \mathrm{TeV}$. This is in the ballpark of the LHC $5 \sigma$ reach given in [370].

Goldstone fermion dark matter can also be produced from the cascade decay of heavier $R$-parity odd particles, such as gluinos or squarks. Due to the small coupling between the MSSM and the Goldstone sectors, the cascade decays will all go through the lightest ordinary supersymmetric particle (LOSP). The decay of LOSP to the Goldstone fermion is determined by the operators connecting the two sectors. In the current setup, there are two types of interactions:

- the anomaly induced coupling $\bar{\chi} G \lambda$, as in (9.28), and
- the kinetic mixing discussed in Section 9.3.2 .

The details of the decay modes depends on the nature of the LSP. For example, a bino-like LOSP will decay to the LSP via the anomaly, $\tilde{B} \rightarrow \chi+\gamma / Z$. A Higgsino-like LOSP would decay instead to the LSP because of the kinetic mixing, $\tilde{h} \rightarrow \chi+h$, and $\tilde{h} \rightarrow \chi+a \rightarrow \chi+2 j$. In the latter decay mode, the reconstruction of the Goldstone boson resonance in the jet final state is difficult if $a$ is below 100 GeV , but it may be possible instead in the diphoton decays of $a$ with sufficient luminosity. These channels yield prompt decays even though they may be suppressed by loops or small mixing angles. For example, the natural width for a pure bino LOSP is around $10^{-5} \mathrm{GeV}$.

Finally, the presence of exotic heavy fermions $\Psi_{i}$ also has interesting implications at colliders. These fermions may be considered to be "fourth generation" quarks which, if they are sufficiently light, can be probed at the early stages of the LHC (see the discussion in [362] for an example).

### 9.8.2 Non-standard Higgs boson decays



Figure 9.8.1: Plots of Higgs boson branching ratios for various parameters.

The largest natural coupling of the Goldstone boson and fermion to the SM is through Higgs boson via the kinetic terms, (9.16). This coupling allows the Higgs to decay into $2 a$ or $2 \chi$ if kinematically allowed. Typical branching ratios are plotted in Figure 9.8.1.

The Higgs boson decay $h \rightarrow 2 a$ gives rise to four light, unflavored jets coming from $a \rightarrow 2 g$. This decay mode is therefore easily 'buried' under the QCD background. Such non-standard Higgs boson decays have recently been investigated in SUSY models where the Higgs boson itself is also a pseudo-Goldstone boson emerging from the spontaneous breaking of a global symmetry ${ }^{4}$. In particular, the spontaneous breaking of $S U(3) \rightarrow S U(2)$ gives rise both to a light Goldstone multiplet $A$ and a light Higgs multiplet [361]. The resulting coupling $c_{h} \approx \sqrt{2}$ is set by the kinetic mixing between the two multiplets which, in turn, is fixed by the scale $f$ of the global symmetry breaking. A more recent example of a 'buried Higgs' in SUSY has been discussed in the context of a spontaneously broken $U(1)$ symmetry where $c_{h}$ depends on couplings in the superpotential because the Higgs is no longer a pseudo-Goldstone boson [362].

Even though these non-standard Goldstone fermion decay modes can dominate, the branching ratio to SM particles is still larger than $\sim 20 \%$ at low Higgs masses and therefore the LEP bound on the Higgs mass cannot be lowered below $\sim 110 \mathrm{GeV}$. Furthermore, while the discovery of a completely buried Higgs is challenging at the LHC [381], this 'partially buried' Higgs would be discovered in SM channels with a missing piece in the total width. The invisible Higgs boson decays ( $\chi$ s leave the detectors) can be probed at the LHC through the missing energy signal [382]. Both the buried and invisible decay modes may have sizeable branching ratios, and the observation of both channels would give strong evidence for this scenario.

### 9.9 CONCLUSIONS

Acceptable dark matter scenarios within the MSSM must become increasingly contrived as the sensitivity of direct detection experiments increases. In order to remain consistent with recent XENON 100 results, neutralino WIMP models must typically invoke accidental mass relations to boost the annihilation cross-section through co-annihilations or strategically placed resonances.

Inspired by this tension, we have explored a general supersymmetric framework compatible with GUT unification in which the LSP is the fermionic component $\chi$ of a Goldstone supermultiplet associated with a $\mathrm{U}(1)$ global symmetry that is spontaneously broken at the TeV scale. Because the Goldstone fermion's couplings to the Standard Model are suppressed by $\sim v_{\mathrm{EW}} m_{\chi} / f^{2}$ (and additional loop factors in some cases), these models are able to avoid direct detection constraints from XENON 100 and indirect detection constraints from Fermi and PAMELA.

The annihilation cross section of a weak-scale Goldstone fermion at freeze out is on the order of 1 pb , with dominant contributions coming from $p$-wave annihilation into Goldstone bosons. Typically subdominant $s$-wave annihilations into gluons arise through anomalies of the new global symmetry. The observed dark matter relic density is obtained with natural values for the model parameters.

This class of models also offers novel and distinctive signatures at colliders. Goldstone fermions can be produced at the LHC in pairs through the anomalous coupling to gluons, leading to monojet signals when there is additional hard QCD radiation from the initial state. Additionally, SUSY cascades are modified by decays of the NLSP to the Goldstone fermion. Examples include the bino decay to a photon and the Goldstone fermion, and the higgsino decay to the Goldstone fermion and the Goldstone boson. The Goldstone multiplet also modifies the phenomenology of the Higgs sector. Interactions with the Goldstone boson allow cascade decays of the Higgs to four jets, $h \rightarrow 2 a \rightarrow 4 j$, analogous to models where the Higgs decays are 'buried' under the QCD background. If kinematically allowed, the Higgs may also have a sizeable fraction of 'invisible' decays, $h \rightarrow \chi \chi$.

## 9.A Explicit models

We present explicit models to demonstrate how one may generate different values of the coupling $b_{1}$, defined in (9.13). In their simplest form, both examples have an unbroken $R$-symmetry which implies that only the $A$-terms generate a mass for the

[^15]Goldstone fermion. It is straightforward to modify these examples to explicitly break the $R$-symmetry without modifying the structure of these theories.

## 9.A. 1 The simplest example

We consider a simple variation of the model considered in [362] with the superpotential $W=y S\left(\bar{N} N-\mu^{2}\right)$. This gives

$$
\begin{equation*}
K=f_{N}^{2} e^{\left(A+A^{\dagger}\right) / f}+f_{\bar{N}}^{2} e^{-\left(A+A^{\dagger}\right) / f} \quad f^{2}=f_{N}^{2}+f_{\bar{N}}^{2} \tag{9.48}
\end{equation*}
$$

so that the tree-level range for $b_{1}$ is

$$
\begin{equation*}
-1 \leq b_{1}=\frac{f_{N}^{2}-f_{\bar{N}}^{2}}{f_{N}^{2}-f_{\bar{N}}^{2}} \leq 1 \tag{9.49}
\end{equation*}
$$

## 9.A. 2 AN EXAMPLE WITH $\left|b_{1}\right| \geq 1$

A perturbative model that may give $\left|b_{1}\right| \geq 1$ is the following:

$$
\begin{equation*}
W=\lambda X Y Z-\mu^{2} Z+\frac{\tilde{\lambda}}{2} Y^{2} N-\tilde{\mu} \bar{N} N \tag{9.50}
\end{equation*}
$$

where the charges are $q_{Z}=0, q_{N}=-q_{\bar{N}}=-2 q_{Y}=2 q_{X}$ and all couplings and masses are non-zero. The resulting supersymmetric minimum

$$
f_{X} f_{Y}=\mu^{2} / \lambda, \quad f_{Z}=f_{N}=0, \quad f_{\bar{N}}=\tilde{\lambda} \frac{f_{Y}^{2}}{2 \tilde{\mu}}
$$

gives vanishing F-terms while the Goldstone chiral multiplet is

$$
\begin{equation*}
A=\sum_{i} \frac{q_{i} f_{i} \psi_{i}}{f}=\frac{q_{Y}}{f}\left(Y f_{Y}-X f_{X}+2 \bar{N} f_{\bar{N}}\right) \quad \quad f^{2}=q_{Y}^{2}\left(f_{Y}^{2}+f_{X}^{2}+4 f_{\bar{N}}^{2}\right) \tag{9.52}
\end{equation*}
$$

The corresponding $b_{1}$ at tree-level is given by

$$
\begin{equation*}
b_{1}=\frac{1}{f^{2}}\left(\sum_{i} q_{i}^{3} f_{i}^{2}\right)=\frac{-f_{X}^{2}+f_{Y}^{2}+8 f_{\bar{N}}^{2}}{f_{X}^{2}+f_{Y}^{2}+4 f_{\bar{N}}^{2}} \tag{9.53}
\end{equation*}
$$

which goes to $b_{1} \rightarrow 2$ when $f_{\bar{N}} \gg f_{X, Y}$.

## 9.B Annihilation cross section

Diagrams for the dominant annihilation channels are presented in Figure 9.B.1.

## 9.B. $1 \quad \chi \chi \rightarrow g g$

The annihilation cross section to gluons (see Figure 9.B.1a) is controlled by the anomalous coupling (9.28) where $c_{\mathrm{an}}=a_{s} q_{\Psi} N_{\Psi} /(8 \pi)$ and the vertex $b_{1} /(2 \sqrt{2} f) \bar{\chi} \gamma^{\mu} \gamma^{5} \chi \partial_{\mu} a$. Away from resonance one finds

$$
\begin{equation*}
\sigma v=\frac{2 a_{s}^{2}}{(8 \pi)^{3}} N_{c} N_{\Psi}^{2}\left(b_{1} m_{\chi}+\delta m_{a}\right)^{2} \frac{s^{2} q_{\Psi}^{2}}{\left(s-m_{a}^{2}\right)^{2} f^{4}} \quad s=\left(p_{1}+p_{2}\right)^{2}=4 E_{\chi}^{2} \tag{9.54}
\end{equation*}
$$



Figure 9.B. 1 : Goldstone fermion annihilation channels.
where $v$ is the relative velocity in the center of mass frame, $N_{c}=8$ is the number of colors in the final state, and $\delta=-(\alpha+\beta) / 2$ is the contribution from the explicit breaking vertex (9.32). Note that this process gives a non-vanishing $s$-wave annihilation component.

## 9.B. $2 \quad \chi \chi \rightarrow a a$

Annihilation into Goldstone bosons proceeds through $t$ - and $u$ - channel diagrams (see Figs. 9.B.1b-9.B.1c) as well as a contact interaction coming from explicit breaking (see Figure 9.B.1d).

## $t$ - AND $u$-CHANNEL

These diagrams give a $p$-wave contribution to the cross section, $\sigma v=a+b v^{2}+\ldots$,

$$
\begin{align*}
a= & 0 \quad z=m_{a} / m_{\chi}  \tag{9.55}\\
b= & \frac{m_{\chi}^{2}}{96 \pi f^{4}\left(z^{2}-2\right)^{4}}\left[b_{1}^{3}\left(b_{1}+4 z \delta\right)\left(3 z^{8}-16 z^{6}+48 z^{4}-64 z^{2}+32\right)\right.  \tag{9.56}\\
& \left.+z^{2} \delta^{2}\left(3 z^{8}-14 z^{6}+46 z^{4}-64 z^{2}+32\right)+16 b_{1} \delta^{3}\left(z^{2}-1\right)\left(b_{1} z^{3}+\delta\left(z^{2}-1\right)\right)\right]
\end{align*}
$$

where $\delta=-(\alpha+\beta) / 2$ is the contribution from the explicit breaking vertex (9.32).

## Explicit breaking vertex

The quartic contribution to the annihilation cross section is also $p$-wave is

$$
\begin{equation*}
\sigma v=\frac{1}{128 \pi} \rho^{2} \frac{m_{a}^{2}}{f^{4}} v_{a}\left(\frac{s-4 m_{\chi}^{2}}{s}\right) \quad v_{a}=\sqrt{1-\frac{4 m_{a}^{2}}{s}} \tag{9.57}
\end{equation*}
$$

where $\rho$ is given in terms of the charges of the explicit breaking operators (9.32), $\rho=\alpha^{2}+\alpha \beta+\beta^{2}$.

## Interference

The contact interaction interferes with the $t$ - and $u$-channel diagrams. Summing the amplitudes and then squaring gives,

$$
\begin{equation*}
b=\frac{m_{\chi}^{2} b_{1}^{2}}{96 \pi f^{4}}\left(2 b_{1}^{2}+8 b_{1} z \delta+z \rho\right)+\frac{m_{\chi}^{2} z^{2}}{1536 \pi f^{4}}\left(3 \rho^{2}+32 b_{1} \delta \rho+128 b_{1}^{2} \delta^{2}-16 b_{1}^{4}\right)+\mathcal{O}\left(z^{3}\right), \quad a=0 \tag{9.58}
\end{equation*}
$$

where $\sigma v=a+b v^{2}+\ldots$ and $z=m_{a} / m_{\chi}$. Note that for all plots in this document we use the full expression for $b$ that is valid for all $z \leq 1$.

## 9.B. 3 Subleading processes

The annihilations to a single Higgs (Figure 9.B.1e) and to two Higgses (Figure 9.B.1f) are subdominant.
$\chi \chi \rightarrow a^{*} \rightarrow a h$
This channel is available when $2 m_{\chi}>m_{a}+m_{h}$. Naively, it should be less important because the cross section has an extra suppression by $\left(v_{\mathrm{EW}} / f\right)^{2}$. On the other hand, this is an $s$-wave contribution and therefore the effect is not completely negligible compared to the $\chi \chi \rightarrow$ aa $p$-wave process. The cross section is given by

$$
\begin{equation*}
\sigma v=\frac{v_{a}}{32 \pi} \frac{\left(b_{1} m_{\chi}+\delta m_{a}\right)^{2} c_{h}^{2} v_{\mathrm{EW}}^{2}}{f^{6}}\left(\frac{m_{a}^{2}-m_{h}^{2}+s}{s-m_{a}^{2}}\right)^{2} . \tag{9.59}
\end{equation*}
$$

$\chi \chi \rightarrow h h$
This channel is allowed when $m_{\chi}>m_{h}$, up to thermal contributions. Because Higgs can be buried under QCD, it is possible to have $m_{h} \sim 90 \mathrm{GeV}$. This process is generated from the contact interaction term $c_{h h}\left(\bar{\chi} i \gamma^{\mu} \partial_{\mu} \chi\right) h^{2} /\left(2 f^{2}\right)$, which follows from the $c_{2}, d_{2}$ coefficients in the Kähler potential.

$$
\begin{equation*}
\sigma v=\frac{v_{a}}{8 \pi} \frac{m_{\chi}^{2} c_{h h}^{2}}{f^{4}}\left(\frac{s-4 m_{\chi}^{2}}{s}\right) \tag{9.60}
\end{equation*}
$$

This is a $p$-wave process.

## 9.C Sommerfeld enhancement from Goldstone boson exchange

Thus far we have calculated the relic density assuming no enhancement due to long-range forces. Here we briefly present the non-relativistic potential between the Goldstone fermions and argue that there could be regions of parameter space with a sizeable Sommerfeld enhancement in the annihilation cross section due to an attractive force between the Goldstone fermions due to the exchange of multiple low-energy Goldstone bosons [365], as depicted in Figure 9.C.1. It is thereby possible to lower the Goldstone boson and fermion mass scales. We emphasize that this enhancement is not necessary to obtain the
correct abundance and sufficiently low direct detection cross sections, but it may open up a different region of the parameter space where the Goldstone fermion mass in the $10-50 \mathrm{GeV}$ range.


Figure 9.C.1: Exchange of multiple soft Goldstone bosons can lead to an attractive force enhancing the annihilation cross section for the Goldstone fermions.

In the non-relativistic limit, the $\chi_{1} \chi_{2} \rightarrow \chi_{1^{\prime}} \chi_{2^{\prime}}$ scattering amplitude gives rise to a spin-spin interaction. The low-energy potential can be written in terms of a traceless tensor and a central piece:

$$
\begin{equation*}
V(r)=V_{\mathrm{T}}(r)\left({ }_{3} \mathbf{S}_{1} \cdot \hat{r} \mathbf{S}_{2} \cdot \hat{r}-\mathbf{S}_{1} \cdot \mathbf{S}_{2}\right)+V_{\mathrm{C}}(r) \mathbf{S}_{1} \cdot \mathbf{S}_{2}, \tag{9.61}
\end{equation*}
$$

where the coefficients are

$$
\begin{equation*}
V_{\mathrm{T}}(r)=\frac{b_{1}^{2}}{8 \pi f^{2}}\left(\frac{1}{r^{3}}+\frac{m_{a}}{r^{2}}+\frac{1}{3} \frac{m_{a}^{2}}{r}\right) e^{-m_{a} r} \quad V_{\mathrm{C}}(r) \quad=\frac{b_{1}^{2}}{8 \pi f^{2}} \frac{1}{3} \frac{m_{a}^{2}}{r} e^{-m_{a} r} \tag{9.62}
\end{equation*}
$$

Note that the leading term for distances $r<m_{a}^{-1}$ is contained in the tensor potential. For total spin $S=0$, the tensor potential averages out to zero, whereas the central part gives an attractive interaction which is independent of the orbital angular momentum

$$
\begin{equation*}
\langle S=0, \ell| V(r)|S=0, \ell\rangle=-\frac{3}{4} V_{\mathrm{C}}(r) . \tag{9.63}
\end{equation*}
$$

This contribution vanishes in the limit $m_{a} \rightarrow 0$ in agreement with [383]. For $S=1, \ell=1$ the central potental is repulsive whereas the tensor is attractive. The net effect is an attractive potential ${ }^{5}$

$$
\begin{equation*}
\langle S=1, \ell=1| V(r)|S=1, \ell=1\rangle=\left(\frac{1}{20}-\frac{1}{4}\right) V_{\mathrm{T}}(r)+\frac{1}{4} V_{\mathrm{C}}(r) . \tag{9.64}
\end{equation*}
$$

The magnitude of this Sommerfeld enhancement was calculated in detail for $s$-wave annihilation processes in [?], where it was found to take values as large as 1000 and as small as 0.1 . For the current model one would only need a factor of few to lower the Goldstone boson and fermion masses to the 10 GeV range. Since most of the leading annihilation channels relevant to this class of models are $p$-wave, the results of [?] are not directly applicable. A dedicated calculation is left for future work.

[^16]
# 10 

# The Effective Non-relativistic Theory of Self-Interacting Dark Matter 

While we know that dark matter does not interact with ordinary matter appreciably, it is possible that dark matter interacts very non-trivially amongst itself. One result of such a scenario would be a modification on the rate at which certain interactions between dark matter and the Standard Model occur. In this chapter we explore the properties of these self interacting effects.

### 10.1 OvERVIEW

We present an effective non-relativistic theory of self-interacting dark matter. We classify the long range interactions and discuss how they can be generated from quantum field theories. Generic dark sectors can generate singular potentials. We show how to consistently renormalize such potentials and apply this to the calculation of the Sommerfeld enhancement of dark matter interactions. We explore further applications of this enhancement to astrophysical probes of dark matter including the core vs. cusp problem.

### 10.2 INTRODUCTION

Less than a quarter of the matter density of the universe is composed of ordinary baryons. The remaining component is called dark matter (DM) and has only been probed through its gravitational interactions at cosmological and astrophysical scales. One appealing class of DM candidates are weakly-interacting massive particles (wIMPs). These are

- stable or long-lived compared to the age of universe
- non-relativistic upon freeze out from thermal equilibrium in the early universe
- electrically neutral and weakly interacting, i.e. with annihilation cross sections in the pb range, so that $\Omega_{\mathrm{DM}} h^{2} \approx 0.1 \mathrm{pb} /\langle\sigma v\rangle$.

These features hint at a possible link between the cosmological properties of DM and the mechanism for electroweak symmetry breaking.

In principle, wIMP annihilations should still occur today in dense regions of our galaxy. The potential for this type of indirect detection has gained attention recently due to possible anomalies in the positron fluxes measured by pamela [98], Fermi [99] and ams-02 [100], and the gamma ray spectrum measured by Fermi [101-105]. Such signals, however, require total wIMP annihilation cross sections well in excess of the thermal value. Nevertheless, there are mechanisms to boost the annihilation rate to the level of experimental sensitivity without spoiling the relic abundance. One possibility is that dm has long range self-interactions mediated by a light force carrier. If this exchange of particles produces an attractive self-interaction, it can effectively increase the annihilation cross section because of Sommerfeld enhancement or resonance scattering [383-389]. The annihilation cross section is thus enhanced by a boost factor, $S \sigma_{\circ}$, with $S \geq 1$, where $\sigma_{\circ}$ is the short-range annihilation cross section.

More recently, self-interacting [390,391] DM has also recently been proposed as a viable solution to possible discrepancies between observations of small scale structures and the predictions from $N$-body simulations based on collisionless cold dM. In particular, dwarf galaxies show flat DM density profiles in halo cores [392,393], whereas collisionless cold DM predicts cusp-like profiles. In addition to this "core vs. cusp problem", there is the "missing satellites problem" and the "too big to fail problem," see e.g. [116] and references therein. While it is possible that these problems could be addressed by including baryonic physics to collisionless DM simulations [394], self-interacting DM offers a viable and motivated alternative scenario that is rich of interesting observational consequences [392,393].

The standard approach to self-interactions and Sommerfeld enhancement is to assume an ultra-light elementary scalar or vector $\varphi$ in the dark sector which mediates a force between the DM particles [ $115,116,395$ ]. In this paper we take a more agnostic approach; we construct an effective theory that only assumes rotationally invariant self-interactions in the dark sector. One can classify the possible potentials in terms of the dm mass $m_{\chi}$, spin $\mathbf{s}$, transferred momentum $\mathbf{q}$, and relative velocity $\mathbf{v}$. We work at the leading order in the exchanged momentum and velocity which is an excellent approximation for cold DM. For example, we show in Section 10.3.2 that the most general long-range $P$ - and $T$-symmetric potential between two DM particles of arbitrary spin, is

$$
V_{\mathrm{eff}}^{P, T}=\frac{1}{4 \pi r}\left[\tilde{g}_{1}(r)+\tilde{g}_{2}(r)\left(\mathbf{s}_{1} \cdot \mathbf{s}_{2}\right)+\frac{\tilde{g}_{3}(r)}{\Lambda^{2} r^{2}}\left(3 \mathbf{s}_{1} \cdot \hat{r} \mathbf{s}_{2} \cdot \hat{r}-\mathbf{s}_{1} \cdot \mathbf{s}_{2}\right)+\frac{\tilde{g}_{7,8}(r)}{\Lambda r}\left(\mathbf{s}_{1} \pm \mathbf{s}_{2}\right)(\hat{r} \times \mathbf{v})\right]
$$

where $\tilde{g}_{i}(r)$ are arbitrary functions that depend only on the the DM separation, and $\Lambda$ is the characteristic interaction scale of the dark sector that we take much larger than the mediator mass. At scales where the mediator mass can be neglected and the theory is weakly coupled, the couplings $\tilde{g}_{i}$ freeze to constants, $\tilde{g}_{i}(r) \rightarrow g_{i}$.

Strongly interacting mediators in the dark sector can generate singular potentials through non-standard propagators, see e.g. [396,397]. Notice, however, that even weakly coupled models can generate potentials that are more singular than the $1 / r^{2}$ centrifugal barrier at short distances. For example, dark matter interactions mediated by a light pseudo-scalar produce a $g_{3}$ term in the potential (10.1) which goes like $1 / r^{3}$. This can be generated, for example, by Goldstone bosons [4]. Another example is DM with dipole interactions generated by charged states. These produce a $g_{3}$ term in the potential. Models based on these magnetic dipole interactions were recently proposed [398] as a way to resolve discrepancies between tentative signals in direct detection experiments. More exotic potentials can be generated by the loop-level exchange of composite operators made of light fields [399-402]. Table 10.1 shows examples of weakly coupled models, preserving $P$ and $T$, that generate the various $g_{i}$ in (10.1).

Such singular potentials must be regularized at short distances and then renormalized by requiring that low-energy observables are cutoff independent. We carry out this renormalization program making possible to extract physical predictions from singular potentials generated by DM self-interactions. In particular, we calculate the Sommerfeld enhancement from a $1 / r^{3}$ potential, extending the analysis in [365] by including wavefunction renormalization ${ }^{1}$. We plot the elastic scattering cross section as a function of the velocity and the mass near the resonance region where the boost factor is large. Astrophysical systems at various scales, from clusters to dwarf galaxies with velocity ranging from $v \sim 10^{-3}$ and $v \sim 10^{-5}$, provide

[^17]| Interaction | $g_{1}$ | $g_{2}$ | $g_{3}$ | $g_{7}$ | $g_{8}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\bar{\chi} \chi \phi$ | $\checkmark$ | X | X | $\checkmark$ | X |
| $\bar{\chi} \gamma^{5} \chi \phi$ | X | X | $\checkmark$ | X | X |
| $i \bar{\chi} \gamma^{\mu} \gamma^{5} \chi \partial_{\mu} \phi$ | X | X | $\checkmark$ | X | X |
| $\bar{\chi} \gamma^{\mu} \chi A_{\mu}$ | $\checkmark$ | X | X | $\checkmark$ | X |
| $i \bar{\chi} \gamma^{5} \gamma^{\mu} \chi A_{\mu}$ | X | X | $\checkmark$ | X | X |
| $i \bar{\chi} \sigma^{\mu \nu} \chi F_{\mu \nu}$ | X | X | $\checkmark$ | X | X |

Table 10.1: Leading order $P$ - and $T$-preserving long-range static potentials in (10.1) from massless real scalar $\phi$, vector gauge boson $A_{\mu}$, or field strength $F_{\mu \nu}=\partial_{[\mu} A_{\nu]}$ mediators. Observe that $g_{2}$ is not generated in the massless limit. $g_{8}$ is not generated because of the spin conservation in $C P$-symmetric theories of spin- $\frac{1}{2}$ DM. See Table 10.1 and 10.2 for more details.
constraints on the DM self-interactions and hence the Sommerfeld enhancement [115, 116,395]. While we leave an investigation of how these bounds may be adapted to singular potentials for future work, we point out that the formalism presented here may be useful to avoid these constraints because of the velocity dependence of the elastic cross section.

Even though Sommerfeld enhancement is typically relevant only for $s$-wave annihilations due to the centrifugal barrier, the self-interacting DM potential (10.1) does not generically conserve orbital angular momentum $\mathbf{L}^{2}$. Interaction channels with different orbital angular momenta, $\ell$, can be coupled. This explains why the $g_{3}$ term in (10.1), which would be averaged to zero because of isotropy of $\ell=0$ states, can still be relevant for Sommerfeld enhancement in $\Delta \ell=2$ transitions [365]. Moreover, spin-spin interactions with $g_{3} \neq \mathrm{o}$ in (10.1) may generate macroscopic long range interactions when the DM spins are polarized (in average) [402], a condition that on galaxy scales may be plausible for these $\mathbf{L}$-violating interactions.

This chapter is organized as follows. In Section 2 we derive an effective long-range, non-relativistic potential for self-interacting dark matter at leading order in wIMP velocity. In Section 3 we present a procedure to renormalize singular potentials and apply this to the calculation of the physical, cutoff-independent Sommerfeld enhancement. In Sections 4 and 5 we present numerical results for a $1 / r^{3}$ potential and discuss the types of astrophysical bounds that such an analysis may be applied to. We conclude in Section 6 and include appendices reviewing the standard procedure for calculating Sommerfeld enhancement for non-singular potentials and a convenient square well approximation for singular potentials.

### 10.3 EfFECTIVE LONG-RANGE POTENTIAL

The elastic scattering amplitude $\mathcal{M}$ from rotationally invariant DM self-interactions is a scalar function of the spins $\mathbf{s}_{i}$, exchanged momentum $\mathbf{q}$, and relative velocity $\mathbf{v}$. It is often convenient to use the Hermitian operators $i \mathbf{q}$ and the velocity transverse to the momentum transfer,

$$
\begin{equation*}
\mathbf{v}_{\perp}=\mathbf{v}-\frac{\mathbf{q}(\mathbf{q} \cdot \mathbf{v})}{\mathbf{q}^{2}}=\mathbf{v}+\mathbf{q} / m_{\chi} \tag{10.2}
\end{equation*}
$$

where the last equality follows from the four-momentum conservation.
In the center of mass frame, the elastic scattering amplitude is

$$
\begin{equation*}
\mathcal{M}=\frac{-1}{\mathbf{q}^{2}+m_{\varphi}^{2}} \sum_{i} g_{i}\left(\mathbf{q}^{2} / \Lambda^{2}, \mathbf{v}_{\perp}^{2}\right) \mathcal{O}_{i}\left(\mathbf{s}_{j} \cdot i \mathbf{q} / \Lambda, \mathbf{s}_{j} \cdot \mathbf{v}_{\perp}, \mathbf{s}_{1} \cdot \mathbf{s}_{2}\right) \tag{10.3}
\end{equation*}
$$

where $\Lambda$ is the heavy scale of the dark sector, e.g. the Dm mass $m_{\chi}$, and $\mathcal{O}_{i}$ are the spin matrix elements. We explicitly pull out a factor associated with the propagator for the light force carrier with mass $m_{\varphi}^{2} \ll \mathbf{q}^{2} \ll \Lambda^{2}$ which acts as an infrared (IR) regulator at large distances. Further, we only consider the leading term in the exchanged momentum $\mathbf{q} / \Lambda$ and DM velocities,
which we assume to be small $v, v_{\perp} \ll 1$. This is a good approximation for cold DM in the phenomenologically interesting regime from dwarf galaxy scales $v \sim 10^{-5}$ to freeze out $v \sim 0.3$. This type of non-relativistic effective theory was recently applied to the direct detection of dark matter in [403, 404]. In order to conserve DM energy (and the total angular momentum) we assume that mediator bremsstrahlung is kinematically suppressed, $m_{\chi} \mathbf{v}^{2} \ll m_{\varphi}$. In other words, we work in the regime

$$
\begin{equation*}
\mathbf{v}^{4} \ll \frac{m_{\varphi}^{2}}{m_{\chi}^{2}} \ll \frac{\mathbf{q}^{2}}{m_{\chi}^{2}} \sim \mathbf{v}^{2} . \tag{10.4}
\end{equation*}
$$

We assume mediators with spin less than 2 since the longitudinal components of massive particles with higher spins spoil the derivative expansion at scales comparable with their mass, $\mathbf{q} \sim m_{\varphi}$.

### 10.3.1 Rotationally invariant non-RELATIVISTIC OPERATORS

Under parity and time reversal velocities, spins and momentum, transform as

$$
\begin{array}{lll}
P: i \mathbf{q} \rightarrow-i \mathbf{q}, & \mathbf{s} \rightarrow+\mathbf{s}, & \mathbf{v}_{\perp} \rightarrow-\mathbf{v}_{\perp}, \\
T: i \mathbf{q} \rightarrow+i \mathbf{q}, & \mathbf{s} \rightarrow-\mathbf{s}, & \mathbf{v}_{\perp} \rightarrow-\mathbf{v}_{\perp} . \tag{10.6}
\end{array}
$$

In turn, one can build the following invariant parity-even operators

$$
\begin{align*}
& \mathcal{O}_{1}=1 \\
& \mathcal{O}_{2}=\mathbf{s}_{1} \cdot \mathbf{s}_{2} \\
& \mathcal{O}_{3}=-\frac{1}{\Lambda^{2}}\left(\mathbf{s}_{1} \cdot \mathbf{q}\right)\left(\mathbf{s}_{2} \cdot \mathbf{q}\right) \\
& \mathcal{O}_{4}=\left(\mathbf{s}_{1} \cdot \mathbf{v}_{\perp}\right)\left(\mathbf{s}_{2} \cdot \mathbf{v}_{\perp}\right)  \tag{10.10}\\
& \mathcal{O}_{5,6}=-\frac{i}{\Lambda}\left[\left(\mathbf{s}_{1} \cdot \mathbf{q}\right)\left(\mathbf{s}_{2} \cdot \mathbf{v}_{\perp}\right) \pm\left(\mathbf{s}_{1} \cdot \mathbf{v}_{\perp}\right)\left(\mathbf{s}_{2} \cdot \mathbf{q}\right)\right]  \tag{10.11}\\
& \mathcal{O}_{7,8}=-\frac{i}{\Lambda}\left[\left(\mathbf{s}_{1} \pm \mathbf{s}_{2}\right) \cdot(\mathbf{q} \times \mathbf{v})\right] \tag{10.12}
\end{align*}
$$

where spin wavefunctions are suppressed for simplicity. Operators $\mathcal{O}_{5,6}$ respect parity but break time reversal. In the following we discard $\mathcal{O}_{4}$ because it is only generated by spin-2 mediators [404]. Relaxing parity invariance introduces eight additional operators [402]: four of those respect time reversal or, equivalently, $C P$

$$
\begin{aligned}
\mathcal{O}_{9} & =-\frac{1}{\Lambda}\left(\mathbf{s}_{1} \times \mathbf{s}_{2}\right) \cdot i \mathbf{q} \\
\mathcal{O}_{10,11} & =\left(\mathbf{s}_{1} \pm \mathbf{s}_{2}\right) \cdot \mathbf{v}_{\perp} \\
\mathcal{O}_{12} & =-\frac{i}{\Lambda}\left[\mathbf{s}_{1} \cdot(\mathbf{q} \times \mathbf{v})\right]\left(\mathbf{s}_{2} \cdot \mathbf{v}_{\perp}\right)+\frac{i}{\Lambda}\left[\mathbf{s}_{2} \cdot(\mathbf{q} \times \mathbf{v})\right]\left(\mathbf{s}_{1} \cdot \mathbf{v}_{\perp}\right),
\end{aligned}
$$

while other four break both $P$ and $C P$

$$
\begin{align*}
& \mathcal{O}_{13,14}=-\frac{1}{\Lambda}\left(\mathbf{s}_{1} \pm \mathbf{s}_{2}\right) \cdot i \mathbf{q}  \tag{10.16}\\
& \mathcal{O}_{15}=\left(\mathbf{s}_{1} \times \mathbf{s}_{2}\right) \cdot \mathbf{v}_{\perp},  \tag{10.17}\\
& \mathcal{O}_{16}=-\frac{1}{\Lambda^{2}}\left(\mathbf{s}_{2} \cdot \mathbf{q}\right)\left[\mathbf{s}_{1} \cdot(\mathbf{q} \times \mathbf{v})\right]+\frac{1}{\Lambda^{2}}\left(\mathbf{s}_{1} \cdot \mathbf{q}\right)\left[\mathbf{s}_{2} \cdot(\mathbf{q} \times \mathbf{v})\right] . \tag{10.18}
\end{align*}
$$

Observe that self-conjugate DM is symmetric under the exchange $1 \leftrightarrow 2$. This is equivalent to invariance under $\left(\mathbf{q}, \mathbf{v}, \mathbf{s}_{1}\right) \leftrightarrow\left(-\mathbf{q},-\mathbf{v}, \mathbf{s}_{2}\right)$, which forbids $\mathcal{O}_{6,8,10,12,13,16}$.

### 10.3.2 THE GENERAL EFFECTIVE POTENTIAL

A more general approach is to replace the free propagator with a general two point function in (10.3). This may include arbitrary negative powers of $\mathbf{q}^{2}$ from non-local interactions mediated by light states that have been integrated out. In an integral representation, the amplitude is

$$
\begin{equation*}
\mathcal{M}=-\int_{0}^{\infty} d \mu^{2} \frac{\rho\left(\mu^{2}\right)}{\mathbf{q}^{2}+\mu^{2}} \sum_{i} g_{i}\left(\mathbf{q}^{2} / \Lambda^{2}, \mathbf{v}_{\perp}^{2}\right) \mathcal{O}_{i}\left(\mathbf{v}_{j} \cdot i \mathbf{q} / \Lambda, \mathbf{s}_{i} \cdot \mathbf{v}_{\perp}, \mathbf{s}_{1} \cdot \mathbf{s}_{2}\right) \tag{10.19}
\end{equation*}
$$

where $\rho\left(\mu^{2}\right)$ is the spectral density of the theory which provides a common language to describe weakly and strongly coupled models. The standard propagator is recovered when $\rho\left(\mu^{2}\right)=\delta\left(\mu^{2}-m_{\varphi}^{2}\right)$.

Since the couplings always appear with the mediator's propagator, we can make the replacement $g_{i}\left(\mathbf{q}^{2} / \Lambda^{2}, \mathbf{v}_{\perp}^{2}\right)=g_{i}\left(-\mu^{2} / \Lambda^{2}, \mathbf{v}_{\perp}^{2}\right)$ after neglecting short-range interactions such as $\delta$-functions. Moreover, for light mediators, the spectral density only has support for $\mu^{2} \ll m_{\chi}^{2}, \Lambda^{2}$ so that we may further write $g_{i}\left(\mathbf{q}^{2} / \Lambda^{2}, \mathbf{v}_{\perp}^{2}\right) \simeq g_{i}(\mathrm{o}, \mathrm{o}) \equiv g_{i}$ unless this order vanishes. In such a case one should go to the first non-vanishing order, $g_{i} \rightarrow\left(-\mu^{2} / \Lambda^{2}\right)^{n} g_{i}^{(n)} / n!$. We have also dropped the velocity dependence because it does not provide the leading contribution unless one fine tunes the coefficients of the UV operators to cancel the velocity-independent contributions [403, 404].

Taking the Fourier transform of the scattering amplitude with respect to $\mathbf{q}$, one obtains the long-range effective potential as a function of the relative distance $\mathbf{r}$ and velocity $\mathbf{v}$. For example, $P$ - and $T$-symmetric interactions result in an effective long-range potential

$$
\begin{equation*}
V_{\mathrm{eff}}^{P, T}=\frac{1}{4 \pi r}\left\{\tilde{g}_{1}(r)+\tilde{g}_{2}(r)\left(\mathbf{s}_{1} \cdot \mathbf{s}_{2}\right)+\frac{\tilde{g}_{3}(r)}{\Lambda^{2} r^{2}}\left[3 \mathbf{s}_{1} \cdot \hat{r} \mathbf{s}_{2} \cdot \hat{r}-\mathbf{s}_{1} \cdot \mathbf{s}_{2}\right]+\frac{\tilde{g}_{7,8}(r)}{\Lambda r}\left(\mathbf{s}_{1} \pm \mathbf{s}_{2}\right)(\hat{r} \times \mathbf{v})\right\} \tag{10.20}
\end{equation*}
$$

where $\tilde{g}_{i}(r)$ are integrals of the Yukawa factor over the spectral density

$$
\begin{align*}
& \tilde{g}_{1}(r)=\int_{0}^{\infty} d \mu^{2} \rho\left(\mu^{2}\right) e^{-\mu r}\left(g_{1}-g_{1}^{(1)} \frac{\mu^{2}}{\Lambda^{2}}\right)  \tag{10.21}\\
& \tilde{g}_{2}(r)=\int_{0}^{\infty} d \mu^{2} \rho\left(\mu^{2}\right) e^{-\mu r}\left[g_{2}+\left(\frac{g_{3}}{3}-g_{2}^{(1)}\right) \frac{\mu^{2}}{\Lambda^{2}}\right]  \tag{10.22}\\
& \tilde{g}_{3}(r)=g_{3} \int_{0}^{\infty} d \mu^{2} \rho\left(\mu^{2}\right) e^{-\mu r}\left[1+\mu r+\frac{1}{3}(\mu r)^{2}\right]  \tag{10.23}\\
& \tilde{g}_{7,8}(r)=g_{7,8} \int_{0}^{\infty} d \mu^{2} \rho\left(\mu^{2}\right) e^{-\mu r}(1+\mu r) \tag{10.24}
\end{align*}
$$

It is understood that working at the leading non-vanishing order, $g_{3,7,8}$ and $g_{1,2}^{(1)}$ should always be dropped unless the $\mathcal{O}\left(\mathbf{q}^{\circ}\right)$ terms like $g_{1,2}$ are vanishing or suppressed. For weakly coupled dark sectors, $\tilde{g}_{1,2}$ are the usual exponential factors while $\tilde{g}_{3,7,8}$ carry additional polynomial corrections in the mediator mass. In general these functions have an arbitrary $r$ dependence, as expected when the mediator is a composite operator. A simple example is a four-fermi operator between spin- $\frac{1}{2} \mathrm{DM}$ particles $\chi$ and a massless neutrino-like species $v$, that is $\mathcal{L}=\sqrt{a}\left[\bar{v} \gamma_{\mu}\left(1-\gamma_{s}\right) v\right]\left[\bar{\chi} \gamma_{\mu}\left(a-b \gamma_{s}\right) \chi\right]$. The mediator is a composite operator made of two light fermions. It generates a singular potential at the loop level [400], $\tilde{g}_{i=1,2} \propto 1 / r^{4}$ and $\tilde{g}_{3} \propto \Lambda^{2} / r^{2}$. Note that the spin structure of the potential is fixed by the quantum numbers of the light mediator.

Note that for spin- $\frac{1}{2}$ DM the particle-antiparticle potential must have $g_{8}=0$ since CPcorresponds to a factor $(-)^{S+1}$ and thus implies the conservation of total spin $S^{2}=\left(\mathbf{s}_{1}+\mathbf{s}_{2}\right)^{2}$ which can only take values o and 1 . In this case, it is convenient to express the potential in the following form

$$
\begin{equation*}
V^{\left(s_{i}=1 / 2\right)}=\frac{1}{4 \pi r}\left\{\left(\tilde{g}_{1}(r)-\frac{3}{4} \tilde{g}_{2}(r)\right)+\frac{1}{2} \tilde{g}_{2}(r) \mathbf{S}^{2}+\frac{\tilde{g}_{3}(r)}{2 \Lambda^{2} r^{2}}\left[3(\mathbf{S} \cdot \hat{r})^{2}-\mathbf{S}^{2}\right]+\frac{2 \tilde{g}_{7}(r)}{m_{\chi} \Lambda r^{2}} \mathbf{S} \cdot \mathbf{L}\right\} \tag{10.25}
\end{equation*}
$$

where $\mathbf{L}=\mathbf{r} \times \mathbf{p}$ is the orbital angular momentum and $\mathbf{p}=m_{\chi} \mathbf{v} / 2$ is the conjugate momentum, $\left[\mathbf{r}^{i}, \mathbf{p}^{j}\right]=i \delta^{i j}$.
At large distances, but smaller than the mediator Compton wavelength, $\Lambda^{-1} \ll r \ll \mu^{-1}$, the functions $\tilde{g}_{i}(r)$ become

| interaction | $\frac{1}{r}$ | $\frac{1}{r}\left(\mathbf{s}_{1} \cdot \mathbf{s}_{2}\right)$ | $\frac{1}{r^{3}}\left[3\left(\mathbf{s}_{1} \cdot \hat{r}\right)\left(\mathbf{s}_{2} \cdot \hat{r}\right)-\mathbf{s}_{1} \cdot \mathbf{s}_{2}\right]$ |
| :--- | :---: | :---: | :---: |
| $\lambda_{s} \bar{\chi} \chi \phi$ | $-\lambda_{s}^{2}$ | 0 | 0 |
| $i \lambda_{p} \bar{\chi} \gamma^{5} \chi \phi$ | 0 | $\frac{\lambda_{p}^{2} m_{\phi}^{2}}{3 m_{\chi}^{2}}$ | $\frac{\lambda_{p}^{2}}{m_{\chi}^{2}} h\left(m_{\phi}, r\right)$ |
| $\overline{1}^{1} \bar{\chi} \gamma^{\mu} \gamma^{5} \chi \partial_{\mu} \phi$ | 0 | $\frac{4 m_{\phi}^{2}}{3 f^{2}}$ | $\frac{4}{f^{2}} h\left(m_{\phi}, r\right)$ |
| $\lambda_{\nu} \bar{\chi} \gamma^{\mu} \chi A_{\mu}$ | $\pm \lambda_{v}^{2}\left(1+\frac{m_{A}^{2}}{4 m_{\chi}^{2}}\right)$ | $\pm \frac{2 \lambda_{v}^{2} m_{A}^{2}}{3 m_{\chi}^{2}}$ | $\mp \frac{\lambda_{v}^{2}}{m_{\chi}^{2}} h\left(m_{A}, r\right)$ |
| $\lambda_{a} \bar{\chi} \gamma^{5} \gamma^{\mu} \chi A_{\mu}$ | 0 | $-\frac{8 \lambda_{a}^{2}}{3}\left(1-\frac{m_{A}^{2}}{8 m_{\chi}^{2}}\right)$ | $\lambda_{a}^{2}\left(\frac{1}{m_{\chi}^{2}}+\frac{4}{m_{A}^{2}}\right) h\left(m_{A}, r\right)$ |
| $\frac{i}{2 \Lambda} \bar{\chi} \sigma^{\mu v} \chi F_{\mu \nu}$ | 0 | $\mp \frac{2 m_{A}^{2}}{3 \Lambda^{2}}$ | $\pm \frac{1}{\Lambda^{2}} h\left(m_{A}, r\right)$ |

Table 10.1: Parity-preserving particle-(anti-)particle (upper/lower sign) long-range, static potentials from scalar $\phi$, gauge boson $A_{\mu}$, and field strength $F_{\mu \nu}=\partial_{[\mu} A_{\nu]}$ mediators. Here $\sigma^{\mu \nu}=\frac{i}{4}\left[\gamma^{\mu}, \gamma^{\nu}\right]$ and $h$ is defined in (10.28). Each term implicitly carries a Yukawa factor $e^{-m_{\varphi} r} / 4 \pi$. Observe that the long-range $\mathbf{s}_{1} \cdot \mathbf{s}_{2}$ is always suppressed by the mediator mass since $\lambda_{a}=m_{A} / f$.
constants and the potential simplifies even further:

$$
\begin{equation*}
V_{\mathrm{eff}}^{P, T}=\frac{1}{4 \pi r}\left[g_{1}+g_{2}\left(\mathbf{s}_{1} \cdot \mathbf{s}_{2}\right)+\frac{g_{3}}{\Lambda^{2} r^{2}}\left(3 \mathbf{s}_{1} \cdot \hat{r} \mathbf{s}_{2} \cdot \hat{r}-\mathbf{s}_{1} \cdot \mathbf{s}_{2}\right)+\frac{g_{7,8}}{\Lambda r}\left(\mathbf{s}_{1} \pm \mathbf{s}_{2}\right)(\hat{r} \times \mathbf{v})\right] \tag{10.26}
\end{equation*}
$$

This is the regime where Sommerfeld enhancement may be effective because the interaction is still long-range compared to the short distance annihilation processes that take place at $r \sim \Lambda^{-1}$.

The expressions for the potentials that break $P$ but respect $T$ are presented in Appendix 10.A.

### 10.3.3 WEAKLY COUPLED EXAMPLES

As an example, consider a dark sector with a weakly coupled, light scalar or vector mediator $\varphi$ with interactions $\lambda \mathcal{O}^{\text {QFT }}$ in Table 10.1. These generate a static potential

$$
\sum_{i} \lambda_{i} \mathcal{O}_{i}^{\mathrm{QFT}} \longrightarrow V_{\mathrm{s}}^{P}=\left[g_{1}+g_{2}\left(\mathbf{s}_{1} \cdot \mathbf{s}_{2}\right)+\frac{g_{3}}{\Lambda^{2} r^{2}} h\left(m_{\varphi}, r\right)\left[3\left(\mathbf{s}_{1} \cdot \hat{r}\right)\left(\mathbf{s}_{2} \cdot \hat{r}\right)-\mathbf{s}_{1} \cdot \mathbf{s}_{2}\right]\right] \frac{e^{-m_{\varphi} r}}{4 \pi r}
$$

where $h$ encodes the dependence on the mediator mass,

$$
\begin{equation*}
h\left(m_{\varphi}, r\right)=\left(1+m_{\varphi} r+\frac{m_{\varphi}^{2} r^{2}}{3}\right) . \tag{10.28}
\end{equation*}
$$

Table 10.1 gives the contributions to each of the coefficients on the right-hand side of (10.27) coming from the corresponding types of QFT interactions.

| mediator | interaction | $\frac{1}{4 \pi r}\left[\mathbf{v}^{2}+\hat{r}(\hat{r} \cdot \mathbf{v}) \mathbf{v}\right]$ | $\frac{1}{4 \pi r^{2}}(\hat{r} \times \mathbf{v}) \cdot\left(\mathbf{s}_{\mathbf{1}}+\mathbf{s}_{2}\right)$ |
| :--- | :--- | :---: | :---: |
| scalar | $\lambda_{s} \bar{\chi} \chi \phi$ | $-\frac{\lambda_{s}^{2}}{8}$ | $\frac{\lambda_{s}^{2}}{4 m_{\chi}}$ |
| vector | $\lambda_{v} \bar{\chi} \gamma^{\mu} \chi A_{\mu}$ | $\pm \frac{\lambda_{v}^{2}}{8}$ | $\mp \frac{3 \lambda_{s}^{2}}{4 m_{\chi}}$ |

Table 10.2: Parity-preserving particle-(anti-)particle (upper/lower sign) long-range, non-static potentials from massless scalars $\phi$ and gauge bosons $A_{\mu}$. Long-range contributions from pseudo-scalars, axial vectors and field strength vanish for massless mediators.

Note that the Dirac DM mass $m_{\chi}$ breaks axial symmetry so that the limit of a massless axial gauge boson mediator is consistent at finite $m_{\chi}$ only when chiral symmetry is broken spontaneously at a scale $f$ so that $m_{A}=\lambda_{a} f$. In this case the transverse components decouple, $\lambda_{a}=m_{A} / f \rightarrow 0$, and only the longitudinal modes contribute to the amplitude with coupling $1 / f$, matching the result from Goldstone boson exchange.

Table 10.2 gives the long-range, non-static potential contributions from massless scalars and gauge bosons. The $\mathbf{v}_{\perp}^{2}$ contribution generates $\mathrm{a} \sim 1 / r\left(\mathbf{v}^{2}+\hat{r}(\hat{r} \cdot \mathbf{v}) \mathbf{v}\right)$ in position space which can be neglected because it is always subleading. Pseudo-scalar, axial-vector and field strength mediators, give vanishing non-static, long-range potentials at this order. Note that these potentials generically need to be complemented by the relativistic corrections to the kinetic energies, $\mathbf{p}^{2} / m_{\chi}^{2}\left(1-\mathbf{p}^{2} /\left(4 m_{\chi}^{2}\right)+\ldots\right)$. In the following sections we neglect these corrections to the kinetic energy since we checked that their contribution is very small.

### 10.4 Renormalization of Singular potentials and Sommerfeld enhancement

The potential $V_{\text {eff }}^{P, T}$ in (10.20) represents the most general long-range interactions between DM particles that preserve parity and time reversal. A standard method for calculating the Sommerfeld enhancement for the non-singular Coulomb and Yukawa potentials is presented in [384] and reviewed in Appendix 10.B. In practice, one determines the boost factor by solving a Schrödinger-like equation with the proper boundary conditions. However, since the terms in $V_{\text {eff }}^{P, T}$ are typically very singular, the usual calculations for the boost factor will generically fail. In this section we show how to overcome these problems by renormalizing the Schrödinger equation. Since a full numerical solution can be computationally intensive for singular potentials, we also provide an algebraic algorithm to estimate the Sommerfeld enhancement for general potentials in Appendix 10.C.

### 10.4.1 WILSONIAN TREATMENT OF DIVERGENCES

Potentials that go to infinity faster than $1 / r^{2}$ at the origin are called singular [405] and generically arise in dark sectors with spinning DM and/or with some strong dynamics. The occurrence of unphysical behavior originating from the infinitely large energies of such potentials are analogous to the infinities of quantum field theory (QFT). These inconsistencies arise when one extrapolates a long-range potential to arbitrarily short distances where ultraviolet physics should be taken into account. In fact, the Schrödinger equation can be renormalized by adopting the Wilsonian renormalization group (RG) methods of QFT [406]: the singular potential is regulated at a short distance $a$ and augmented with a series of local operators that parametrize the unknown Uv physics,

$$
V(r) \longrightarrow V(r) \theta(r-a)+c_{0}(a) \delta^{3}(r)+c_{2}(a) a^{2} \nabla^{2} \delta^{3}(r)+\ldots
$$

The short-distance part of this effective potential is a derivative expansion that can be truncated to the desired order as long as the typical momenta $q$ are much smaller than the cutoff scale $\Lambda=a^{-1}$. This given order in $q$ determines the finite set of coupling constants $c_{i}(a)$ which can be determined by low-energy data.

### 10.4.2 RENORMALIZED POTENTIAL

Singular potentials diverge at the origin so that further care is required to impose boundary conditions. The Schrödinger equation for an $\ell$-wave state is conveniently expressed using the dimensionless coordinate $x=p r$, the product of the dark matter relative momentum and separation:

$$
\begin{equation*}
-\Phi_{p, \ell}^{\prime \prime}(x)+\left(\mathcal{V}(x)+\frac{\ell(\ell+1)}{x^{2}}-1\right) \Phi_{p, \ell}(x)=0 \tag{10.30}
\end{equation*}
$$

where the dimensionless potential is rescaled by the momentum $p$ and reduced mass $M=m_{\chi} / 2$,

$$
\begin{equation*}
\mathcal{V}(x)=\frac{2 M}{p^{2}} V\left(\frac{x}{p}\right) \tag{10.31}
\end{equation*}
$$

We regulate the potential at $x_{\text {cut }}=a p$ with a square well of height $\mathcal{V}_{0}$ encoding the $u v$ data of the relativistic completion,

$$
\begin{equation*}
\mathcal{V}_{\mathrm{reg}}(x)=\mathcal{V}(x) \theta\left(x-x_{\mathrm{cut}}\right)+\frac{1}{x_{\mathrm{cut}}^{2}} \mathcal{V}_{\mathrm{o}} \theta\left(x_{\mathrm{cut}}-x\right) \tag{10.32}
\end{equation*}
$$

In practice, we simulate the local counter-terms with a short-distance square well potential which makes the calculations much easier [407]. We stress, however, that any other choice or deformation of the counter-terms is allowed and physically equivalent as long as it changes only the UV behavior of the interactions [406].

Observe that the centrifugal barrier is left uncut since it is non-singular and unrelated to the $u v$ physics. Once $\mathcal{V}_{0}$ is known, one may integrate the Schrödinger equation subject to the usual boundary condition at zero

$$
\begin{equation*}
\lim _{x \rightarrow 0} \Phi_{p, \ell}(x)=x^{\ell+1} \tag{10.33}
\end{equation*}
$$

and then extract the Sommerfeld enhancement from the asymptotic solution. In the regulated region $x<x_{\mathrm{cut}}$, the Schrödinger equation can be solved explicitly in the approximation $x_{\text {cut }} \ll 1$,

$$
\begin{equation*}
\Phi_{p, \ell}\left(x<x_{\mathrm{cut}}\right)=\Gamma\left(\ell+\frac{3}{2}\right)\left(\frac{2 x_{\mathrm{cut}}}{\mathcal{V}_{o}^{1 / 2}}\right)^{\ell+1 / 2} x^{1 / 2} J_{\ell+1 / 2}\left(\mathcal{V}_{o}^{1 / 2} \frac{x}{x_{\mathrm{cut}}}\right) . \tag{10.34}
\end{equation*}
$$

The value $\mathcal{V}_{\circ}$ that appears in the Schrödinger equation is determined by requiring that a low energy observable is independent of the particular choice of the cutoff, $x_{\text {cut }}$. It is thus meaningful to define $\mathcal{V}_{0}\left(x_{\mathrm{cut}}\right)$ with respect to the value of a physical observable, which can be conveniently chosen to be the scattering phase $\delta_{\ell}$ of the elastic dark matter scattering process that generates this enhancement.

For the region $x>x_{\text {cut }}$, recall that the general solution to the Schrödinger equation is a linear combination of two independent solutions,

$$
\begin{equation*}
\Phi_{p, \ell}\left(x>x_{\mathrm{cut}}\right)=A f(x)+B g(x) . \tag{10.35}
\end{equation*}
$$

Asymptotically far from the origin, these independent solutions are combinations of sines and cosines. The scattering phase is related to the shift in the argument when the asymptotic solution is written as a pure sine. Thus the $\delta_{\ell}$ has a one-to-one relation to the ratio $A / B$.

In order to determine $\mathcal{V}_{\mathrm{o}}\left(x_{\mathrm{cut}}\right)$ subject to a fixed scattering phase, we may match the logarithmic derivatives of the two
piecewise solutions at $x_{\text {cut }}$. Comparing (10.34) with (10.35),

$$
\begin{equation*}
-\frac{\ell}{x_{\mathrm{cut}}}+\frac{\mathcal{V}_{o}^{1 / 2}\left(x_{\mathrm{cut}}\right)}{x_{\mathrm{cut}}} \frac{J_{\ell-1 / 2}\left(\mathcal{V}_{o}^{1 / 2}\left(x_{\mathrm{cut}}\right)\right)}{J_{\ell+1 / 2}\left(\mathcal{V}_{0}^{1 / 2}\left(x_{\mathrm{cut}}\right)\right)}=\frac{\frac{A}{B} f^{\prime}\left(x_{\mathrm{cut}}\right)+g^{\prime}\left(x_{\mathrm{cut}}\right)}{\frac{A}{B} f\left(x_{\mathrm{cut}}\right)+g\left(x_{\mathrm{cut}}\right)} . \tag{10.36}
\end{equation*}
$$

Observe that matching the logarithmic derivative gives an expression that depends on $A / B$ which is cutoff independent and directly related to our low-energy observable [407]. Once $\mathcal{V}_{0}\left(x_{\text {cut }}\right)$ is determined, (10.32) is the correct non-singular low-energy potential for the problem with the given cutoff.

Due to the oscillatory nature of the Bessel function, there can be multiple solutions to the transcendental equation (10.36). All of these solutions are physically equivalent. To simplify our calculations we choose the first quadrant so that $\mathcal{V}_{0}\left(x_{\text {cut }}\right)$ can take values in the range $\left(-\infty, \mathcal{V}_{\text {max }}\right)$ where $\mathcal{V}_{\text {max }}$ is given by the first positive solution of

$$
\begin{equation*}
J_{\ell+1 / 2}\left(\mathcal{V}_{\max }^{1 / 2}\right)=0 \tag{10.37}
\end{equation*}
$$

For $\ell=0, \mathcal{V}_{\max }=\pi^{2}$.

### 10.4.3 WAVEFUNCTION RENORMALIZATION

Since $\mathcal{V}_{\text {reg }}$ in (10.32) is manifestly non-singular, one may proceed to solve the Schrödinger equation (10.30) subject to (10.33) following the procedure outlined in Appendix 10.B. The resulting Sommerfeld enhancement, $S^{(\circ)}$, appears to depend on the choice of $x_{\text {cut }}$. This residual cutoff dependence is not physical and is removed by including wavefunction renormalization, $Z_{\ell}$ :

$$
\begin{equation*}
S_{\ell}=Z_{\ell} S_{\ell}^{(0)} \tag{10.38}
\end{equation*}
$$

$Z_{\ell}$ is fixed by using the observation that at relativistic speeds the Sommerfeld enhancement factor should go to one,

$$
\begin{equation*}
Z_{\ell}=\frac{1}{S_{\ell}^{(o)}(v \rightarrow 1)} \tag{10.39}
\end{equation*}
$$

### 10.4.4 Comparison to Coulomb potential

We now verify that the above procedure matches the usual result for the non-singular Coulomb potential, $V(r)=-\alpha / r$. The wavefunction in the region $x>x_{\text {cut }}$ is

$$
\begin{equation*}
\Phi_{p, \ell}\left(x>x_{\mathrm{cut}}\right)=A x^{1 / 2} J_{2 \ell+1}\left(2 \sqrt{\frac{x a}{v}}\right)+B x^{1 / 2} Y_{2 \ell+1}\left(2 \sqrt{\frac{x a}{v}}\right) . \tag{10.40}
\end{equation*}
$$

One can check that the Sommerfeld enhancement is indeed independent of the choice $x_{\text {cut }}$. For different choices of $A / B$, one can obtain different Sommerfeld enhancements, as seen by the different lines on the left plot of Figure 10.4.1. Of these, one line (black) corresponds to the analytical formulae found in the literature [384]; this corresponds to picking a scattering phase that is consistent with a relativistic completion that includes a massless boson. In other words, this is the choice that is consistent with a theory where the non-relativistic Coulomb potential is completed by a relativistic field theory resembling QED. Other choices correspond to theories whose non-relativistic limit is Coulomb but whose local interactions differ from pure qED.

### 10.5 NUMERICAL RESULTS

The general DM potential considered here does not generally conserve orbital angular momentum $\mathbf{L}^{2}$ so that a coupled channel analysis between different $\ell$-wave annihilation modes is required. This implies that the $g_{3}$ contribution in (10.1) can


Figure 10.4.1: Cutoff-dependence of $s$-wave Sommerfeld enhancement using the procedure described in the text. Low energy data is encoded by the ratio $A / B$ of solutions to the homogeneous Schrödinger equation in (10.35). We take relative velocity $v=10^{-3}$. Left: Coulomb potential with $\alpha / v=e^{2} / 4 \pi v=10$. The unique phase $(A / B=0)$ given by a QED-like UV completion is indicated by the black line. RIGHT: $r^{-3}$ potential with $\tilde{a}=2 M^{2} v a / f^{2}=10^{-3}$, for $\alpha$ defined in (10.46).

| $J$ | $S$ | $P$ | $\ell$ |
| :---: | :---: | :---: | :---: |
| 0 | 0 | - | 0 |
| 0 | 1 | + | 1 |
| 1 | 0 | + | 1 |
| 1 | 1 | + | 1 |
| 1 | 1 | - | 0,2 |

Table 10.1: Low total angular momentum, $J$, DM scattering states labelled by spin, $S$, parity, $P$, and orbital angular momentum $\ell$. J, S, and $P$ are conserved by the Hamiltonian and are used to label states.
still be relevant for Sommerfeld enhancement via $\Delta \ell=2$ transitions even though it averages to zero for $\ell=0$ states [365]. This is contrary to the common belief that Sommerfeld enhancement is relevant only for $s$-wave annihilations due to the centrifugal barrier. For some states $\mathbf{L}^{2}$ is a well-defined quantum number once the total angular momentum $J$, the total spin $S$ and parity $P= \pm$ are specified. In these cases the calculation of the boost factor reduces to a standard single-channel Schrödinger problem as discussed above. Table 10.1 shows the quantum numbers for fermionic dm for low total angular momenta. Among the $\ell=\mathrm{o}$ states, $(J=\mathrm{o}, S=\mathrm{o}, P=-)$ gives a single channel problem with arbitrary potential $V_{0}(r)$, whereas $(J=1, S=1, P=-)$ requires a coupled channel analysis between $\ell=0$ and $\ell=2$.

Assuming parity conservation, the effective potential $V_{\text {eff }}=\langle$ out $| V(r) \mid$ in $\rangle+\ell(\ell+1) /\left(2 M r^{2}\right)$ for each channel is obtained


Figure 10.5.1: Sommerfeld enhancement for a singular $r^{-3}$ potential and orbital angular momentum $\ell=0$ (LEFT) and $\ell=1$ (RIGHT) for relative velocity $v=10^{-3}$ and various values of $\tilde{a}=2 M^{2} v a / f$, with $\alpha$ defined in (10.46).
by sandwiching (10.25) with the centrifugal term between the appropriate $|J S P\rangle$ states,

$$
\begin{align*}
& \left\lvert\, \begin{array}{lll}
\circ & \circ & -\rangle
\end{array} \rightarrow \quad V_{\text {eff }}=\left(\tilde{g}_{1}(r)-\frac{3}{4} \tilde{g}_{2}(r)\right) \frac{1}{4 \pi r}\right.  \tag{10.41}\\
& \left\lvert\, \begin{array}{lll}
0 & 1 & +\rangle
\end{array} \rightarrow \quad V_{\text {eff }}=\frac{1}{M r^{2}}+\left(\tilde{g}_{1}(r)+\frac{1}{4} \tilde{g}_{2}(r)-\frac{\tilde{g}_{3}(r)}{2 \Lambda^{2} r^{2}}-\frac{2 \tilde{g}_{7}(r)}{M \Lambda r^{2}}\right) \frac{1}{4 \pi r}\right.  \tag{10.42}\\
& \begin{array}{|lll}
1 & 0 & +\rangle
\end{array} \rightarrow \quad V_{\text {eff }}=\frac{1}{M r^{2}}+\left(\tilde{g}_{1}(r)-\frac{3}{4} \tilde{g}_{2}(r)\right) \frac{1}{4 \pi r}  \tag{10.43}\\
& \left\lvert\, \begin{array}{lll}
1 & 1 & +\rangle
\end{array} \rightarrow \quad V_{\text {eff }}=\frac{1}{M r^{2}}+\left(\tilde{g}_{1}(r)+\frac{1}{4} \tilde{g}_{2}(r)+\frac{\tilde{g}_{3}(r)}{4 \Lambda^{2} r^{2}}-\frac{\tilde{g}_{7}(r)}{M \Lambda r^{2}}\right) \frac{1}{4 \pi r}\right.  \tag{10.44}\\
& \begin{array}{|lll}
1 & 1 & -\rangle
\end{array} \rightarrow \quad V_{\text {eff }}=\frac{1}{M r^{2}}\left(\begin{array}{ll}
0 & 0 \\
0 & 3
\end{array}\right)+\left(\begin{array}{cc}
\tilde{g}_{1}(r)+\frac{\tilde{g}_{2}(r)}{4} & \frac{\tilde{g}_{3}(r)}{2(\sqrt{2})} \\
\frac{\tilde{g}_{3}(r)}{2 \sqrt{2} \Lambda^{2} r^{2}} & \tilde{g}_{1}(r)+\frac{\tilde{g}_{2}(r)}{4}-\frac{\tilde{y}_{3} r^{2}}{4 \Lambda^{2} r^{2}}-\frac{3 \tilde{g}^{2}(r)}{M \Lambda r^{2}}
\end{array}\right) \frac{1}{4 \pi r} \quad \text { (10.45) }
\end{align*}
$$

where the $\ell=0$ and $\ell=2$ channels are coupled in (10.45). If the $\tilde{g}_{i}$ are constant, then at leading order these channels are effectively non-singular and Coulomb-like. However, if $\tilde{g}_{1}+\tilde{g}_{2} / 4=0$, such as for pseudo-scalar exchange, then some of these channels are dominated by the singular $V \sim 1 / r^{3}$ term. Moreover, one can also consider scenarios-for example, the exchange of multiple light particles [399-402] —in which $\tilde{g}_{1,2} \sim 1 / r^{3}$ so that even the $\tilde{g}_{1}$ and $\tilde{g}_{2}$ terms are singular with $\ell=0$. Thus one may in principle generate a singular potential for any partial wave. For simplicity, we shall consider a simple $1 / r^{3}$ potential for both $\ell=0$ and $\ell=1$. The coupled channel in (10.45), however, requires a more careful analysis that we leave for future work.

In Figure 10.5.1 we plot the Sommerfeld enhancement for a potential

$$
\begin{equation*}
V(r)=-\frac{a}{f^{2} r^{3}} \tag{10.46}
\end{equation*}
$$

as a function of the IR observable $\cot \delta$ for $\ell=0,1$. When comparing these, note that the $\ell=1$ cross section has an additional factor of $v^{2}$ relative to $\ell=0$. The resonance is located at $\cot \delta=o$ because this is where the cross section is maximal. These plots can be used to give an upper bound on Sommerfeld enhancement for various couplings. Note that while it is true that the resonance is larger for smaller couplings, it requires more tuning from the $u v$ to reach the resonance for a smaller coupling. Moreover, while $\cot \delta$ contains data about uv physics, it also depends on the IR coupling in such a way that reducing the


Figure 10.5.2: Resonances in Sommerfeld enhancement for a singular $r^{-3}$ potential and orbital angular momentum $\ell=\mathrm{o}($ LEFT $)$ and $\ell=1$ (RIGHt) for a range of relative velocities and $a / f^{2}=\mathrm{TeV}^{-2}$ with $a$ defined in (10.46). The large enhancements can be understood from the box approximation, see the Appendix 10.C. For simplicity the height of the regulated potential is fixed by continuity with the long range piece.
coupling would not increase the Sommerfeld unless one simultaneously increases the height of the square well potential $V_{0}$.
Figure $\mathbf{1 0 . 5 . 2}$ presents an exploration of these resonances as a function of the dark matter reduced mass. As described in the procedure above, the physical Sommerfeld enhancement for a singular potential requires information from an IR observable such as the scattering phase $\delta$. As a reasonable estimate for natural uv models, we regulate the theory at a length scale $r_{0}$ where the non-relativistic description breaks down, $V\left(r_{0}\right)=M$. We then fix the height of the cutoff by continuity with the singular long-range part, $V_{\circ}=V\left(r_{0}\right)=M$. Notice that for a $V(r)=-a /\left(f^{2} r^{3}\right)$ potential with $f=1 \mathrm{TeV}$, the dark matter mass necessary to reach a significant enhancement is about 1 TeV . If the dark matter mass is sufficiently large one may also need to consider the $\ell=1$ contribution.

### 10.6 Phenomenology

While the collisionless cold DM paradigm successfully accounts for the large scale structure of the universe, it faces tension at smaller scales where $N$-body simulations present some discrepancies with observations. In particular, dwarf galaxies show flat core DM densities profiles in the central part of the halos, whereas collisionless cold DM predicts cusp-like profiles [106-109]. While this discrepancy may be due to unaccounted baryonic physics [408-410], it may alternately be taken as a motivation for dark matter self-interactions [392, 393, 411]. A related astrophysical motivation for self interactions is the "too big to fail problem," in which the brightest observed dwarf spheroidal satellites in the Milky Way appear to be incompatible with the central densities of subhalos predicted by collisionless DM [110-112]. A third suggestion for self interactions is the "missing satellites problem"; collisionless DM predictions for the number the satellite galaxies expected in the Milky Way appears to disagree with observations [113, 114]. See, e.g. $[115,116]$ and references therein for critical discussions.

To solve the core vs. cusp problem, the dark matter self interaction must have a sufficiently large cross section, $\sigma / m_{\chi} \sim 0.1-10 \mathrm{~cm}^{2} / \mathrm{g}$, for velocities typical of dwarf galaxies, $v \sim 10^{-5}$, while having a smaller cross section for galaxy cluster velocities, $v \sim 10^{-3}$, where collisionless DM results are in good agreement. There are additional upper bounds on the cross section coming from astrophysical observations sensitive to the velocities characteristic of galaxy clusters [115, 395]. One of the most stringent bounds, for example, comes from the ellipticity of galaxy clusters [395,412,413]. The most recent simulations have softened this bound to $\sigma / m_{\chi}=0.1 \mathrm{~cm}^{2} / \mathrm{g}[392,393]$. Further, the cosmic microwave background (CMB) sets an upper bound on Sommerfeld enhancement from the effect of DM annihilation after recombination [414-416]. Though


Figure 10.6.1: Core vs. cusp problem. left: Sommerfeld enhancement (upper) and scattering cross section (lower) as a function of relative velocity for a range of low energy parameters $A / B$ as discussed below (10.35) and $2 \alpha M^{2} / f^{2}=1$. RIGHT: Total dark matter cross section as a function of velocity. Red: velocity dependent with $2 a M^{2} / f^{2}=1, M=5.85 \mathrm{TeV}, A / B=-10^{-3}$, and an additional short distance interaction, $M^{2} \sigma_{\text {short }}=500$. Blue: velocity independent cross section with no new short range interaction and $M=\mathrm{TeV}, 2 \alpha M^{2} / f^{2}=0.1$, $A / B=-6 \times 10^{-4}$.
a constant cross section $\sigma / m_{\chi} \lesssim 0.5 \mathrm{~cm}^{2} / \mathrm{g}$ may account for these effects, this velocity dependence is also suggestive of a Sommerfeld enhanced cross section [417]. We leave a more thorough investigation of the astrophysical and cosmological bounds on the enhancement of singular potentials for future work.

As an example for how to apply Sommerfeld enhancement to address the dwarf galaxy scale astrophysical puzzles while simultaneously avoiding the bounds from galaxy cluster scale observations, we consider dark matter self interactions mediated by a light force carrier that generates a singular potential,

$$
\begin{equation*}
V(r)=\frac{-a}{f^{2}} \frac{1}{r^{3}} \tag{10.47}
\end{equation*}
$$

The left side of Figure 10.6 .1 shows the Sommerfeld enhancement (upper) and the total cross section (lower) from such a model with a choice of parameters near the resonance. Observe that even for very small $A / B$, that is small cot $\delta$ or large scattering phase, the cross section is saturated between the characteristic galaxy cluster velocities $v \sim 10^{-5}$ and dwarf galaxy velocities $v \sim 10^{-3}$. For $A / B \sim 10^{-5}$, as indicated by the red line in the lower figure, this saturates to $\sigma / m_{\chi} \sim 10^{-2} \mathrm{~cm}^{2} / \mathrm{g}$ for $m_{\chi} \sim \mathrm{TeV}$. This saturation occurs over the range of velocities where we would like a stronger velocity-dependence to avoid cluster scale bounds. In order to do this, we assume the existence of a short range interaction that contributes to the elastic scattering process with cross section $\sigma_{\text {short }}^{(\circ)}$. The long range mediators Sommerfeld enhance this cross section by the factor shown in the upper plot; observe that this enhancement decreases exponentially as one increases from dwarf galaxy velocities to galaxy cluster velocities. The total cross section is roughly (ignoring cross terms for simplicity),

$$
\begin{equation*}
\sigma_{\text {tot }}(v) \sim \sigma_{\text {elast }}(v)+S(v) \sigma_{\text {short }}^{(\circ)} \tag{10.48}
\end{equation*}
$$

Since the enhancement factors can be fairly large, the additional short range interaction can be weakly coupled, e.g. $\sigma_{\text {short }}^{(\circ)} M^{2} \sim 10^{4}$ so that $\sigma_{\circ} \sim 10^{6} \mathrm{pb}$ for a TeV scale dark matter particle. The right side of Figure 10.6 .1 compares the velocity-dependence of this type of solution to another solution without enhanced short range physics.

Finally, we remark on the use of Sommerfeld enhancement for generating indirect signals of dark matter through positrons
and gamma rays [383, 387]. The excess of cosmic positrons observed by pamela [98] and later confirmed by Fermi [99] and AMS-02 [100] is a potential signal for dark matter annihilation. Since the cross section required to produce these signals is much larger than the required cross section for thermal relics, DM models that realize the positron excess typically require large Sommerfeld enhancements [418]. A study for non-singular dark sectors with Yukawa interactions was performed in [395, 419]; an investigation of how these bounds change for singular potentials is left for future work.

A recent speculative signal of indirect DM detection is the 135 GeV line in the Fermi gamma ray spectrum [101-105]. Indeed, gamma ray signatures were the original motivation for investigating Sommerfeld enhancement in dark matter [387]. The cross section required for the line is about $10^{-27} \mathrm{~cm}^{3} / \mathrm{s}$ which generically points toward a large boost factor, $S \approx 10^{4}$. It is possible to get such a large enhancement with a singular potential $V(r)=-\alpha /\left(f r^{3}\right)$, but since the dark matter mass must be 135 GeV this requires a low scale $f \approx 100 \mathrm{GeV}$ to avoid tuning in the Uv. Dark matter models can generate such a feature, though these typically generate an unobserved continuum contribution to the spectrum [420]. Ways around difficulty were explored in [421-425].

### 10.7 CONCLUSION

We have presented the effective non-relativistic theory of self-interacting dark matter parameterized to leading order in the relative velocity, $v$, and the exchanged momentum, $q / \Lambda$. The resulting potentials generically include singular terms which must be regulated and renormalized so that the resulting predictions are cutoff independent. We have shown how this effective theory can be applied to calculate the Sommerfeld enhancement generated by singular potentials.

Using a simple toy model with a $1 / r^{3}$ potential, we have found that on resonance one can generate enhancements as large as $S \sim 10^{6}$ at velocities on the order of $v \sim 10^{-3}$. This opens up promising directions for the astrophysical phenomenology of general self-interacting dark matter models. For example, extant astrophysical puzzles such as the core vs. cusp problem can be addressed with this velocity-dependent enhancement. A more thorough investigation and implications for specific uv models of these bounds is left for future work.

## 10.A CP-PRESERVING POTENTIAL

In Section 10.3, we presented a list of $P$ - and $T$-preserving operators in the non-relativistic potential for DM self-interactions. In this appendix we present the additional terms in the effective potential that are generated when parity invariance is relaxed. In addition to $\mathcal{O}_{1,2,3,4,7,8}$, the four operators $\mathcal{O}_{9,10,11,12}$ in (10.13)-(10.15) preserve $C P$ but break parity. For simplicity we consider only the case of self-conjugate dM so that $\mathcal{O}_{10,12}$ are forbidden.

The $\mathcal{O}_{9}$ term contains no $\mathbf{v}_{\perp}$ factors and the corresponding potential is

$$
\begin{equation*}
V_{9}=\frac{\tilde{g}_{9}(r)}{4 \pi r^{3} \Lambda}\left(\mathbf{s}_{1} \times \mathbf{s}_{2}\right) \mathbf{r} \tag{10.49}
\end{equation*}
$$

where $\tilde{g}_{9}(r)$ is defined analogously to (10.24).
In order to determine $V_{11}$ we need the Fourier transform of the propagator along the direction tranverse to the exchanged momentum

$$
\int \frac{d^{3} \mathbf{q}}{(2 \pi)^{3}} e^{i \mathbf{q} \cdot \mathbf{r}}\left[\delta^{i j}-\frac{\mathbf{q}_{i} \mathbf{q}_{j}}{\mathbf{q}^{2}}\right] \frac{1}{\left(\mathbf{q}^{2}+\mu^{2}\right)}=\frac{e^{-\mu r}}{4 \pi r}\left[\frac{2}{3} \delta^{i j}+\frac{1}{\mu^{2} r^{2}}\left(3 \hat{r}^{i} \hat{r}^{j}-\delta^{i j}\right)\left(e^{\mu r}-1-\mu r-\frac{\mu^{2} r^{2}}{3}\right)\right] .
$$

Contracting this expression with $\left(\mathbf{s}_{1}-\mathbf{s}_{2}\right)^{i}$ and $\mathbf{v}^{j}$ gives $V_{11}$. Since the final result is quite involved, we focus on two interesting limits. At distances smaller than the mediator Compton wavelength, $\Lambda^{-1} \ll r \ll \mu^{-1}$, the expression greatly simplifies because

$$
\begin{equation*}
\left.\lim _{\mu \rightarrow 0} \int \frac{d^{3} \mathbf{q}}{(2 \pi)^{3}} e^{i \mathbf{q} \cdot \mathbf{r}}\left[\delta^{i j}-\frac{\mathbf{q}_{\mathbf{i}}^{\mathbf{q}}}{j}\right] \frac{1}{\mathbf{q}^{2}}\right] \frac{1}{\left(\mathbf{q}^{2}+\mu^{2}\right)}=\frac{1}{8 \pi r}\left(\delta^{i j}+\hat{r}^{i} \hat{r} j\right), \tag{10.50}
\end{equation*}
$$

and hence

$$
\begin{equation*}
V_{11}=\frac{1}{8 \pi r}\left[\left(\mathbf{s}_{1}-\mathbf{s}_{2}\right) \cdot \mathbf{v}+\left(\mathbf{s}_{1}-\mathbf{s}_{2}\right) \cdot \hat{r}(\hat{r} \cdot \mathbf{v})\right] \tag{10.51}
\end{equation*}
$$

On the other hand, at scales where the mediator mass is important, $r \gg \mu^{-1}$, we have

$$
\begin{equation*}
V_{11}=\frac{1}{4 \pi r^{3} m^{2}}\left[3\left(\mathbf{s}_{1}-\mathbf{s}_{2}\right) \cdot \hat{r}(\hat{r} \cdot \mathbf{v})-\left(\mathbf{s}_{1}-\mathbf{s}_{2}\right) \cdot \mathbf{v}\right] . \tag{10.52}
\end{equation*}
$$

where $m^{2}=\int d \mu^{2} \rho\left(\mu^{2}\right) / \mu^{2}$.
We stress that the ordering of the various operators in the non-static part of the potential is generically important since $\mathbf{p}=m_{\chi} \mathbf{v} / 2$ is the conjugate coordinate associated with the relative distance, $\left[\mathbf{r}^{i}, \mathbf{p}^{j}\right]=i \delta^{i j}$.

## 10.B Sommerfeld enhancement for non-Singular potentials

Let us first briefly review the general method to obtain the Sommerfled enhancement [384,385]. Consider two particles of mass $m_{\chi}$ and center-of-mass momentum $\mathbf{p}$. The $\ell$-wave amplitude $A_{\ell}(\mathbf{p})$ for the annihilation of these two particles under an attractive central potential $V(r)$ can be expressed as a function of a bare amplitude $A_{\circ}, \ell(\mathbf{q})=a_{\mathrm{o}}, \ell q^{\ell}$ and a wavefunction $\varphi_{\mathbf{p}}(\mathbf{r})$,

$$
\begin{equation*}
A_{\ell}(\mathbf{p})=\int d \mathbf{r} \varphi_{\mathbf{p}}^{*}(\mathbf{r}) \int d \mathbf{q} e^{i \mathbf{q} \cdot \mathbf{r}} A_{o, l}(\mathbf{q}) \tag{10.53}
\end{equation*}
$$

The wavefunction $\varphi_{\mathbf{p}}(\mathbf{r})$ satisfies the Schrödinger equation,

$$
\begin{equation*}
\left(-\frac{1}{2 M} \partial^{2}+V(r)-\frac{p^{2}}{2 M}\right) \varphi_{\mathbf{p}}(\mathbf{r})=0 \tag{10.54}
\end{equation*}
$$

where $M=m_{\chi} / 2$ is the reduced mass and $p=M v$ is the non-relativistic momentum. In general, the potential $V(r)$ can be matrix valued in the space of partial waves, in which case the Schrödinger equation is then a system of coupled differential equations. To solve this equation we decompose the wavefunction $\varphi_{\mathbf{p}}(\mathbf{r})$ in partial waves

$$
\begin{equation*}
\varphi_{\mathbf{p}}(\mathbf{r})=\frac{(2 \pi)^{3 / 2}}{4 \pi p} \sum_{\ell}(2 \ell+1) e^{i \delta_{\ell}} R_{p, \ell}(r) P_{\ell}(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}}) \tag{10.55}
\end{equation*}
$$

such that the radial part, $R_{p, \ell}(r)$, satisfies

$$
\begin{equation*}
\frac{-1}{2 M}\left(\frac{d^{2} R_{p, \ell}}{d r^{2}}+\frac{2}{r} \frac{d R_{p, \ell}}{d r}-\frac{\ell(\ell+1)}{r^{2}} R_{p, \ell}\right)-\left(\frac{p^{2}}{2 M}-V(r)\right) R_{p, \ell}=0 \tag{10.56}
\end{equation*}
$$

with the completeness relation

$$
\begin{equation*}
\int_{0}^{\infty} d p R_{p, \ell}(r) R_{p, \ell}\left(r^{\prime}\right)=\frac{\delta\left(r-r^{\prime}\right)}{r^{2}} \tag{10.57}
\end{equation*}
$$

Plugging the partial wave decomposition (10.55) into (10.53) along with $\varphi_{\mathbf{p}}^{\circ}(\mathbf{r})=e^{i \mathbf{p} \cdot \mathbf{r}}$ gives

$$
\begin{equation*}
A_{l}(p)=\frac{1}{p} \int_{0}^{\infty} r^{2} d r R_{p, \ell}(r) \int_{0}^{\infty} q d q R_{q, \ell}^{\circ}(r) A_{\mathrm{o}, \ell}(q) \tag{10.58}
\end{equation*}
$$

From the free solution $R_{p, \ell}^{\circ}$ we know that

$$
\begin{equation*}
\frac{d^{\ell}}{d r^{\ell}} R_{q, \ell}^{\circ}(r=0)=\sqrt{\frac{2}{\pi}} \frac{\ell!q^{\ell+1}}{(2 \ell+1)!!} . \tag{10.59}
\end{equation*}
$$

Applying the completeness relation (10.58) gives

$$
\begin{equation*}
A_{\ell}\left(p, p^{\prime}\right)=\sqrt{\frac{\pi}{2}} \frac{(2 \ell+1)!!}{\ell!} \frac{1}{p} \frac{d^{\ell}}{d r^{\ell}} R_{p, \ell}(r=o) a_{0}, \ell \tag{10.60}
\end{equation*}
$$

such that the Sommerfeld enhancement for a the $\ell^{\text {th }}$ partial wave is

$$
\begin{equation*}
S_{l}=\left|\sqrt{\frac{\pi}{2}} \frac{(2 \ell+1)!!}{\ell!} \frac{1}{p^{\ell+1}} \frac{d^{\ell}}{d r^{\ell}} R_{p, \ell}(r=0)\right|^{2} \tag{10.61}
\end{equation*}
$$

We thus see that the Sommerfeld enhancement is given by the solution of the Schrödinger equation at the origin.

## 10.B. 1 NUMERICAL ALGORITHM

Refs. [384,385] provide a method to numerically evaluate the enhancement factor $S$. The completeness relation (10.57) is valid at long distances,

$$
\begin{equation*}
\left.R_{p, \ell}(r)\right|_{r \rightarrow \infty} \rightarrow \sqrt{\frac{2}{\pi}} \frac{\sin \left(p r-\ell \pi / 2+\delta_{\ell}\right)}{r} . \tag{10.62}
\end{equation*}
$$

For simplicity, let us work with the dimensionless variable $x=p r$ and the rescaled wavefunction $\Phi_{p, \ell}(x)=\frac{x R_{p}, \ell(x)}{N p}$ where $N$ is an arbitrary normalization. Using these variables, the Schrödinger equation takes the form

$$
\begin{equation*}
-\Phi_{p, \ell}(x)^{\prime \prime}+\left(\mathcal{V}(x)+\frac{\ell(\ell+1)}{x^{2}}-1\right) \Phi_{p, \ell}(x)=0 \tag{10.63}
\end{equation*}
$$

where $\mathcal{V}(x)=\frac{2 M}{p^{2}} V(x / p)$ and we impose the initial conditions

$$
\begin{equation*}
\lim _{x \rightarrow 0} \Phi_{p, \ell}(x)=x^{\ell+1} \tag{10.64}
\end{equation*}
$$

From (10.63) and the fact that $\lim _{x \rightarrow \infty} \mathcal{V}(x)=0$, it is clear that in the asymptotically far away region,

$$
\begin{equation*}
\left.\Phi_{\ell}(x)\right|_{x \rightarrow \infty} \rightarrow C \sin \left(x-\ell \pi / 2+\delta_{\ell}\right) \tag{10.65}
\end{equation*}
$$

Moreover, to satisfy the asymptotic normalization of $R_{p, \ell}(r)$, we need to fix the normalization $N=\sqrt{\frac{2}{\pi}} \frac{1}{\mathrm{C}}$. We can then use $R_{p, \ell}=N p \Phi_{l} / x$ in (10.61) along with the initial condition to obtain

$$
\begin{equation*}
A_{\ell}(p)=\frac{(2 \ell+1)!!}{C} p^{\ell} a_{\mathrm{o}, \ell}=\frac{(2 \ell+1)!!}{C} A_{\mathrm{o}, \ell}(p) \tag{10.66}
\end{equation*}
$$

so that the Sommerfeld factor is

$$
\begin{equation*}
S=\left(\frac{(2 \ell+1)!!}{C}\right)^{2} \tag{10.67}
\end{equation*}
$$

We thus reduce the calculation of the Sommerfeld enhancement $S$ to the determination of $C$. This is obtained by numerically solving (10.63) with the initial condition (10.64) and

$$
\begin{equation*}
C^{2}=\left.\left(\Phi_{l}(x)^{2}+\Phi_{l}(x-\pi / 2)^{2}\right)\right|_{x \rightarrow \infty} \tag{10.68}
\end{equation*}
$$



Figure 10.B.1: Numerical evaluation of the Sommerfeld enhancement factor as a function of the dark matter reduced mass $M$ for a range of relative velocities. The mediator mass is fixed to 90 GeV and $\alpha=1 / 30$.

## 10.B. 2 Coulomb and Yukawa example

For the Coulomb potential $V(r)=-\alpha / r$, one can obtain an analytic expression for the Sommerfeld enhancement [384,385],

$$
\begin{equation*}
S_{\ell}=\frac{e^{\pi \alpha / v} \pi \alpha}{v \sinh (\pi a / v) \ell!^{2}} \prod_{s=1}^{\ell}\left(s^{2}+\frac{a^{2}}{v^{2}}\right) \approx \frac{2 \pi}{\ell!^{2}}\left(\frac{\alpha}{v}\right)^{2 \ell+1} \tag{10.69}
\end{equation*}
$$

where the approximation holds for large $\alpha / v$. There exists no simple analytical expression for the enhancement from a Yukawa potential $V(r)=-\alpha e^{-\mu r} / r$, but one can easily evaluate it numerically using the method presented, see Figure (1o.B.1). The presence of resonances can be explained by bound states [388].

## 10.C Box approximation

We have shown that bound state resonances can generate large Sommerfeld enhancements. In this appendix we adapt the procedure used in [388] to quantitatively understand these resonances. In [388], it was shown that the a reasonable approximation for the Yukawa potential is a flat potential well whose width is determined by the characteristic length scale of the interaction, $r_{0}=1 / m_{\phi}$,

$$
\begin{equation*}
V_{\mathrm{box}}(r)=-U_{\circ} \Theta\left(r_{\mathrm{o}}-r\right) \tag{10.70}
\end{equation*}
$$

The depth of the rectangular well $U_{\mathrm{o}}$ is fixed by requiring that the box approximation matches the Yukawa potential at $r=r_{\mathrm{o}}$,

$$
\begin{equation*}
V_{\mathrm{box}}(r)=-\frac{a m}{e} \Theta\left(\frac{1}{m}-r\right) . \tag{10.71}
\end{equation*}
$$

This approximate is constructed to capture only the qualitative behavior of the full potential and is not a detailed matching to an effective theory. Observe that this analysis agrees with the fact that the Coulomb limit ( $m_{\phi} \rightarrow 0$ ) does not have resonances: this potential has no natural length scale for constructing the rectangular well.

## 10.C. 1 Application to $V \sim r^{-3}$

We adapt this procedure to the singular $1 / r^{3}$ potential,

$$
\begin{equation*}
V(r)=\frac{-a}{f^{2}} \frac{1}{r^{3}} \tag{10.72}
\end{equation*}
$$

The natural length scale of the problem is the dimensionful scale of the coupling, $r_{0}=\sqrt{\alpha} / f$. In principle there is also a scale set from the exponential term $e^{-m_{\phi} r}$, but for UV models with $m_{\phi} \ll f$ this contribution is negligible. This reflects the fact that the resonant behavior of singular potentials in this limit do not depend strongly on the specific value of the mediator mass $m_{\phi}$.

This simple box potential approximation provides an estimate for the upper bound of Sommerfeld enhancement coming from resonances in a singular potential. The solution to the $\ell=0$ Schrödinger equation inside the box $\left(r<r_{0}\right)$ is

$$
\begin{equation*}
\left.\varphi\left(p r<p r_{0}\right)\right|_{p}=\frac{\sin (\kappa p r)}{\kappa} \tag{10.73}
\end{equation*}
$$

where $\kappa p=\sqrt{p^{2}+2 U_{0} M}$. Outside the box, $r>r_{0}$, there is effective no potential so that

$$
\begin{equation*}
\varphi\left(p r>p r_{\circ}\right)_{p}=C \sin (p r+\delta) . \tag{10.74}
\end{equation*}
$$

$C$ is determined by requiring continuity at $r_{0}$ so that the enhancement is

$$
\begin{align*}
S & =\left[\cos ^{2}\left(\kappa p r_{0}\right)+\frac{\sin ^{2}\left(\kappa p r_{0}\right)}{\kappa^{2}}\right]^{-1} \\
& \approx\left[\cos ^{2}\left(r_{0} \sqrt{2 U_{0} M}\right)+\frac{p^{2}}{2 M U_{0}} \sin ^{2}\left(r_{0} \sqrt{2 U_{0} M}\right)\right]^{-1} \tag{10.75}
\end{align*}
$$

where we use the non-relativistic approximation $p^{2} \ll U_{0} M$. Observe that the prefactor of the sine term is small so that $S$ becomes large when the cosine vanishes. In other words, this expression maximized when $r_{0} \sqrt{2 U_{0} M}=(2 n+1) \pi / 2$ with

$$
\begin{equation*}
S_{\max } \approx \frac{2 M U_{0}}{p^{2}}=\frac{(2 n+1)^{2} \pi^{2}}{4 r_{\circ}^{2} p^{2}} . \tag{10.76}
\end{equation*}
$$

This peak is exactly the resonance when the pair of dark matter particles forms a bound state. Note that this approximation is independent of the depth of the rectangular well, $U_{0}$.

It is straightforward to generalize these expressions for an arbitrary orbital angular momenta, $\ell$, by including the angular barrier to the box potential and applying the appropriate boundary conditions. One obtains

$$
S_{\ell}=\left(\frac{\pi\left[\left(2 \ell^{\prime}\right)!!\right]^{2} \tilde{\kappa}^{2 \ell^{\prime}}}{2^{2 \ell^{\prime}+1} \Gamma\left(\ell^{\prime}+1\right)^{2}}\right) \frac{\left[Y_{\ell^{\prime}}\left(p r_{o}\right)-\cot (\delta) J_{\ell^{\prime}}\left(p r_{o}\right)\right]^{2}}{\left[1+\cot ^{2}(\delta)\right] J_{\ell^{\prime}}^{2}\left(\tilde{\kappa} p r_{\mathrm{o}}\right)}
$$

where $\ell^{\prime}=\ell+\frac{1}{2}$ and $\tilde{\kappa}^{2}=2 M U_{0} / p^{2}$. The qualitative scaling behavior of the resonance can be seen by setting $\cot (\delta)=0$, and assuming that $p r_{0} \ll 1$ so that

$$
\begin{equation*}
S_{\max } \sim \frac{1}{\left(p^{2} r_{0}^{2}\right)^{2 \ell+1}} \sim \frac{1}{v^{\ell \ell+2}} . \tag{10.78}
\end{equation*}
$$

## 10.C. 2 Dimensional analysis

To estimate the Sommerfeld enhancement off resonance one must estimate $U_{0}$. We use the assumption that the uv physics encoded in $U_{\circ}$ does not significantly change the IR potential so that the height of the square well $U_{\circ}$ is well approximated by the value of the singular potential at the cutoff scale,

$$
\begin{equation*}
U_{0} \sim \frac{f}{a^{1 / 2}} \sim \frac{1}{r_{0}} \tag{10.79}
\end{equation*}
$$

so that for $\ell=0$, the Sommerfeld enhancement is approximately

$$
\begin{equation*}
S \approx\left[\cos ^{2}\left(\sqrt{\frac{2 M \alpha^{1 / 2}}{f}}\right)+\frac{p^{2} \alpha^{1 / 2}}{2 M f} \sin ^{2}\left(\sqrt{\frac{2 M \alpha^{1 / 2}}{f}}\right)\right]^{-1} \tag{10.80}
\end{equation*}
$$

An estimate for the parameters required to hit a resonance without tuning is thus

$$
\begin{equation*}
M_{\mathrm{res}} \sim \frac{1}{r_{\mathrm{o}}} \sim \frac{f}{a^{1 / 2}}, \tag{10.81}
\end{equation*}
$$

which, for most cases, lies at the boundary of the range of the theory's validity.

## 11 <br> RPV gluinos

$R$-PARITY VIOLATING MODELS OF SUPERSYMMETRY are able to avoid stringent experimental constraints on the supersymmetry mass scale by relaxing the requirement that the lightest superpartner must be stable against decay. In this chapter we present a search for this scenario which is otherwise rather difficult to probe.

### 11.1 Overview

The lack of observation of superpartners at the Large Hadron Collider so far has led to a renewed interest in supersymmetric models with R-parity violation (RPV). In particular, imposing the Minimal Flavor Violation (MFV) hypothesis on a general RPV model leads to a realistic and predictive framework. Naturalness suggests that stops and gluinos should appear at or below the TeV mass scale. We consider a simplified model with these two particles and MFV couplings. The model predicts a significant rate of events with same-sign dileptons and $b$-jets. We re-analyze a recent CMS search in this channel and show that the current lower bound on the gluino mass is about 800 GeV at $95 \%$ confidence level, with only a weak dependence on the stop mass as long as the gluino can decay to an on-shell top-stop pair. We also discuss how this search can be further optimized for the RPV/MFV scenario, using the fact that MFV stop decays often result in jets with large invariant mass. With the proposed improvements, we estimate that gluino masses of up to about 1.4 TeV can be probed at the 14 TeV LHC with a 100 $\mathrm{fb}^{-1}$ data set.

### 11.2 INTRODUCTION

Supersymmetry (SUSY) remains one of the most compelling ideas for extending the Standard Model (SM). While SUSY is clearly broken in nature, naturalness of electroweak symmetry breaking strongly suggests that it should be restored at an energy scale $\lesssim 1 \mathrm{TeV}$. This would require the SUSY partners of the SM particles to appear at that scale. However, experiments conducted in 2010-2012 at the Large Hadron Collider (LHC) have seen no evidence for such superpartners, placing lower bounds on the masses of some of them, squarks and gluinos, well in excess of 1 TeV . This apparent contradiction led many theorists to question the assumptions underlying the LHC searches. One of the most important assumptions is R-parity conservation, which implies that the lightest superpartner (LSP) is stable. A stable LSP in turn implies that each event with
superpartner production contains either missing transverse energy (MET) or exotic charged tracks, either of which provides a good handle to distinguish such events from the SM backgrounds. Most LHC searches make extensive use of such handles. If there is no conserved R-parity, these searches would not be applicable and the LHC bounds would be weakened significantly, removing conflict with naturalness.

From the theoretical point of view, R-parity is not required by SUSY: it is an additional discrete symmetry. The motivation for introducing this extra symmetry is purely phenomenological: it forbids baryon (B) and lepton (L) number violating operators that would otherwise induce rapid proton decay. However, proton decay and other tightly constrained B-and L-violating processes may be forbidden or suppressed to acceptable levels without introducing R-parity. An interesting proposal along these lines has been made recently by Csaki, Grossman and Heidenreich [128] (see also [426]). The authors start with a minimal SUSY model without R-parity. They then impose the Minimal Flavor Violation (MFV) hypothesis, which is strongly motivated by flavor physics constraints on SUSY, on the full superpotential, including B-and L-violating operators. The MFV hypothesis in effect imposes an accidental approximate R-parity on the first two generations and greatly suppresses dangerous operators such as those that induce proton decay. At the same time, there are non-trivial R-parity violating (RPV) couplings involving the third generation which are sufficient to render the LSP unstable on collider time scales and weaken the LHC bounds. This is the framework that we focus on in this paper. ${ }^{1}$

As for any SUSY model, the collider phenomenology of MFV SUSY depends sensitively on the superpartner spectrum. This, in turn, is determined by the details of the SUSY breaking sector and mediation, for which many possible models have been proposed. In this paper, we focus on a simple scenario motivated by bottom-up naturalness considerations. It is well known that the only superpartners required to be light $(\lesssim 1 \mathrm{TeV})$ by naturalness are the stops $\tilde{t}_{1,2}$, the Higgsino $\tilde{H}$, and the gluino $\tilde{g}$ : see, for example, Ref. [430] for a clear and careful explanation of this point. Of these, $\tilde{H}$ has a suppressed production rate due to its weak coupling. Thus, it will not have a considerable impact on phenomenology as long as it is not the LSP. We will therefore consider a simplified model [431] with just two states: a gluino $\tilde{g}$ and a stop $\tilde{t}$. All other SUSY particles are assumed to be either too heavy or too weakly coupled to be relevant at the LHC. ${ }^{2}$ We assume that the stop is the LSP, as motivated by naturalness considerations, and that $m_{\tilde{g}}>m_{\tilde{t}}+m_{t}$. We focus on gluino pair-production, $p p \rightarrow \tilde{g} \tilde{g}$, followed by a cascade decay:

$$
\begin{equation*}
\tilde{g} \rightarrow \tilde{t}, \quad \tilde{t} \rightarrow \bar{b} \bar{s} \quad \text { or } \quad \tilde{g} \rightarrow \tilde{t}^{*} t, \quad \tilde{t}^{*} \rightarrow b s \tag{11.1}
\end{equation*}
$$

The branching ratio for each of these channels is $50 \%$, assuming a purely Majorana gluino. With probability of $50 \%$, the gluino pair will produce a same-sign top pair $(t t$ or $\overline{t t})$. If each top decays leptonically, the final state will contain two same-sign leptons: $e^{ \pm} e^{ \pm}, \mu^{ \pm} \mu^{ \pm}$, or $e^{ \pm} \mu^{ \pm}$. Such "same-sign dilepton" (SSDL) events are very rare in the SM, and the SSDL signature already plays a prominent role in the LHC SUSY searches. Typically, these searches demand substantial MET in addition to SSDL, reducing their sensitivity to the RPV cascades (11.1) where the only sources of MET are neutrinos from leptonic top decays. However, the SSDL signature by itself is so striking that searches may be conducted even with no (or very low) MET cut, making them sensitive to RPV SUSY [432-435]. ${ }^{3}$ The first goal of this paper is to estimate the current bounds on our simplified model using the latest publicly available CMS search for the SSDL signature [444]. This search uses $10.5 \mathrm{fb}^{-1}$ of data collected at $\sqrt{s}=8 \mathrm{TeV}$ in the 2012 LHC run.

While the current SSDL searches already place interesting bounds on RPV SUSY, they are not optimized for this class of models. The second goal of this paper is to suggest ideas for optimizing this search that may be implemented by the experiments in the future. SSDL events in RPV SUSY have at least 6 parton-level jets. This high jet multiplicity can, by itself, provide an additional handle to suppress backgrounds. Moreover, two pairs of these jets come from stop decays. Depending on the gluino and stop masses, two regimes are possible. If $m_{\tilde{g}}-m_{\tilde{t}} \sim m_{t}$, the stops are typically non-relativistic in the lab frame and the two jets are well separated. In this regime, one simply needs to look for a resonance in the dijet invariant mass.

[^18]|  | SRo | SR1 | SR2 | $\mathrm{SR}_{3}$ | SR4 | SR5 | SR6 | $\mathrm{SR}_{7}$ | SR8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| No. of jets | $\geq 2$ | $\geq 2$ | $\geq 2$ | $\geq 4$ | $\geq 4$ | $\geq 4$ | $\geq 4$ | $\geq 3$ | $\geq 4$ |
| No. of $b$-tags | $\geq 2$ | $\geq 2$ | $\geq 2$ | $\geq 2$ | $\geq 2$ | $\geq 2$ | $\geq 2$ | $\geq 3$ | $\geq 2$ |
| $\ell$ charges | $++/--$ | $++/--$ | ++ | $++/--$ | $++/--$ | $++/--$ | $++/--$ | $++/--$ | $++/--$ |
| $E_{T}^{\text {miss }}$ | $>\mathrm{OGeV}$ | $>30 \mathrm{GeV}$ | $>30 \mathrm{GeV}$ | $>120 \mathrm{GeV}$ | $>50 \mathrm{GeV}$ | $>50 \mathrm{GeV}$ | $>120 \mathrm{GeV}$ | $>50 \mathrm{GeV}$ | $>0 \mathrm{Gev}$ |
| $H_{T}$ | $>80 \mathrm{GeV}$ | $>80 \mathrm{GeV}$ | $>80 \mathrm{Gev}$ | $>200 \mathrm{GeV}$ | $>200 \mathrm{GeV}$ | $>320 \mathrm{GeV}$ | $>320 \mathrm{GeV}$ | $>200 \mathrm{GeV}$ | $>320 \mathrm{GeV}$ |

Table 11.1: Event characteristics required in the 9 signal regions (SRs) used in the CMS SSDL + MET $+b$ analysis [444]. Note that the number of jets on the first line of the table includes both $b$-tagged and non- $b$-tagged jets. For the predicted background rates and the observed rates in each region, see Table 2 of Ref. [444].

The case $m_{\tilde{g}} \gg m_{\tilde{t}}$ is more interesting. In this case, the stops are predominantly relativistic, and their decay products are boosted in the direction of their motion. The two parton showers would typically be merged in a single jet, and the signatures of their "stoppy" origin are hidden in the substructure of the jet. Recently, much work has been done on exploring observables sensitive to jet substructure (for a review, see [445]). We will show how some of these techniques can be used to further enhance the sensitivity of the SSDL search for RPV SUSY.

The rest of the paper is organized as follows. The current bounds on RPV SUSY derived from the recently published CMS search in the SSDL channel are presented in Section 11.3. Additional cuts that can be used to improve the sensitivity of this search specifically in the RPV SUSY case are discussed in Section 11.4. Section 11.5 contains brief conclusions and outlook, while some of the details of the procedure used to recast the CMS search are presented in Appendix 11.A.

### 11.3 Current Bounds: Recasting the CMS SSDL Search

Both CMS and ATLAS perform searches for the SSDL signature, accompanied by MET and jets (with or without $b$-tag requirement), as part of their standard search strategy to look for R-parity conserving (RPC) SUSY with light gluinos and stops. These analyses have non-trivial sensitivity to the RPV SUSY cascade (11.1) since leptonic top decays contain neutrinos which provide genuine MET, typically in the few tens of GeV range. While most RPC SUSY searches must impose a MET cut of at least 100 GeV to suppress SM backgrounds, the SSDL signature by itself is very rare in the SM so that such a strong MET cut is not required. The CMS collaboration recently published bounds based on a number of signal regions (SRs) with either no MET cut or sufficiently low MET cuts ( $30-50 \mathrm{GeV}$ ) that are easily exceeded by the top-induced MET [444]. While the CMS paper interprets the results in terms of RPC SUSY, it is straightforward to "recast" their published data to provide limits on the RPV case. ${ }^{4}$

The cuts imposed by the CMS analysis are summarized in Table 11.1. The acceptance cuts are $p_{T}>40 \mathrm{GeV},|\eta|<2.4$ for jets (both $b$-tagged and non- $b$-tagged), and $p_{T}>20 \mathrm{GeV},|\eta|<2.4$ for electrons and muons. Events with a third lepton are vetoed if they contain an opposite-sign lepton pair with invariant mass below 12 GeV , or between 76 and 106 GeV , to avoid contamination from $Z$ decays. For more details on the CMS analysis, see Ref. [444].

In all nine signal regions, the data is consistent with the SM expectation, so an upper bound on the number of signal events can be set. We simulated the process $p p \rightarrow \tilde{g} \tilde{g}$, followed by the decays (11.1) and the leptonic top decay on both sides, using Pythia 8.162 [446], for a large set of ( $m_{\tilde{g}}, m_{\tilde{t}}$ ) points. The leading order (LO) cross section provided by Pythia is multiplied by the NLO K-factor computed with Prospino 2.1 [447] for normalization. To compute the efficiency of the CMS cuts on the signal, we essentially follow the procedure described in the CMS report [444] and its predecessors [444, 448]. For details, see Appendix 11.A. The only non-trivial deviation from the CMS prescriptions concerns the treatment of lepton selection efficiencies. These have two factors: identification (ID) efficiency and the efficiency of the lepton isolation cut. CMS only published the combined lepton selection efficiency for a benchmark RPC SUSY point

[^19]

Figure 11.3.1: 95\% CL exclusion of the RPV SUSY simplified model parameter space, based on the 4 most sensitive search regions (SRs) from the CMS SSDL+MET $+b$ search [444] with $10.5 \mathrm{fb}^{-1}$ of data collected at the 8 TeV LHC.

LM9 [449]. However, the RPV SUSY signal is expected to have a significantly different lepton isolation efficiency: there is more hadronic activity, and, in some parts of the parameter space, the tops are boosted, resulting in a $b$-jet in close proximity to the lepton. To take this into account, we estimate the lepton isolation cut efficiency from our signal MC, at each ( $m_{\tilde{g}}, m_{\tilde{t}}$ ) point, and multiply by the lepton ID efficiency estimated by a separate simulation of the LM9 RPC SUSY signal. The cross section, acceptance and efficiency are then used to compute the number of expected signal events at each ( $m_{\tilde{g}}, m_{\tilde{t}}$ ) point. Comparing this number with the background prediction and data provided by CMS and using the $C L_{s}$ method [450] yields the expected $95 \%$ confidence level (CL) exclusion.

The results of this analysis are summarized by Figure 11.3 .1 , which shows the $95 \%$ CL exclusion contours from the four most sensitive signal regions. We conclude that the current bound on the gluino mass is about 800 GeV . The bound is approximately independent of the stop mass as long as an on-shell decay $\tilde{g} \rightarrow \tilde{t} t$ is kinematically allowed. Note that this bound is somewhat stronger than the bound recently obtained in Ref. [435] by recasting the ATLAS SSDL+MET $+j$ search [451]. The difference is especially pronounced in the region of relatively small gluino/stop mass splitting, where the ATLAS analysis loses sensitivity due to the large MET required ( $\geq 150 \mathrm{GeV}$ ). The remaining differences are accounted for by the slightly higher integrated luminosity of the CMS search, as well as the additional requirement of $b$-tagged jets imposed by CMS.


Figure 11.4.1: Lab-frame angular separation between the two quarks from a stop decay. The stops are produced in the gluino cascade (11.1), following gluino pair-production at a 14 TeV LHC. We assume $m_{\tilde{g}}=1.2 \mathrm{TeV}$, and vary the stop mass: $m_{\tilde{t}}=200,400,600$ and 800 GeV distributions are shown in red, orange, green and blue, respectively. The distributions were calculated using MadGraph 5 [452].

### 11.4 Future Searches: Optimizing for the RPV

While the current SSDL+MET $+b$ searches already provide meaningful bounds on RPV SUSY, they are clearly not optimized for this model. In this section, we suggest ways to enhance their sensitivity to the RPV model, and demonstrate the improvements with a Monte Carlo analysis.

The key observation is that in a large section of the available parameter space, the stops produced in the gluino decays are relativistic. The stop boost in the gluino rest frame is given by

$$
\begin{equation*}
\gamma=\frac{1}{\sqrt{1-\beta^{2}}}=\frac{m_{\tilde{g}}^{2}+m_{\tilde{t}}^{2}-m_{t}^{2}}{2 m_{\tilde{g}} m_{\tilde{t}}} \tag{11.2}
\end{equation*}
$$

so that stops are relativistic when $m_{\tilde{g}}>m_{\tilde{t}}$. For example, $m_{\tilde{g}}=1.2 \mathrm{TeV}$ and $m_{\tilde{t}}=200 \mathrm{GeV}$ yields $\beta \approx 0.9$. Since gluinos themselves are mostly produced with non-relativistic speeds in the lab frame, such stops are typically also relativistic in the lab frame. In this regime, the two quarks produced in the stop decay are boosted in the same direction and have a small angular separation as can be seen in Figure 11.4.1. The showers produced by the neighboring quarks tend to be merged into a single jet. Such "stoppy" jets can be distinguished from regular QCD jets, as we will discuss in detail below, giving an extra handle that can be used to suppress the background and improve the search reach.

To assess the potential improvement, we performed a Monte Carlo study for the 14 TeV LHC. For this study, we simulated the signal, $p p \rightarrow \tilde{g} \tilde{g}$, using Pythia 8.162 [446], for a large set of $\left(m_{\tilde{g}}, m_{\tilde{t}}\right)$ points. The leading order (LO) cross section provided by Pythia is multiplied by the NLO K-factor for normalization. Gluino, top and $W$ decays are also treated in Pythia, as are QCD initial radiation, showering and hadronization. Jet reconstruction is modeled with Fast Jet [453] using the anti- $k_{T}$ clustering algorithm. The dominant irreducible backgrounds, $\bar{t} W$ and $\bar{t} Z$, were simulated using the same tools.


Figure 11.4.2: Estimated 95\% CL expected exclusion (left panel) and $5 \sigma$ expected discovery (right panel) reach in the RPV SUSY simplified model parameter space at the 14 TeV LHC with $100 \mathrm{fb}^{-1}$. Red/green lines: reach of the analysis identical to the one in Ref. [444], for signal regions SR6/SR8. Black/gray: reach of the analysis with the SR8 cuts and an additional requirement of one/two jets with $M_{j}>175 \mathrm{GeV}$. In the gray shaded region, the decay $\tilde{g} \rightarrow \tilde{t} t$ is kinematically forbidden.

The cross sections for these processes are also normalized with NLO K-factors [454, 455].
To set a benchmark point against which improvements can be judged, we estimated the reach of the searches currently performed by CMS [444] at the 14 TeV LHC with $L_{\mathrm{int}}=100 \mathrm{fb}^{-1}$. For this estimate, we implemented the cuts corresponding to the CMS signal regions listed in Table 11.1 (with the exception of $\mathrm{SR}_{7}$, which would require a separate analysis due to an additional $b$-tagged jet requirement) on both signal and background samples. We modeled $b$-tagging by applying a $p_{T}$-dependent tagging efficiency for the CSVM tagger [456] to all the jets that can be traced back to a $b$-hadron. The cut efficiencies for the signal and the background are listed in Table 11.1. We then estimated the instrumental background. The two dominant sources are "fake leptons" from sources such as heavy-flavor decays and misidentified hadrons, and "charge flips", events with opposite-sign leptons where one of the charges is mismeasured. The ratio of the instrumental background to the irreducible component reported in Ref. [444] is roughly between $1: 1$ and $2: 1$, depending on the signal region. This indicates that instrumental backgrounds will play an important role at 14 TeV as well. Unfortunately, detailed modeling of these backgrounds requires either detector simulation or data-based techniques. However, a rough estimate may be obtained as follows. Since the physical process primarily responsible for the instrumental backgrounds is top pair-production ${ }^{5}$, it is reasonable to expect that the rates scale approximately with the total $\bar{t}$ cross section when the collision energy is increased from 8 to 14 TeV . Using this scaling and the instrumental background rates in various signal regions quoted in Ref. [444], we obtained corresponding estimates at 14 TeV . We found that the irreducible and instrumental background components scale by similar factors when going to 14 TeV : for example, our estimate of the instrumental/irreducible ratio at 14 TeV for the signal region SR6 is 0.86 , while for SR 8 it is 1.62 , quite close to the ratios at 8 TeV .

Combining the irreducible and instrumental backgrounds, we computed the exclusion levels expected under the

[^20]| process | $\sigma$ (total) | Eff(SR8) | $\sigma$ (SR8) | Eff( 1 HMJ ) | $\sigma(\mathrm{SR} 8+1 \mathrm{HMJ})$ | Eff( 2 HMJ ) | $\sigma$ (SR8+2HMJ) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| signal (1200, 200) | 113 | 0.41 | 0.46 | 86 | 0.40 | 40 | 0.18 |
| $(1200,500)$ | 114 | 0.44 | 0.50 | 64 | 0.32 | 24 | 0.12 |
| $(1200,800)$ | 114 | 0.45 | 0.52 | 70 | 0.36 | 31 | 0.16 |
| $(1300,200)$ | 63 | 0.36 | 0.23 | 89 | 0.20 | 40 | 0.09 |
| $(1300,500)$ | 63 | 0.48 | 0.30 | 71 | 0.22 | 22 | 0.07 |
| $(1300,800)$ | 63 | 0.45 | 0.28 | 75 | 0.21 | 31 | 0.09 |
| $(1300,1100)$ | 62 | 0.30 | 0.19 | 81 | 0.15 | 43 | 0.08 |
| $(1400,200)$ | 35 | 0.39 | 0.14 | 95 | 0.13 | 48 | 0.07 |
| $(1400,500)$ | 35 | 0.44 | 0.15 | 73 | 0.11 | 27 | 0.04 |
| $(1400,800)$ | 35 | 0.43 | 0.15 | 78 | 0.12 | 34 | 0.05 |
| $(1400,1000)$ | 35 | 0.45 | 0.16 | 81 | 0.13 | 43 | 0.07 |
| $(1400,1200)$ | 35 | 0.29 | 0.1 | 80 | 0.08 | 40 | 0.04 |
| background $t \bar{t} W$ | 590 | 0.07 | 0.38 | 4.7 | 0.02 | 0.3 | 0.001 |
| $\bar{t} \bar{\square}$ | 910 | 0.03 | 0.30 | 7.9 | 0.02 | 0.6 | 0.002 |

Table 11.1: Cross sections (in fb) and efficiencies (in \%) of signal and background processes, at the 14 TeV LHC. The signal points are labeled by $\left(m_{\tilde{g}}, m_{\tilde{t}}\right)$, both in GeV . The selection cuts are labeled as follows: SR8 refers to the cuts imposed by the CMS analysis [444] in signal region 8 (see Table 11.1); 1HMJ means requiring at least one "high-mass" jet ( $M_{j}>175 \mathrm{GeV}$ ); similarly, 2 HMJ requires at least 2 jets with $M_{j}>175 \mathrm{GeV}$. The 1 HMJ and 2HMJ cuts are applied to the events that pass all SR8 cuts.
assumption that the data exactly matches the background prediction, as well as the discovery reach defined by requiring at least a $5 \sigma$ difference between the signal+background and background-only predictions. The estimated exclusion and discovery reach contours are shown in Figure 11.4.2 for the two most sensitive signal regions: SR6 (red contour) and SR8 (green contour).

To identify the merged jets from stop decays, we first reclustered the samples, setting the jet opening angle to $\Delta R=1.0$, as opposed to $\Delta R=0.5$ used by the CMS analysis. Such "fat" jets are already being used by experimental analyses involving jet substructure (see, for example, Refs. [457,457]). We then computed the invariant mass $M_{j}$ of each jet. The distributions of the largest $M_{j}$ in each event, for both the signal and the (irreducible) background samples, are shown in Figure 11.4.3. It is obvious that $M_{j}^{\max }$ is an excellent signal/background discriminator. For the case $m_{\tilde{g}} \gg m_{\tilde{t}}$, illustrated in the left panel of the figure, the reason is obvious: the high-mass jets in the signal are due to boosted stop decays, and their masses peak around $m_{\tilde{t}}$. However, somewhat more surprisingly, this discriminator continues to work well in the regime $m_{\tilde{g}} \sim m_{\tilde{t}}$, as illustrated by the right panel of the figure. The reason for this is simply the large jet multiplicity in the signal, which at parton level has 6 quarks in the final state. In this situation, two independent parton showers (from different stops, or from a stop and a top) often get accidentally merged into a single jet which is more likely to have a large invariant mass than a single-parton QCD jet. (This phenomenon was previously noticed in [458].) As a result, requiring massive jet(s) improves the reach of the search throughout the parameter space, and not just for large $m_{\tilde{g}} / m_{\tilde{t}}$.

The improvement of the reach with the jet mass cut is shown by the black and gray lines in Figure 11.4.2. This analysis imposes all of the SR8 cuts with the additional requirement of at least one or two high-mass jets with $M_{j}>175 \mathrm{GeV}$. The efficiencies of these cuts, and cross sections after all cuts, are listed in Table 11.1. For the reach estimate, we assumed that the efficiency of the jet invariant mass cuts on the instrumental and irreducible backgrounds are the same (which seems reasonable since both contain QCD jets of similar energies). We found that gluinos up to $1.4-1.45 \mathrm{TeV}$ can be excluded at the $95 \% \mathrm{CL}$, while gluinos up to $1.3-1.35 \mathrm{TeV}$ can be discovered at the $5 \sigma$ level at the 14 TeV LHC with $100 \mathrm{fb}^{-1}$. The dependence of the reach on the stop mass is quite weak, especially when the analyses with $\geq 1$ and $\geq 2$ high-mass jets are combined.

An even stronger separation of signal and background can be achieved by noticing that the high-mass jets in the background are primarily due to boosted, fully hadronic tops. Such jets have three hard partons. In contrast, the signal jets typically have two hard partons from a two-body stop decay. To exploit this, we used the $N$-subjettiness technique proposed


Figure 11.4.3: Distributions of the largest jet invariant mass $M_{j}^{\max }$, in the signal (blue) and irreducible background (red) events passing SR8 cuts at the 14 TeV LHC. The signal is simulated for $\left(m_{\tilde{g}}, m_{\tilde{t}}\right)=(1200,200) \mathrm{GeV}$ (left panel) and (1200, 800) GeV (right panel). The background includes the SM $\bar{t} W$ and $\bar{t} Z$ processes.
by Thaler and Van Tilburg [459]. In this approach, observables $\tau_{N}$ are defined with $N=1,2, \ldots$. A low value of the ratio $\tau_{N} / \tau_{N-1}$ indicates that the jet likely has an $N$-pronged substructure. For example, the distributions of jets with $M_{j}>175 \mathrm{GeV}$ in $\tau_{2} / \tau_{1}$ and $\tau_{3} / \tau_{2}$ observables are shown in Figure 11.4.4, where in the signal simulation we assumed $\left(m_{\tilde{g}}, m_{\tilde{t}}\right)=(1400,200)$ GeV , and used the onepass_kt_axes minimization scheme and $\beta=1.1$. As expected, low values of $\tau_{2} / \tau_{1}$ are favored in the signal, while low values of $\tau_{3} / \tau_{2}$ are favored in the background. It should be noted that with the $100 \mathrm{fb}^{-1}$ data set, the reach of the jet-mass based searches shown in Figure 11.4.2 is already statistics-limited, so no further improvement can be achieved by cutting on the $N$-subjettiness observables. However, they can be useful for larger data sets, or as a part of more globally optimized set of cuts.

Since no detector simulation could be performed for this study, our instrumental background estimate is clearly very crude and has a large uncertainty. To illustrate how this uncertainty affects the reach of the proposed search, we define

$$
\begin{equation*}
\zeta=\frac{\text { Total BG Rate }}{\text { Irreducible BG Rate }}, \tag{11.3}
\end{equation*}
$$

where both rates include all the cuts imposed in a particular analysis. Figure 11.4.5 shows the variation of the reach for values of $\zeta$ between 1 and 10 , for the same analysis as in Figure 11.4 .2 (SR8 plus $\geq 1$ or $\geq 2$ jets with $M_{j}>175 \mathrm{GeV}$ ). The reach estimates are relatively robust with respect to the uncertainty in the instrumental background estimate, due to a strong dependence of the signal rates on $m_{\tilde{g}}$.

### 11.5 Discussion and Conclusions

The main results of this paper can be summarized as follows:

- The current CMS searches for anomalous events with SSDL and $b$-jets place a lower bound of about 800 GeV on the gluino mass in the gluino-stop simplified model of RPV/MFV SUSY. The bound is only weakly sensitive to the stop mass, as long as an on-shell decay $\tilde{g} \rightarrow \tilde{t}$ is kinematically allowed.
- A search identical to the current CMS search, implemented at the 14 TeV LHC with $100 \mathrm{fb}^{-1}$ of data, is estimated to have the sensitivity to exclude gluino masses up to about 1.3 TeV at the $95 \% \mathrm{CL}$, and a $5 \sigma$ discovery reach of about 1.2 TeV . Again, these are largely insensitive to the stop mass.


Figure 11.4.4: Distributions of $N$-subjettiness observables, $\tau_{2} / \tau_{1}$ (left) and $\tau_{3} / \tau_{2}$ (right), for the high-mass jets ( $M_{j}>175 \mathrm{GeV}$ ) in the signal (blue) and irreducible background (red) events passing SR8 cuts. The signal is simulated for $\left(m_{\tilde{g}}, m_{\tilde{t}}\right)=(1400,200) \mathrm{GeV}$. All distributions are normalized to unit area.

- An addition of a cut on the jet invariant mass improves the $95 \% \mathrm{CL}$ exclusion reach and the $5 \sigma$ discovery reach to approximately 1.45 TeV and 1.35 TeV , respectively. While the improvement in terms of the gluino mass is only about $10 \%$ in both cases, it is still very significant since the gluino cross section drops very rapidly with mass.

While the motivation for our analysis comes primarily from the MFV SUSY model [128], the results apply quite generally to RPV models with a stop LSP, decaying via a UDD-type operator. (See, for example, Ref. [460] for a recent discussion of such models.) A non-MFV flavor structure of the stop decay operator may result in fewer $b$-jets, but since top quarks still provide two genuine $b$-jets per event, even in this case the efficiencies of the cuts should not be strongly degraded.

For our signature to work, it is crucial that the gluino be a Majorana particle. If the gluino is Dirac, no SSDL signature is possible, and other handles must be used to suppress the SM background. However, high-mass jets from stop decays are still present in this situation, and can provide a useful discriminant [435]. It would be interesting to see if, in addition to stop jets, massive jets formed by the boosted SM tops produced from the same gluino decays can be useful in this context. (The utility of boosted top-jets in searching for the gluino-stop cascade decays in R-parity conserving SUSY has been pointed out in [433].) We leave this possibility for future study.

## 11.A Details of the Recasting Procedure

To recast the CMS SSDL+MET $+b$ analysis in terms of the RPV SUSY model, we follow closely the instructions provided by CMS in [444] and its predecessors [444, 448]. The only significant difference is in the treatment of leptons. The instructions recommend analyzing leptons at parton level, by taking the leptons that pass the kinematic cuts and applying the selection efficiencies given in Section 7 of [444]. These selection efficiencies, which account for lepton identification efficiencies, isolation cuts, and detector effects, had been computed from Monte Carlo studies of simplified model A1 ( $p p \rightarrow \tilde{g} \tilde{g}$, $\tilde{g} \rightarrow t t \tilde{\chi}^{0}$ ) at the RPC SUSY benchmark point LM9. However, because the leptons in the RPV SUSY signal process may come from boosted tops, there is extra hadronic activity near the leptons, and the LM9 selection efficiencies do not properly model the isolation cuts for the RPV signal. Therefore, we extract the isolation cut efficiencies for RPV from our signal MC. To do so, we impose a lepton isolation cut on the hadronized signal MC events. Following [448], Iso $(\hat{\ell})$ is defined as a scalar sum of the lepton $p_{T}$ 's and photon and hadron $E_{T}$ 's within a cone of size $\Delta R \equiv \sqrt{(\Delta \eta)^{2}+(\Delta \varphi)^{2}}<0.3$ about the lepton, not including


Figure 11.4.5: Estimated discovery reach in the RPV SUSY simplified model parameter space, at the 14 TeV LHC with $100 \mathrm{fb}^{-1}$ of data, for a range of assumptions concerning the instrumental background. The selection cuts are SR8, plus $\geq 1$ (left) or $\geq 2$ (right) jets with $M_{j}>175 \mathrm{GeV}$. The value $\zeta=2.62$ is the estimate obtained by rescaling from 8 TeV and used in Figure 11.4.2. In the gray shaded region, the decay $\tilde{g} \rightarrow \tilde{t} t$ is kinematically forbidden.
the $p_{T}$ of the lepton itself:

$$
\begin{equation*}
\operatorname{Iso}(\hat{\ell}) \equiv \frac{\sum_{\Delta R<0.3} p_{T}(\ell \neq \hat{\ell})+\sum_{\Delta R<0.3} E_{T}(\gamma)+\sum_{\Delta R<0.3} E_{T}(h)}{p_{T}(\hat{\ell})} \tag{11.4}
\end{equation*}
$$

To pass the isolation cut, the lepton must have have $\operatorname{Iso}(\hat{\ell})<0.1$. On top of the isolation cut, we impose the identification efficiency, which we assume to be independent of $p_{T}, \eta$, and the physical process: $73 \%$ for electrons and $84 \%$ for muons. The identification efficiency for each lepton species is extracted by simulating the A1 LM9 benchmark model at hadron level, computing the lepton isolation cut efficiency Eff(Iso) for this sample using (11.4), and dividing the total selection efficiency reported by CMS by Eff(Iso).

The rest of the lepton analysis emulates [444] as closely as possible. From the set of selected leptons, we choose the "SSDL pair": the same-sign pair with the highest $p_{T}$ and a pair invariant mass of at least 8 GeV . We then apply the dilepton trigger efficiency: $96 \%$ for $e e, 93 \%$ for $e \mu$, and $88 \%$ for $\mu \mu$. We veto events where a third lepton (with $p_{T}>10 \mathrm{GeV}$, the normal $|\eta|$ cuts, and $\operatorname{Iso}\left({ }_{3}\right)<0.2$ ) forms an opposite-sign same-flavor pair with one of the SSDL pair leptons, with a pair invariant mass between 76 and 106 GeV . We also veto events where a third lepton (with $p_{T}>5 \mathrm{GeV}$, the normal $|\eta|$ cuts, and $\left.\operatorname{Iso}^{( }{ }_{3}\right)<0.2$ ) forms an opposite-sign same-flavor pair with one of the SSDL pair leptons, with a pair invariant mass below 12 GeV .

The remaining physics objects are handled at parton level, following the instructions. The number of jets is a count of colored partons passing the kinematic cuts: $p_{T}>40 \mathrm{GeV}$ and $|\eta|<2.4$. To count $b$-tagged jets, we apply a $p_{T}$-dependent tagging efficiency, parameterized in Section 7 of [444], to all the $b$ quarks that pass the jet kinematic cuts. To implement the cuts on $H_{T}$ and $E_{T}$, we compute "generator-level" quantities gen- $H_{T}$ and gen- $\dot{E}_{T}$, and use the turn-on efficiency curves parameterized in Section 7 of [448] to get efficiencies for the cuts. gen $-H_{T}$ is the scalar sum of $p_{T}$ 's of the jets that pass the
kinematic cuts, and gen- $E_{T}$ is the magnitude of the vector sum of the $\mathbf{p}_{T}$ 's of non-interacting final-state particles.

## A

## Conventions

One of the subtle difficulties in particle physics is keeping up with the plethora of notational and mathematical conventions that one must pick.

## A. 1 Units and signs

As civilized particle physicists, we use natural units where

$$
\begin{equation*}
c=\hbar=1 . \tag{A.1}
\end{equation*}
$$

Typically dimensionful quantities will be expressed in GeV or TeV . Cross sections are expressed in pico- and femtobarns which have the property:

$$
\begin{equation*}
\mathrm{pb}=10^{3} \mathrm{fb}=10^{-36} \mathrm{~cm}^{2}=2.57 \times 10^{-9} \mathrm{GeV}^{-2} \tag{A.2}
\end{equation*}
$$

We use the 'mostly-minus' ("West coast" or particle physicist's) metric,

$$
\eta_{\mu \nu}=\left(\begin{array}{cccc}
1 & & &  \tag{A.3}\\
& -1 & & \\
& & -1 & \\
& & & -1
\end{array}\right)
$$

where the Minkowski norm of a physical four-momentum is positive semidefinite.
In 4D, we note that the $\gamma^{5}$ matrix is defined $\gamma_{4 \mathrm{D}}^{5}=i \gamma^{0} \gamma^{1} \gamma^{2} \gamma^{3}$ so that the chiral projection operators are $\left(1 \pm \gamma^{5}\right) / 2 \operatorname{In}{ }_{5} \mathrm{D}$, the natural choice of $\gamma^{5}$ obeys $\left\{\gamma^{5}, \gamma^{5}\right\}=\eta^{55}=-1$ so that $\gamma^{5}=-i \gamma_{4 \mathrm{D}}^{5}$.

## A. 2 Indices and character ornaments

We will tend to adhere to the following conventions for indices and ornamentation, though occasionally we may sacrifice these conventions in favor of simplicity.

- Matrices with a hat, $\hat{m}$, are diagonal in that basis.
- Lower case indices from the second quarter of the Roman alphabet (e.g. $i, j, k, \ell, m$ ) refer to flavor indices.
- Lower case indices from the middle of the Greek alphabet (e.g. $\mu, v, \rho, \sigma$ ) refer to vectorial Minkowski space indices
- Lower case indices from the beginning of the Greek alphabet (e.g. $\alpha, \beta$ ) typically refer to spinor indices. Dotted indices are used to distinguish 4D spinor representations.
- Capital Roman letters from the middle of the Roman alphabet, $M, N, P, Q$
- Bold face quantities are spatial three-vectors ( $\mathbf{v}$ ) or matrices (M). Their identity should be clear from context.

Analogous to the usual slash notation for four-component objects, $/ p=p_{\mu} \gamma^{\mu}$, we define slashes for two-component spinors: $\vec{\phi}=v_{\mu} \bar{\sigma}^{\mu}, \notin=v_{\mu} \sigma^{\mu}$.

## A. 3 BARS, DAGGERS, DOTS, AND ALL THAT

See Appendix 7.B-7.B for a pedagogical review of the Poincaré algebra that highlights our conventions for bars, daggers, dots, and all that.

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# Colophon 

I1 HIS THESIS WAS TYPESET using $\mathrm{X}_{\mathrm{H}} \mathrm{H}_{\mathrm{E}} \mathrm{X}$, developed by Jonathan Kew, and based on $\mathrm{BT}_{\mathrm{E}} \mathrm{X}$, originally developed by Leslie Lamport which in turn was based on Donald Knuth's $\mathrm{T}_{\mathrm{E}} \mathrm{X}$. The body text is set in 11 point Arno Pro, designed by Robert Slimbach in the style of book types from the Aldine Press in Venice, and issued by Adobe in 2007. This document is based on a template by Jordan Suchow released under the mit ( $\mathrm{x}_{11}$ ) license. It was modified by Andy Leifer and later by Flip Tanedo. Feynman diagrams and most figures were prepared using the PGF/TikZ packages by Till Tantau. Some more involved images were prepared using Adobe Illustrator. Plots were prepared using the computer algebra systems Mathematica and Matlab.


[^0]:    ${ }^{1}$ What is actually spontaneously broken is the global $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$ symmetry since the local symmetry is a gauge redundancy. In the broken phase the broken generators can no longer be associated with redundancies since these shift the vacuum.

[^1]:    ${ }^{2}$ During the preparation of this document, Ken Wilson passed away at the age of 77. His imprint on modern physics is indelible and he will be remembered for teaching us all how to understand quantum field theory.

[^2]:    ${ }^{1}$ We thank Martin Beneke, Paramita Dey, and Jürgen Rohrwild for pointing this out.

[^3]:    ${ }^{2}$ The finiteness of dipole operators has been investigated in gauge-higgs unified models where a higher-dimensional gauge invariance can render these terms finite [256]. Here we do not assume the presence of such additional symmetries.

[^4]:    ${ }^{3}$ Further discussion of these points can be found in the appendix of [3].

[^5]:    ${ }^{4}$ We thank Martin Beneke, Paramita Dey, and Jürgen Rohrwild for pointing this out.

[^6]:    ${ }^{1}$ The impact of flavor changing neutral gauge bosons on the operators $Q_{, \ldots, 6}$ has recently been studied in [267]. Since the relevant contributions in RS are suppressed both by the KK scale and the RS GIM mechanism, the contributions are expected to be small and will be neglected in this paper.

[^7]:    ${ }^{2}$ An alternate method of including the entire KK tower based on residue theorems was presented in [268], though it obfuscates the physical intuition presented below.

[^8]:    ${ }^{3}$ Significant progress has recently been made on the form factor predictions in the large $q^{2}$ region [301-304]; nevertheless we will not consider this kinematic regime since it is less sensitive to $N P$ entering $C_{7}^{(\prime)}$ than the low $q^{2}$ region.

[^9]:    ${ }^{4}$ These integrands differ by $L \leftrightarrow R$, but the integrals are approximately the same.

[^10]:    ${ }^{5}$ We thank K. Agashe, J. Hubisz, and G. Perez for discussions on these subtleties.

[^11]:    ${ }^{1}$ It is notoriously difficult to plot in Mathematica; see homework 5 from Hitoshi Murayama's Physics 229 C course for suggestions: http://hitoshi.berkeley.edu/229C/index.html.

[^12]:    ${ }^{1}$ We thank Y. Nomura for pointing out the role of $R$-symmetry.

[^13]:    ${ }^{2}$ The hypercharge normalization is fixed if there are no exotic electric charges.

[^14]:    ${ }^{3}$ Since most direct detection events occur at low recoil energy, it is standard to parameterize the cross section in terms of a zero momentum transfer part and a form factor which encodes the momentum and target dependence. See, for example, [344].

[^15]:    ${ }^{4}$ For early attempts of this idea in SUSY see [379]. More recently, SUSY and little Higgs models motivated by the little hierarchy problem have been proposed [380].

[^16]:    ${ }^{5}$ The potential becomes repulsive for $r m_{a} \gtrsim 13$. However, this contribution is cut off by the exponential decay of the Yukawa interaction so that the energy barrier is extremely small $\approx 10^{-11} \times m_{a}^{3} / f^{2}$.

[^17]:    ${ }^{1}$ We note that wavefunction renormalization is essential the cutoff independence of Sommerfeld enhancement. The numerical results in Section 10.5 match [365] within an order of magnitude for a specific choice of renormalization conditions.

[^18]:    ${ }^{1}$ For recent work on complete SUSY models realizing this framework, see Refs. [427-429].
    ${ }^{2}$ We do not include a left-handed sbottom $\tilde{b}_{L}$ in our simplified model even though its presence at the same mass scale as the stop is well motivated. In MFV SUSY, the dominant sbottom decays typically involve the top quark, $\tilde{b} \rightarrow t c$ or $\tilde{b} \rightarrow t \tilde{\chi}^{-}$, so that gluino cascades via sbottoms can still produce the same-sign dilepton signature. Thus we expect that the bounds derived here would qualitatively apply to most MFV SUSY models with $m_{\tilde{g}}>m_{\tilde{b}}$ as well.
    ${ }^{3}$ Other signatures of RPV SUSY with light stops and gluinos have been discussed in Refs. [436-442]. SSDL signature from resonant slepton production has been discussed in [443].

[^19]:    ${ }^{4}$ Previous recasts of the LHC SSDL searches in terms of RPV SUSY have appeared in [432,435]. These searches use smaller data sets than the one considered here.

[^20]:    ${ }^{5}$ We are grateful to Frank Wuerthwein for clarifying this point.

