SOLUTIONS

Assignment 2

Due date: Wednesday, February 6

1. H&F 1.9

2. H&F 1.14 The term "velocity-dependent potential" is a relic of a more old-fashioned mechanics, where all the parts of the Lagrangian beyond the free particle kinetic energy are by default "some kind of potential". The modern view is that it is really the Lagrangian that is fundamental and there is no physics in designating certain of its terms as "potentials". As we will see in the next chapter, it is the time integral of the Lagrangian, or action, that really matters. In this problem the time integral of the $e\,\mathbf{v}\cdot\mathbf{A}$ term is just a line integral along the particle trajectory: there is nothing velocity-dependent about it!

3. H&F 1.19 You may use the fact from Physics 1112/1116 that the kinetic energy of a rigid body is equal to the sum $mv^2/2 + I\omega^2/2$, where m is the total mass, v the speed of the center of mass, I the moment of inertia about the center of mass, and ω the angular velocity of rotation about the center of mass.

4. H&F 1.22 In this and the next problem you do not need to solve the equations of motion.

5. H&F 1.23

6. The generalization of rigid-body transformations, called *conformal transformations*, that is applied to the Cornell seal on the course website is given by the following formula:

$$z \to b(t)\left(\frac{z - a(t)}{1 - a^*(t)z/r^2}\right)$$

Here $z = x + iy$ is a complex number[1] representation of a point in the plane and the formula applies only to points inside the disk of radius $|z| = r$. When the transformation's two time-dependent complex parameters satisfy the conditions

$$|a(t)| < r \qquad |b(t)| = 1$$

the inside of the disk is mapped one-to-one to itself.

(a) Based on the last statement, how many degrees of freedom do these conformal transformations have?

[1] $a^*(t)$ is the complex conjugate of $a(t)$

(b) Something singular happens whenever $|a(t)| = r$. What exactly does the transformation do then? Try a special case, like $a(t) = r$ or $a(t) = ir$.

Suppose we have a disk that is massless in its interior and has mass M uniformly distributed around its rim. Further suppose that the disk can move not just by rigid-body rotations but by conformal transformations (carrying the mass along the rim with it). The Lagrangian of such a disk will be just the kinetic energy of the mass on the rim; it has the formula

$$L = Mr^2 \left(\dot{\beta}^2/2 + (\tan^2 \gamma)\dot{\alpha}^2 + \dot{\gamma}^2\right)$$

where the angles α, β, γ are related to the parameters of the conformal transformation by

$$a(t) = re^{i\alpha(t)} \sin \gamma(t) \qquad b(t) = e^{i\beta(t)}.$$

(c) Obtain the equations of motion for α, β, and γ. The conjugate momentum for two of these is conserved; write down the corresponding first-order differential equations.

make use of the result of the previous problem to show this. With this approach we prove the equivalence of inertial frames from the form of the Lagrangian, instead of postulating this equivalence at the start, which is the usual way of doing things.

c) Instead of proving it, adopt the equivalence of inertial frames as a postulate, in addition to the Euler–Lagrange equations. Explain why this means that

$$L'(v + V_0) = L(v) + \frac{dF(x,t)}{dt}. \tag{1.92}$$

$L(v)$ is an unknown function for the free particle that we are trying to determine from these principles. (Work in one dimension to make things easier.) Let V_0 be an infinitesimal quantity. Expand the left side of Equation (1.92) in a Taylor series and keep only the first two terms. From this prove $L(v) \sim v^2$.

Problem 8: *(Potentials with scaling properties)* Let $V(\vec{r}_1, \ldots, \vec{r}_M)$ be the potential energy of a system of M massive particles which has the scaling property

$$V(\alpha\vec{r}_1, \ldots, \alpha\vec{r}_M) = \alpha^k V(\vec{r}_1, \ldots, \vec{r}_M) \tag{1.93}$$

(k is usually an integer, α an arbitrary constant.) Prove that, if the Lagrangian is to remain invariant (except for multiplication by a constant), and all distances are scaled by a factor α, then the time must be scaled by a factor $\beta = \alpha^{1-\frac{k}{2}}$. Applications of this include:

a) If $k = 1$, the force is constant, like gravity. Prove that distances scale like t^2.

b) If $k = 2$, the force is like that of a harmonic oscillator or a system of harmonic oscillators coupled to each other. Prove that the frequency or frequencies of such a system are independent of the amplitude of oscillation.

c) If $k = -1$, we have the Kepler problem (inverse square force law). Prove Kepler's third law from this scaling law above. (That is, prove $d^3 = t^2$, where d could be any distance in the problem. Normally it is the mean distance of a planet from the sun.)

Hamiltonian Concept/Energy

Problem 9: *(Quadratic forms)* Prove that, if the constraints are scleronomic (i.e., time-independent), T is a *quadratic* function (*quadratic form*) of the generalized velocities. Then prove this implies

$$\sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = 2T. \tag{1.94}$$

Assuming that the kinetic energy *is* a quadratic form in the generalized velocities so that the formula above is correct, prove that the Hamiltonian H (Equation (1.65)) is the total energy ($H = T + V = E$).

Problem 10: *(Bead on a wire of arbitrary shape)* A bead slides without friction down a wire that has the shape $y = f(x)$ (Y is vertical, X is horizontal).

a) Prove that the EOM is

$$(1 + f'^2)\ddot{x} + f'f''\dot{x}^2 + gf' = 0 \tag{1.95}$$

(where $f' \equiv \frac{df}{dx}$, $f'' \equiv \frac{d^2f}{dx^2}$).

b) Since the Hamiltonian is a constant in this problem, it always equals its value at $t = 0$. Use this fact to solve for $\dot{x}(t)$.

c) Let τ be the time to slide down the wire between two heights $y_1 = f(1)$ and $y_0 = f(0)$. Show that this leads to a solution of the form $\sqrt{g}\,\tau = \int_0^1 h(x)\,dx$, where $h(x)$ is the function you should find in terms of $f(x)$ and its derivatives.

Problem 11: *(Comparing H and E)* Invent a concrete example of each type of the situation described below:

a) H is conserved, but $H \neq E$.

b) $H = E$, but $\frac{dH}{dt} \neq 0$, so H is not conserved.

Lagrangian/EOM

Problem 12: *(L for free particle in plane polar coordinates)* Express the Lagrangian for a free particle moving in a plane in plane polar coordinates. From this prove that, in terms of radial and tangential components, the acceleration in polar coordinates is

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\mathbf{e}}_\theta \tag{1.96}$$

(where $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ are unit vectors in the positive radial and tangential directions).

Problem 13: *(Bead on a wire)* Discuss the motion of the bead according to the equation of motion (1.38) as completely as you can. Find explicit solutions. What will happen if the bead is slightly displaced from the point where it has no acceleration?

Problem 14*: *(L for charged particle in a magnetic field)* The Lagrange method ~~does work~~ for some velocity-dependent potentials. A very important case is a charged particle moving in a magnetic field. The magnetic field $\vec{B}$ can be represented as the curl of a "vector potential" $\vec{A}$: $\vec{B} = \vec{\nabla} \times \vec{A}$. A uniform magnetic field $\vec{B}_0$ corresponds to a vector potential $\vec{A} = \frac{1}{2}\vec{B}_0 \times \vec{r}$.

a) Check that $\vec{B}_0 = \vec{\nabla} \times \vec{A}$.

b) From the Lagrangian

$$L = \frac{1}{2}mv^2 + e\vec{v}\cdot\vec{A} \quad \text{(MKSI units)} \tag{1.97}$$

(where e is the charge and m is the mass) show that the EOM derived from the Euler–Lagrange equations is identical with the result from Newtonian mechanics

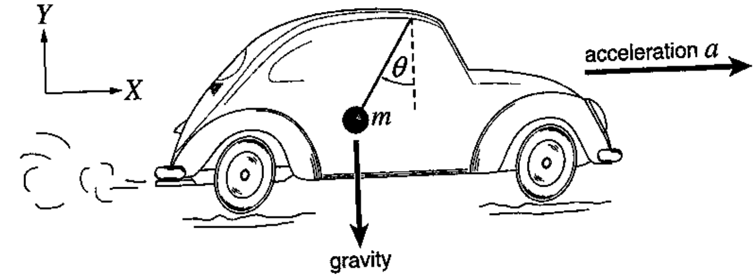


FIGURE 1.10

that pivot at the top and bottom so that the angle θ is a function of the angular speed ω of the shaft. Find the function $\theta(\omega)$ using Lagrangian methods. The ring mass M at the bottom can slide up and down on the shaft.

Problem 19: *(Inclined stick on a table)* A stick is initially held at a vertical angle θ as shown in part B of Figure 1.9. First consider a table where the bottom of the stick is fixed (with a frictionless bearing) to the table top. Is this problem holonomic? Solve for the motion of the stick after it is released. Next assume that the bottom end is on a frictionless table instead. How many degrees of freedom are there? Is the problem holonomic? Again solve for the motion of the stick after it is released.

Problem 20: *(Pendulum in an accelerated reference frame)* A pendulum with a weightless string of length D and mass m is attached to a moving car, as shown in Figure 1.10. The car is continuously accelerated along a horizontal track with constant acceleration a, starting from an initial horizontal velocity v_0. Gravity acts in the vertical direction with acceleration g.

Assume that the (x, y) coordinate system shown is at rest with respect to the ground (not located in the car). *The car is not an inertial frame of reference.* There is one degree of freedom. Use θ (see Figure 1.10, which shows $\theta < 0$) as the generalized coordinate/dynamical variable. The goal is to find out how the acceleration of the pendulum support affects the motion of the pendulum as seen by a person in the car.

a) Find the components of the velocity of the pendulum bob in the *laboratory frame* (i.e., the frame at rest with respect to the moving car). Find the kinetic energy as a function of θ, $\dot{\theta}$, and the other variables, all of which are known functions.

b) Find the Lagrangian $L(\theta, \dot{\theta}, t)$. Does L depend explicitly on the time?

c) Prove that the equation of motion for the pendulum is

$$\ddot{\theta} + \frac{g}{D}\sin\theta + \frac{a}{D}\cos\theta = 0. \tag{1.101}$$

Notice that the velocity of the car is not detectable by observing the pendulum from inside the car, but the car acceleration *is* detectable by the "tilt" of the pendulum when it is at rest with respect to its support point. This is in accordance with the Galilean principle of relativity. Explain why this is true.

d) What is the angle of the pendulum when it remains at rest in stable equilibrium? Give an expression for the tangent of this angle (call it θ_{eq}).

e) Set $\theta(t) = \theta_{\text{eq}} + \eta(t)$, that is, measure the motion with respect to the equilibrium point. Use a Taylor series to find the equation of motion for η for small oscillations around θ_{eq}. If η is sufficiently small, show that the equation for η is

$$\ddot{\eta} + \omega^2 \eta = 0. \tag{1.102}$$

The solutions to this equation are $\sin \omega t$ and $\cos \omega t$, which means that the pendulum makes simple harmonic oscillations about the equilibrium angle with frequency ω. Prove that the angular frequency of these small oscillations is

$$\omega = \sqrt{\frac{\sqrt{a^2 + g^2}}{D}}.$$

Problem 21: *(Simple pendulum with driven support)* A simple pendulum with a point mass m suspended from a weightless rod of length l has its support point driven rapidly up and down with an amplitude of vertical motion

$$A \cos \omega t, \tag{1.103}$$

where A and ω are independently adjustable constants. Find the Lagrangian for this system using θ, the angle the pendulum makes with the vertical, as the generalized coordinate. Is H constant? Is H the total energy?

Problem 22*: *(Box sliding horizontally)* A box of mass M slides horizontally on a frictionless surface. The distance of the box's center of mass from the origin is denoted by X. Suspended from inside the center of the box is a pendulum of length l at the bottom of which is a mass m. All the motion takes place in the XY plane. What is the Lagrangian for this system? What are the EOMs?

Problem 23: *(Bead on a rotating circular hoop)* Imagine a vertical circular hoop of radius R rotating about a vertical axis with constant angular velocity Ω as shown in Figure 1.11. A bead of mass m is threaded on the hoop, so that it can move without friction, but is confined to move on the hoop. (Define the angle θ to be the angle from the vertical line through the center of the hoop to the bead.) Find the Lagrangian and the equations of motion. Find the Hamiltonian H explicitly. Is H a constant of the motion? Is energy $(T + V)$ constant?

Conjugate Momentum/Routhian

Problem 24: *(Physical pendulum)* For the physical pendulum (see Figure 1.8, using θ as the generalized coordinate, what is the canonically conjugate momentum to θ, i.e., p_θ? What is another name for p_θ in this case?

P31B, HW #2 SOLUTIONS DUE: WED, FEB 6

1. (H&F 1.9) HAMILTONIAN SCLERONOMIC : $\vec{r}_i = \vec{r}_i(q_1, \dots, q_N)$ (H&F eq. 1.40)

The KINETIC ENERGY, T, IS GIVEN BY $T = \sum_{i=1} \frac{1}{2} m_i |\dot{\vec{r}}_i|^2$
SINCE $\vec{r}_i$ ONLY DEPENDS ON t VIA THE DEPENDENCE OF THE GENERALIZED COORDINATES, $q(t)$, WE HAVE

$$\dot{\vec{r}}_i = \frac{d\vec{r}_i}{dt} = \sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k$$

THEN, PLUGGING INTO THE EXPRESSION FOR T:

$$T = \sum_i \frac{1}{2} m_i \left| \sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k \right|^2$$

$$= \underbrace{\sum_i \frac{1}{2} m_i \sum_{l,k} \left(\frac{\partial \vec{r}_i}{\partial q_k} \cdot \frac{\partial \vec{r}_i}{\partial q_l} \right)}_{\text{call this all } T_{lk}} \dot{q}_l \dot{q}_k \boxed{= \sum_{l,k} T_{lk} \dot{q}_l \dot{q}_k} \checkmark$$

QUADRATIC FORM IN $\dot{q}$'s.

TAKE THE DERIVATIVE OF $T = \sum_{lk} T_{lk} \dot{q}_l \dot{q}_k$ w/rt $\dot{q}_n$

relabel dummy indices

$$\frac{\partial T}{\partial \dot{q}_n} = \sum_{l,k} T_{lk} \delta_{nl} \dot{q}_k + \sum_{l,k} T_{lk} \dot{q}_l \delta_{nk}$$

$$\Rightarrow \sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = \sum_k \dot{q}_k \left(\sum_{ij} T_{ij} \delta_{ik} \dot{q}_j + \sum_{ij} T_{ij} \delta_{jk} \dot{q}_i \right)$$

$$= \sum_{ij} T_{ij} \dot{q}_i \dot{q}_j + \sum_{ij} T_{ij} \dot{q}_j \dot{q}_i \qquad \sum_k \dot{q}_k \delta_{ik} = \dot{q}_i$$

$$= \boxed{2T} \checkmark$$

(PROBLEM 1, CONTINUED)

$$H = \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L$$

$\searrow$ INDEPENDENT OF $\dot{q}_k$

$$= \sum_k \dot{q}_k \frac{\partial}{\partial \dot{q}_k}(T - \cancel{V}) - (T - V) \qquad \cancel{V} \to 0 \text{ since } V = V(\vec{r}(q_1, \ldots q_N), \ldots)$$

$$= 2T - T + V$$

$$= \boxed{T + V}$$

✓ GIVES TOTAL ENERGY

(but recall counter examples from lecture!)

2. (H&F 1.14) <u>LAGRANGIAN OF A CHARGED PARTICLE IN A MAG. FIELD</u>

WE ARE GIVEN $\vec{A} = \frac{1}{2}\vec{B}_0 \times \vec{r}$

a) IT HELPS TO USE A HANDY PRODUCT IDENTITY:

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A})$$

BECAUSE $\vec{B}_0$ IS CONSTANT $\left(\partial/\partial x_i (B_0)_j = 0 \quad \forall\, i, j\right)$, THIS EXPRESSION SIMPLIFIES:

$$\vec{\nabla} \times \left(\tfrac{1}{2}\vec{B}_0 \times \vec{r}\right) = -\tfrac{1}{2}(\vec{B}_0 \cdot \vec{\nabla})\vec{r} + \tfrac{1}{2}\vec{B}_0(\vec{\nabla} \cdot \vec{r})$$

write $\vec{B}_0 = B_0 \hat{z} \longrightarrow$ (CHOICE OF COORDINATES)

$$= -\tfrac{1}{2} B_0 \frac{\partial}{\partial z}\vec{r} + \tfrac{1}{2} B_0 \hat{z}\left(\sum_{i=x,y,z} \frac{\partial}{\partial x_i} x_i\right)$$

$$= -\tfrac{1}{2} B_0 \hat{z} + \tfrac{3}{2} B_0 \hat{z}$$

$$= B_0 \hat{z}$$

$$= \boxed{\vec{B}_0}$$

alternate derivation

remark: VECTOR IDENTITIES ARE TEDIOUS! OFTEN IT IS EASER TO USE INDICES, THEY ENCODE THE VECTORIAL INFO & ALLOW YOU TO JUST WORK WITH "SCALARS".

recall: $(\vec{A}\times\vec{B})_i = \varepsilon_{ijk} A_j B_k$ ← implied sum over repeated indices

$$
\begin{aligned}
\left(\vec{\nabla}\times\left(\tfrac{1}{2}\vec{B}_0\times\vec{r}\right)\right)_i &= \varepsilon_{ijk}\,\partial_j\left(\varepsilon_{kmn}\tfrac{1}{2}(B_0)_m x_n\right)\\
&= \tfrac{1}{2}\varepsilon_{ijk}\varepsilon_{kmn}\,\partial_j\left((B_0)_m x_n\right)\\
&= \tfrac{1}{2}\varepsilon_{ijk}\varepsilon_{kmn}(B_0)_m\,\delta_{jn}\\
&= \tfrac{1}{2}\underbrace{\varepsilon_{ijk}\varepsilon_{kmj}}(B_0)_m
\end{aligned}
$$

use $\partial_j x_n = \delta_{jn}$ & $(B_0)_m$ CONSTANT

WHAT CAN THIS BE? THE ONLY ROTATIONALLY COVARIANT OBJECT WITH 2 FREE INDICES IS δ_{im}. THEN: $\varepsilon_{ijk}\varepsilon_{kmj} = a\,\delta_{im}$ for some constant a. FIXING $i=m=1$ (for eg.) AND DOING THE SUM (remembering that ε is totally antisymmetric) STRAIGHTFORWARDLY GIVES $a=2$.

$$
\begin{aligned}
&= \tfrac{1}{2}(2\delta_{im})(B_0)_m\\
&= (B_0)_i
\end{aligned}
$$

$$\Rightarrow \boxed{\vec{\nabla}\times\left(\tfrac{1}{2}\vec{B}_0\times\vec{r}\right) = \vec{B}_0} \quad ✓$$

(problem 2, continued)

b) $L = \frac{1}{2}mv^2 + e\vec{v}\cdot\vec{A}$ ← $\vec{v} = \dot{\vec{r}}$

IT WILL BE VERY USEFUL TO USE INDICES RATHER THAN VECTORS:

$$L = \tfrac{1}{2}m(v_i v_i) + e v_i A_i$$

$$\frac{\partial}{\partial v_j} v_i = \delta_{ij}$$

$$\frac{\partial L}{\partial v_j} = m v_j + e A_j$$

$$\frac{d}{dt}\frac{\partial L}{\partial v_j} = m\dot{v}_j + e\dot{A}_j$$

$$= m\ddot{x}_j + e\left(\partial A_j/\partial t + \partial A_j/\partial x_k \underbrace{\partial x_k/\partial t}_{v_k}\right)$$

$$= m\ddot{x}_j + e\underbrace{\cancel{\frac{\partial A_j}{\partial t}}}_{0 \text{ since } \vec{A} = \frac{1}{2}\vec{B}_0\times\vec{r}} + e\frac{\partial A_j}{\partial x_k}v_k$$

$$\frac{\partial L}{\partial x_j} = e v_i \frac{\partial A_i}{\partial x_j}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial v_j} - \frac{\partial L}{\partial x_j} = m\ddot{x}_j + e\left(\frac{\partial A_j}{\partial x_k}v_k - v_i\frac{\partial A_i}{\partial x_j}\right)$$

$$= m\ddot{x}_j + e\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)v_i$$

$$= m\ddot{x}_j + e v_i\left(\partial_i \varepsilon_{jab}\tfrac{1}{2}(B_0)_a x_b - \partial_j \varepsilon_{iab}\tfrac{1}{2}(B_0)_a x_b\right)$$

$$= m\ddot{x}_j + \tfrac{1}{2}e v_i\left(\varepsilon_{jab}(B_0)_a\delta_{ib} - \varepsilon_{iab}\tfrac{1}{2}(B_0)_a\delta_{jb}\right)$$

$$= \underbrace{(\varepsilon_{jai} - \varepsilon_{iaj})}_{=2\varepsilon_{jai}}(B_0)_a$$

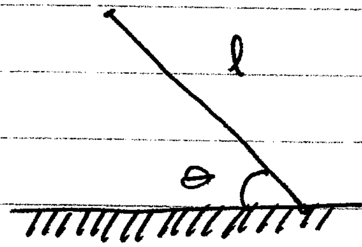
$$= m\ddot{x}_j + e\varepsilon_{jai}(B_0)_a v_i$$

$$\Rightarrow 0 = m\ddot{\vec{x}} + e(\vec{B}\times\vec{v}) \Rightarrow \boxed{m\ddot{\vec{x}} = e\vec{v}\times\vec{B}}$$

USUAL LORENTZ FORCE FOR THIS SYSTEM.

3. (H&F 1.19) INCLINED STICK ON A TABLE

CASE: FIXED BOTTOM OF THE STICK



THE ORIENTATION OF THE STICK IS COMPLETELY DETERMINED BY THE POSITION OF ITS CENTER OF MASS (CM)

$$\left.\begin{aligned} x_{cm} &= -\frac{l}{2}\cos\theta \\ y_{cm} &= \frac{l}{2}\sin\theta \end{aligned}\right\}$$ so this system is $\boxed{\text{HOLONOMIC}}$

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$v = \frac{l}{2}\dot{\theta}$ $\quad$ $I = \frac{1}{12}ml^2$ $\quad$ $\omega = \dot{\theta}$

w.r.t. CENTER OF ROD

$$V = mg\frac{l}{2}\sin\theta$$

$$\Rightarrow L = \frac{1}{2}\left(m\frac{l^2}{4} + I\right)\dot{\theta}^2 - \frac{1}{2}mgl\sin\theta$$
$$= \frac{1}{6}ml^2\dot{\theta}^2 - \frac{1}{2}mgl\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{3}ml^2\ddot{\theta} \qquad \frac{\partial L}{\partial \theta} = +\frac{1}{2}mgl\cos\theta$$

EOM: $\frac{1}{3}ml^2\ddot{\theta} - \frac{1}{2}mgl\cos\theta = 0$

SOLVE EOM: note that we can eliminate m from the problem! ([INVERTED] PENDULUM – INDEP. of MASS!)

$$2 \cdot l \ddot{\theta} = +3 \cdot g \cos\theta$$
$$\ddot{\theta} = +\frac{2}{3}\left(\frac{g}{l}\right)\cos\theta$$

Note the similarity to the pendulum equation.
(THIS IS AN INVERTED PENDULUM WHERE WE'RE USING A DIFFERENT ANGLE THAN THE USUAL TO PARAMETERIZE THE MOTION)

LET US WRITE THIS AS: $\ddot{\theta} = +A\cos\theta$
THE TRICK TO ATTACKING THIS PROBLEM IS TO MAKE USE OF ENERGY CONSERVATION. LET θ_0 BE THE INITIAL ANGLE OF THE STICK. THEN

$$E = \frac{1}{2} mgl \sin\theta_0 = \frac{1}{6} ml^2 \dot{\theta}^2 + \frac{1}{2} mgl\sin\theta$$

for simplicity, define the also-conserved quantity $\tilde{E} = (\frac{1}{2}mgl)^{-1} E$ so that:

$$\tilde{E} = \sin\theta_0 = \frac{1}{3}\frac{l}{g}\dot{\theta}^2 + \sin\theta$$
$$= \frac{1}{2A}\dot{\theta}^2 + \sin\theta$$

$$\Rightarrow \left(\frac{d\theta}{dt}\right)^2 = 2A(\sin\theta_0 - \sin\theta)$$

THIS ALLOWS US TO WRITE:

$$-d\theta/dt = \sqrt{2A\,(\sin\theta_0 - \sin\theta)}$$

$$dt = \frac{d\theta}{\sqrt{2A(\sin\theta_0 - \sin\theta)}}$$

$$\Rightarrow \int_0^t d\tilde{t} = \int_{\theta_0}^{\theta(t)} \frac{d\theta}{\sqrt{2A\,(\sin\theta_0 - \sin\theta)}}$$

$$t = f(\theta, \theta_0)$$ some function given by the above integral

THIS is (in principle at least) INVERTIBLE TO GIVE $\theta(t, \theta_0)$. YOU CAN DO THIS IN Mathematica OR USING A TABLE of ELLIPTIC INTEGRALS. FOR OUR PURPOSES IT IS SUFFICIENT TO STOP HERE: t IS THE INTEGRAL OF A CLOSED-FORM EXPRESSION FOR θ.

(problem 3, continued)

CASE: FRICTIONLESS TABLE (bottom of stick may move)

What do we expect? WITH NO CONSTRAINT FIXING THE BOTTOM OF THE STICK, WE EXPECT FREE FALL OF THE STICK CENTER OF MASS.

BECAUSE THE BOTTOM OF THE STICK IS FREE TO MOVE, THE PROBLEM HAS $\boxed{\text{TWO}}$ DEGREES OF FREEDOM.

$$\left.\begin{aligned} x_{cm} &= \underbrace{-\tfrac{l}{2}\cos\theta}_{} + \underbrace{x_b}_{\text{POSITION OF STICK BOTTOM}} \\ y_{cm} &= \underbrace{\tfrac{l}{2}\sin\theta}_{\text{AS BEFORE}} \end{aligned}\right\} \text{ system is } \boxed{\text{HOLONOMIC}}$$

$$T = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 \qquad I = \tfrac{1}{3}Ml^2,\ \omega = \dot\theta \text{ AS BEFORE}$$

$$\begin{aligned} v^2 &= \dot x_{cm}^2 + \dot y_{cm}^2 \\ &= \left(\tfrac{l}{2}\sin\theta\,\dot\theta + \dot x_b\right)^2 + \left(\tfrac{l}{2}\cos\theta\,\dot\theta\right)^2 \\ &= \tfrac{l}{2}\dot\theta^2 + l\sin\theta\,\dot\theta\dot x_b + \dot x_b^2 \end{aligned}$$

$$T = \underbrace{\tfrac{1}{2}\left(M\tfrac{l^2}{4} + I\right)\dot\theta^2}_{T \text{ IN PVS PART}} + \underbrace{\tfrac{1}{2}Ml\sin\theta\,\dot\theta\dot x_b + \tfrac{1}{2}m\dot x_b^2}_{\text{NEW TERMS}}$$

POTENTIAL ENERGY IS UNCHANGED.

$$L = \frac{1}{2}\cdot\frac{1}{3}m\ell^2\dot\theta^2 + \frac{1}{2}m\ell\sin\theta\,\dot\theta\,\dot x_b + \frac{1}{2}m\dot x_b^{\,2} - \frac{1}{2}mg\ell\sin\theta$$

<u>EQUATIONS OF MOTION</u> :
$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = 0$$
$$\frac{d}{dt}\frac{\partial L}{\partial\dot x_b} - \frac{\partial L}{\partial x_b} = 0$$

$$\theta:\ \frac{1}{3}m\ell^2\ddot\theta + \frac{1}{2}m\ell\cos\theta\,\dot\theta\,\dot x_b + \frac{1}{2}m\ell\sin\theta\,\ddot x_b - \frac{1}{2}m\ell\cos\theta\,\dot\theta\,\dot x_b + \frac{1}{2}mg\ell\cos\theta = 0$$

0

$$\frac{1}{3}m\ell^2\ddot\theta + \frac{1}{2}m\ell\sin\theta\,\ddot x_b + \frac{1}{2}mg\ell\cos\theta = 0$$

$$x:\ \frac{1}{2}m\ell\sin\theta\,\ddot\theta + \frac{1}{2}m\ell\cos\theta\,\dot\theta^2 + m\ddot x_b$$

... anyway, once again a direct attack using the EOM is cumbersome. WE CAN USE ENERGY CONSERVATION AS BEFORE:

$$E = \frac{1}{2}mg\ell\sin\theta_0 = \frac{1}{6}m\ell^2\dot\theta^2 + \frac{1}{2}m\ell\sin\theta\,\dot\theta\,\dot x_b + \frac{1}{2}m\dot x_b^{\,2} + \frac{1}{2}mg\ell\sin\theta$$

... but now there are annoying factors of $\dot x_b$! THAT'S OKAY, WE CAN USE MOMENTUM CONSERVATION SINCE $\partial L/\partial x_b = 0$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot x_b}\right) = 0 \Rightarrow P_b = \frac{\partial L}{\partial\dot x_b} = m\dot x_b + \frac{1}{2}m\ell\sin\theta\,\dot\theta \quad , \text{ CONSERVED}$$

FURTHER, WE HAVE THE INITIAL CONDITION $P_b = 0$, SO $P_b = 0$ AT ALL TIMES.

$$P_b = 0 \Rightarrow \dot{x}_b = -\frac{1}{2} l \sin\theta \, \dot{\theta}$$

We can now plug this into the energy equation to get an ODE for only θ:

$$\frac{1}{2} mgl\sin\theta_0 = \underbrace{\frac{1}{6} ml^2\dot{\theta}^2}_{=\frac{1}{2}(m\frac{l^2}{4}+I)\dot{\theta}^2} \underbrace{- \frac{1}{4} ml^2 \sin^2\theta\,\dot{\theta}^2 + \frac{1}{8} ml^2\sin^2\theta\,\dot{\theta}^2}_{-\frac{1}{8} ml^2\sin^2\theta\,\dot{\theta}^2} + \frac{1}{2} mgl\sin\theta$$

$$= \underbrace{\frac{1}{8} ml^2\dot{\theta}^2 - \frac{1}{8} ml^2\sin^2\theta\,\dot{\theta}^2}_{= \frac{1}{8} ml^2 \cos^2\theta\,\dot{\theta}^2 = \frac{1}{2} m \dot{y}_{cm}^2} + \underbrace{\frac{1}{2} I \dot{\theta}^2}_{\text{SPINNING}} + \underbrace{\frac{1}{2} mgl\sin\theta}_{\text{GRAVITY}}$$

FALLING VERTICALLY

__REMARK__: OBSERVE HOW WE'VE FOUND OURSELVES W/ A SYSTEM DESCRIBING FREE FALL OF THE CM & SPINNING ABOUT CM. WE COULD HAVE SEEN THIS MORE DIRECTLY:

$$P_b = 0 \Rightarrow \underbrace{\dot{x}_b + \frac{1}{2} l \sin\theta \, \dot{\theta}}_{\text{THIS IS PRECISELY } \dot{x}_{cm}!} = 0$$

SO $\dot{x}_{cm}$ DROPS OUT OF THE EXPRESSION FOR T.

__NOTE__: IF $I = 0$ (all mass @ cm, rest of stick massless), THEN THIS IS JUST A SIMPLE FALLING MASS, AS WE EXPECT!

BACK TO OUR E CONSERVATION EXPRESSION:

$$\sin\theta_0 = \frac{1}{4}\frac{l}{g}\cos^2\theta\,\dot{\theta}^2 + \frac{1}{mgl}\overbrace{\left(\frac{1}{3}ml^2\right)}^{I}\dot{\theta}^2 + \sin\theta$$

$$= \left(\frac{1}{4}\cos^2\theta + \frac{1}{3}\right)\frac{l}{g}\dot{\theta}^2 + \sin\theta$$

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{g}{l}\,\frac{\sin\theta_0 - \sin\theta}{\frac{1}{4}\cos^2\theta + 1/3}$$

$$\int_0^t d\tilde{t} = \int_{\theta_0}^{\theta(t)} d\theta \sqrt{\frac{l}{g}\,\frac{\frac{1}{4}\cos^2\theta + \frac{1}{3}}{\sin\theta_0 - \sin\theta}}$$

$$t = \tilde{f}(\theta, \theta_0)$$

AS BEFORE, THIS CAN IN PRINCIPLE BE SOLVED, WE'LL LEAVE THE SOLUTION LIKE THIS

<u>Remark</u>: once we saw that x_{cm} DROPS OUT, WE COULD HAVE ALSO SOLVED THIS USING y_{cm} $(\equiv y)$ INSTEAD OF θ. REWRITING OUR ENERGY CONSERVATION EQUATION GIVES:

$$mg(y_0 - y) = \frac{1}{2} m\dot{y}^2 + \frac{1}{2}\underset{\uparrow\, I = \frac{1}{12}m\ell^2}{I}\sin^{-1}(2\,y/\ell)$$

$$\frac{1}{2} m\dot{y}^2 = mg(y_0 - y) - \frac{1}{24} m\ell^2 \sin^{-1}(2y/\ell)$$

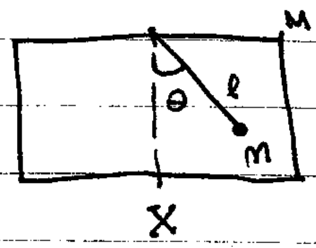
$$\dot{y}^2 = 2g(y_0 - y) - \frac{1}{12}\ell^2 \sin^{-1}(2y/\ell)$$

$$dt = \frac{dy}{\sqrt{2g(y_0 - y) - \frac{1}{12}\ell^2 \sin^{-1}(2y/\ell)}}$$

$$t = \int_{y_0}^{y(t)} \frac{dy}{\sqrt{2g(y_0 - y) - \frac{1}{12}\ell^2 \sin^{-1}(2y/\ell)}}$$

1.22

4. (H&F) <u>BOX SLIDING HORIZONTALLY</u>



USE GEN. COORDS X & θ

<u>Remark</u>: AS LONG AS WE INCLUDE THE BOX POSITION IN THE PARAMETERIZATION of THE PENDULUM POSITION, WE CAN TREAT THE BOX AS A POINT MASS CONSTRAINED TO MOVE IN THE x-DIRECTION.

PENDULUM POSITION:
$$x = l\sin\theta + X$$
$$y = -l\cos\theta$$

$$T = \frac{1}{2}m\left[(l\sin\theta\,\dot\theta + \dot X)^2 + (l\cos\theta\,\dot\theta)^2\right] + \frac{1}{2}M\dot X^2$$
$$= \frac{1}{2}ml^2\dot\theta^2 + ml\sin\theta\,\dot\theta\dot X + \frac{1}{2}(m+M)\dot X^2$$

$$\boxed{L = \frac{1}{2}ml^2\dot\theta^2 + ml\sin\theta\,\dot\theta\dot X + \frac{1}{2}(m+M)\dot X^2 + mgl\cos\theta}$$

observe that L is independent of X

<u>EQUATIONS of MOTION</u>

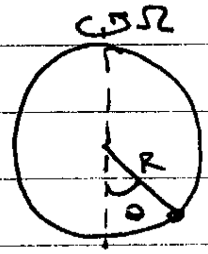
$$\theta:\quad 0 = ml^2\ddot\theta + ml\cos\theta\,\dot\theta\dot X + ml\sin\theta\,\ddot X + mgl\sin\theta$$

$$\Rightarrow \boxed{0 = \frac{l}{g}\ddot\theta + \frac{1}{g}\cos\theta\,\dot\theta\dot X + \frac{1}{g}\sin\theta\,\ddot X + \sin\theta}$$

$$X: \boxed{0 = \tfrac{1}{2}(m+M)\ddot{X} + ml\cos\theta\,\dot{\theta}^2 + ml\sin\theta\,\ddot{\theta}}$$

Note: this is another pendulum problem where the easiest path to a general (not $\theta \ll 1$) solution is to use conservation of energy & $p = \partial L/\partial \dot{x}$, AS WE DID ON THE PREVIOUS PROBLEM.

5. (H&F 1.23) BEAD ON A ROTATING HOOP



PICK COORDINATES SUCH THAT:

$$\left.\begin{aligned} x &= R\sin\theta\cos\varphi \\ y &= R\sin\theta\sin\varphi \\ z &= -R\cos\theta \end{aligned}\right\} \quad \varphi = \Omega t$$

$$L = \tfrac{1}{2}mv^2 - mgz$$

$$\dot{\varphi} = \Omega$$

$$\begin{aligned} v^2 &= \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \\ &= (R\cos\theta\cos\varphi\,\dot{\theta} - R\sin\theta\sin\varphi\,\dot{\varphi})^2 \\ &\quad + (R\cos\theta\sin\varphi\,\dot{\theta} + R\sin\theta\cos\varphi\,\dot{\varphi})^2 \\ &\quad + (R\sin\theta\,\dot{\theta})^2 \end{aligned}$$

$$\begin{aligned} &= R^2\big[\cos^2\theta\cos^2\varphi\,\dot{\theta}^2 - 2\sin\theta\cos\theta\sin\varphi\cos\varphi\,\dot{\theta}\Omega + \sin^2\theta\sin^2\varphi\,\Omega^2 \\ &\quad + \cos^2\theta\sin^2\varphi\,\dot{\theta}^2 + 2\sin\theta\cos\theta\sin\varphi\cos\varphi\,\dot{\theta}\Omega + \sin^2\theta\cos^2\varphi\,\Omega^2 \\ &\quad + \sin^2\theta\,\dot{\theta}^2\big] \end{aligned}$$

(problem 5, continued)

$$v^2 = R^2\left[\cos^2\theta\,\dot\theta^2 + \sin^2\theta\,\Omega^2 + \sin^2\theta\,\dot\theta^2\right]$$

$$= R^2(\underbrace{\dot\theta^2}_{\text{velocity in }\theta\text{ direction}} + \underbrace{\Omega^2\sin^2\theta}_{\text{velocity in }\varphi\text{ direction}})$$

$$\boxed{L = \frac{1}{2}mR^2(\dot\theta^2 + \Omega^2\sin^2\theta) + mgR\cos\theta}$$

$$H = \dot\theta\frac{\partial L}{\partial\dot\theta} - L$$

$$= mR^2\dot\theta^2 - \frac{1}{2}mR^2(\dot\theta^2 + \Omega^2\sin^2\theta) - mgR\cos\theta$$

$$= \boxed{\frac{1}{2}mR^2\dot\theta^2 - \frac{1}{2}mR^2\Omega^2\sin^2\theta - mgR\cos\theta} \neq T+V$$

H IS A CONSTANT OF MOTION BECAUSE L DOES NOT DEPEND EXPLICITLY ON TIME.

NOTE, HOWEVER, THAT $H \neq E$ (CHECK THE SIGN OF THE Ω^2 TERM) AND FURTHER THAT $E = T+V$ IS <u>NOT</u> CONSERVED (CHECK BY EXPLICIT CALCULATION OF $\dot E$). WHY IS THIS? AS $\sin\theta$ INCREASES, IT TAKES MORE ENERGY TO MAINTAIN A CONSTANT φ VELOCITY, Ω.

<u>REMARK</u>: COMPARE THIS TO THE CASE WHERE φ IS A FREE PARAMETER (DOF) WITH INIT. CONDITION $\dot\varphi = \Omega$. THE MOMENTUM $P_\varphi = mR^2\dot\varphi\sin^2\theta$ IS CONSERVED, <u>NOT</u> $\dot\varphi$. ALSO NOTE THAT IN THIS CASE $H = E$.

6. CONFORMAL TRANSFORMATIONS OF THE DISK

$$z \mapsto b\left(\frac{z-a}{1-a^*z/r^2}\right)$$

where a, b depend on time
$|a| < r, \ |b| = 1$

a) $|a| < r \Rightarrow a = \rho e^{i\alpha}$ w/ $\rho < 1$
$|b| = 1 \Rightarrow b = e^{i\beta}$

$\boxed{\text{3 DEGREES OF FREEDOM}}$

b) WHEN $|a| = r$, THERE ARE SINGULAR POINTS ON THE RIM OF THE DISK.
WRITE $a = re^{i\alpha}$ AND $z = re^{i\theta}$; SET $b = 1$.

$$\text{then: } z \longrightarrow r\left(\frac{e^{i\theta}-e^{i\alpha}}{1-e^{i(\theta-\alpha)}}\right) = re^{i\alpha}\,\frac{e^{i(\theta-\alpha)}-1}{1-e^{i(\theta-\alpha)}}$$
$$= \boxed{-re^{i\alpha}}$$

NOTE THAT ALL θ INFORMATION HAS BEEN LOST, THE MAP IS NOT INVERTIBLE, ALL PHASE INFO IS COLLAPSED ONTO A SINGLE DIRECTION, α.

c) $L = Mr^2\left(\frac{1}{2}\dot\beta^2 + (\tan^2\gamma)\dot\alpha^2 + \dot\gamma^2\right)$

EQUATIONS of MOTION

α: $2Mr^2 \frac{d}{dt}\left[\dot\alpha \tan^2\gamma\right] = 0$

recall $\frac{d}{dx}\tan x = \sec^2 x$

$$\boxed{2Mr^2\left(\ddot\alpha + 2\dot\alpha\dot\gamma \sec^2\gamma \tan\gamma\right) = 0}$$

β: $\boxed{Mr^2\ddot\beta = 0}$

γ: $\boxed{2Mr^2\ddot\gamma - Mr^2\dot\alpha^2 \cdot 2\tan\gamma \sec^2\gamma = 0}$

in fact, Mr^2 is an overall prefactor:

$$\boxed{\begin{aligned} 0 &= \ddot\alpha + 2\dot\alpha\dot\gamma\sec^2\gamma\tan\gamma \\ 0 &= \ddot\beta \\ 0 &= \ddot\gamma - \dot\alpha\tan\gamma\sec^2\gamma \end{aligned}}$$

note that P_α & P_β are conserved:

$$\boxed{\begin{aligned} P_\alpha &= 2Mr^2\dot\alpha\tan^2\gamma \\ P_\beta &= Mr^2\dot\beta \end{aligned}}$$