(c) $\mathbf{F}_1 = \frac{\mu_0}{4\pi}\frac{q_1 q_2}{r^2} v^2 \mathbf{r}_0 = -\mathbf{F}_2.$

Mutual attraction.

1.6 GRADIENT, ∇

Suppose that $\varphi(x, y, z)$ is a scalar point function, that is, a function whose value depends on the values of the coordinates (x, y, z). As a scalar, it must have the same value at a given fixed point in space, independent of the rotation of our coordinate system, or

$$\varphi'(x_1', x_2', x_3') = \varphi(x_1, x_2, x_3). \tag{1.51}$$

By differentiating with respect to x_i' we obtain

$$\begin{aligned}\frac{\partial\varphi'(x_1', x_2', x_3')}{\partial x_i'} &= \frac{\partial\varphi(x_1, x_2, x_3)}{\partial x_i'} \\ &= \sum_j \frac{\partial\varphi}{\partial x_j}\frac{\partial x_j}{\partial x_i'} = \sum_j a_{ij}\frac{\partial\varphi}{\partial x_j}\end{aligned} \tag{1.52}$$

by the rules of partial differentiation and Eq. 1.16. But comparison with Eq. 1.17, the vector transformation law, now shows that we have *constructed* a vector with components $\partial\varphi/\partial x_j$. This vector we label the gradient of φ.

A convenient symbolism is

$$\nabla\varphi = \mathbf{i}\frac{\partial\varphi}{\partial x} + \mathbf{j}\frac{\partial\varphi}{\partial y} + \mathbf{k}\frac{\partial\varphi}{\partial z} \tag{1.53}$$

or

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \tag{1.54}$$

$\nabla\varphi$ (or del φ) is our gradient of the scalar φ, whereas ∇ (del) itself is a vector differential operator (available to operate on or to differentiate a scalar φ). It should be emphasized that this operator is a hybrid creature that must satisfy both the laws for handling vectors and the laws of partial differentiation.

EXAMPLE 1.6.1 The Gradient of a Function of r.

Let us calculate the gradient of $f(r) = f(\sqrt{x^2 + y^2 + z^2})$.

$$\nabla f(r) = \mathbf{i}\frac{\partial f(r)}{\partial x} + \mathbf{j}\frac{\partial f(r)}{\partial y} + \mathbf{k}\frac{\partial f(r)}{\partial z}.$$

Now $f(r)$ depends on x through the dependence of r on x. Therefore[1]

[1] This is a special case of the chain rule of partial differentiation:

$$\frac{\partial f(r, \theta, \varphi)}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \varphi}\frac{\partial \varphi}{\partial x}.$$

Here $\partial f/\partial\theta = \partial f/\partial\varphi = 0$, $\partial f/\partial r \to df/dr$.

$$\frac{\partial f(r)}{\partial x} = \frac{df(r)}{dr} \cdot \frac{\partial r}{\partial x}.$$

From r as a function of x, y, z

$$\frac{\partial r}{\partial x} = \frac{\partial (x^2 + y^2 + z^2)^{1/2}}{\partial x} = \frac{x}{(x^2 + y^2 + z^2)^{1/2}} = \frac{x}{r}.$$

Therefore

$$\frac{\partial f(r)}{\partial x} = \frac{df(r)}{dr} \cdot \frac{x}{r}.$$

Permuting coordinates ($x \to y,\ y \to z,\ z \to x$) to obtain the y and z derivatives, we get

$$\begin{aligned} \nabla f(r) &= (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)\frac{1}{r}\frac{df}{dr} \\ &= \frac{\mathbf{r}}{r}\frac{df}{dr} \\ &= \mathbf{r}_0 \frac{df}{dr}. \end{aligned}$$

Here $\mathbf{r}_0$ is a unit vector ($\mathbf{r}/r$) in the *positive* radial direction. The gradient of a function of r is a vector in the (positive or negative) radial direction. In Section 2.5 $\mathbf{r}_0$ is seen as one of the three orthonormal unit vectors of spherical polar coordinates.

A GEOMETRICAL INTERPRETATION

One immediate application of $\nabla\varphi$ is to dot it into an increment of length

$$d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz. \tag{1.55}$$

Thus we obtain

$$\begin{aligned} (\nabla\varphi) \cdot d\mathbf{r} &= \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz \\ &= d\varphi, \end{aligned} \tag{1.56}$$

the change in the scalar function φ corresponding to a change in position $d\mathbf{r}$. Now consider P and Q to be two points on a surface $\varphi(x, y, z) = C$, a constant. These points are chosen so that Q is a distance $d\mathbf{r}$ from P. Then moving from P to Q, the change in $\varphi(x, y, z) = C$ is given by

$$\begin{aligned} d\varphi &= (\nabla\varphi) \cdot d\mathbf{r} \\ &= 0, \end{aligned} \tag{1.57}$$

since we stay on the surface $\varphi(x, y, z) = C$. This shows that $\nabla\varphi$ is perpendicular to $d\mathbf{r}$. Since $d\mathbf{r}$ may have any direction from P *as long as it stays in the surface* φ, point Q being restricted to the surface, but having arbitrary direction, $\nabla\varphi$ is seen as normal to the surface $\varphi = $ constant (Fig. 1.16).

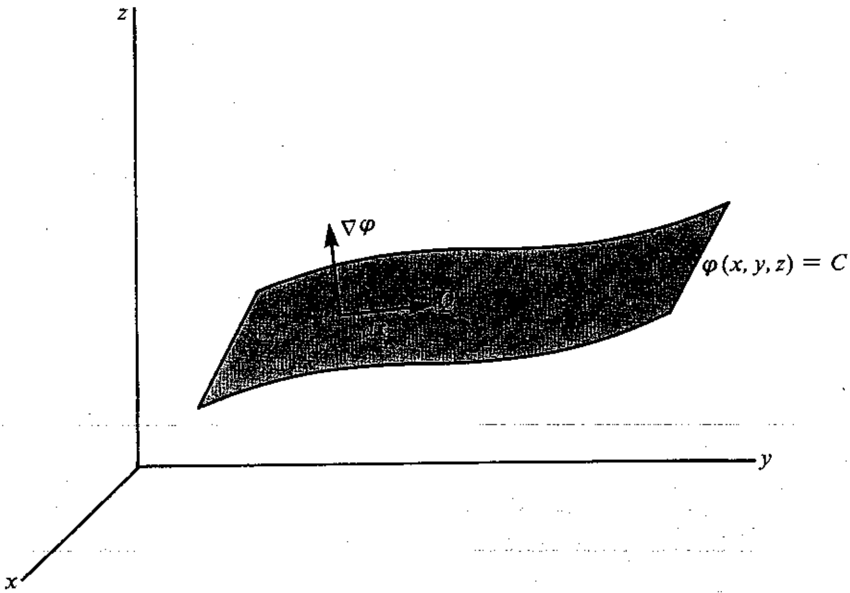

FIG. 1.16 The length increment $d\mathbf{r}$ is required to stay on the surface $\varphi = C$.

If we now permit $d\mathbf{r}$ to take us from one surface $\varphi = C_1$ to an adjacent surface $\varphi = C_2$ (Fig. 1.17*a*),

$$\begin{aligned} d\varphi &= C_2 - C_1 = \Delta C \\ &= (\nabla\varphi) \cdot d\mathbf{r}. \end{aligned} \tag{1.58}$$

For a given $d\varphi$, $|d\mathbf{r}|$ is a minimum when it is chosen parallel to $\nabla\varphi$ ($\cos\theta = 1$); or, for a given $|d\mathbf{r}|$, the change in the scalar function φ is maximized by choosing $d\mathbf{r}$ parallel to $\nabla\varphi$. *This identifies* $\nabla\varphi$ *as a vector having the direction of the maximum space rate of change of* φ, an identification that will be useful in Chapter 2 when we consider noncartesian coordinate systems.

This identification of $\nabla\varphi$ may also be developed by using the calculus of variations subject to a constraint, Exercise 17.6.9.

EXAMPLE 1.6.2

As a specific example of the foregoing, and as an extension of Example 1.6.1, we consider the surfaces consisting of concentric spherical shells, Fig. 1.17*b*. We have

$$\varphi(x, y, z) = (x^2 + y^2 + z^2)^{1/2} = r_i = C_i,$$

where r_i is the radius equal to C_i, our constant. $\Delta C = \Delta\varphi = \Delta r_i$, the distance between two shells. From Example 1.6.1

$$\nabla\varphi(r) = \mathbf{r}_0 \frac{d\varphi(r)}{dr} = \mathbf{r}_0.$$

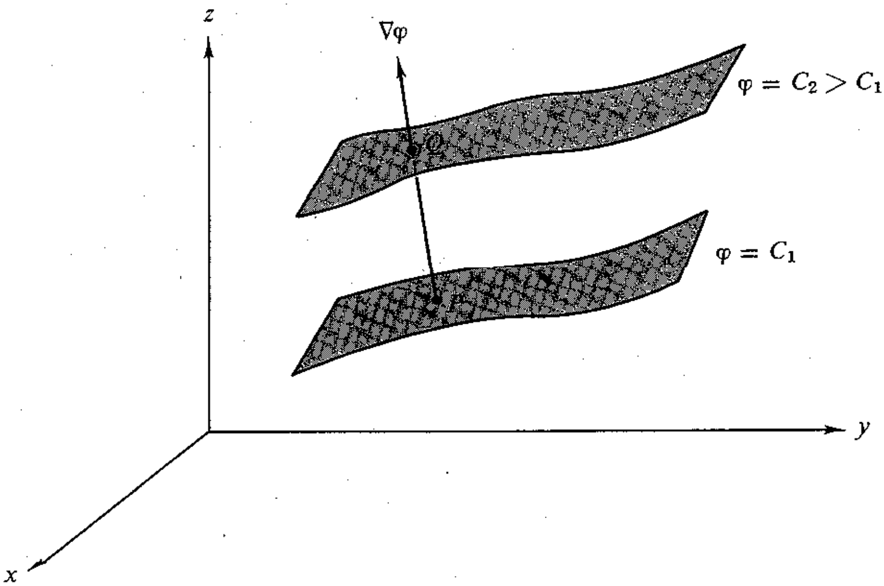

FIG. 1.17*a* Gradient

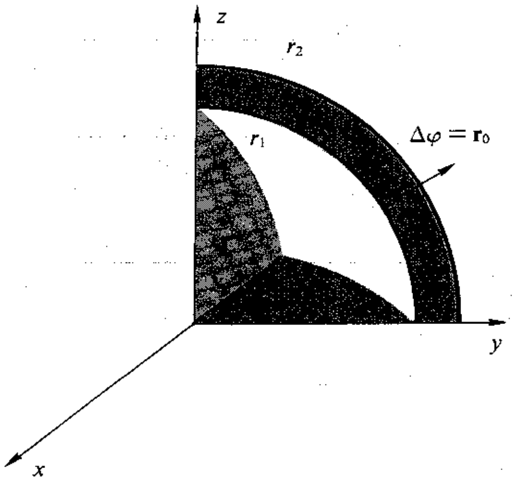

FIG. 1.17*b* Gradient for $\varphi(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$, spherical shells: $(x^2 + y^2 + z^2)^{1/2} = r_2 = C_2$, $(x^2 + y^2 + z^2)^{1/2} = r_1 = C_1$

The gradient is in the radial direction and is normal to the spherical surface $\varphi = C$.

The gradient of a scalar is of extreme importance in physics in expressing the relation between a force field and a potential field.

$$\text{force} = -\nabla\,(\text{potential}). \tag{1.59}$$

This is illustrated by both gravitational and electrostatic fields. among others. Readers should note that the minus sign in Eq. 1.59 results in water flowing downhill rather than uphill! We reconsider Eq. 1.59 in a broader context in Section 1.13.

EXERCISES

1.6.1 If $S(x, y, z) = (x^2 + y^2 + z^2)^{-3/2}$, find
(a) ∇S at the point $(1, 2, 3)$;
(b) the magnitude of the gradient of S, $|\nabla S|$ at $(1, 2, 3)$;
and
(c) the direction cosines of ∇S at $(1, 2, 3)$.

1.6.2 (a) Find a unit vector perpendicular to the surface

$$x^2 + y^2 + z^2 = 3$$

at the point $(1, 1, 1)$.
(b) Derive the equation of the plane tangent to the surface at $(1, 1, 1)$.

ANS. (a) $(\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$.
(b) $x + y + z = 3$.

1.6.3 Given a vector $\mathbf{r}_{12} = \mathbf{i}(x_1 - x_2) + \mathbf{j}(y_1 - y_2) + \mathbf{k}(z_1 - z_2)$, show that $\nabla_1 r_{12}$ (gradient with respect to x_1, y_1, and z_1 of the magnitude r_{12}) is a unit vector in the direction of $\mathbf{r}_{12}$.

1.6.4 If a vector function $\mathbf{F}$ depends on both space coordinates (x, y, z) and time t, show that

$$d\mathbf{F} = (d\mathbf{r} \cdot \nabla)\mathbf{F} + \frac{\partial \mathbf{F}}{\partial t} dt.$$

1.6.5 Show that $\nabla(uv) = v\nabla u + u\nabla v$, where u and v are differentiable scalar functions of x, y, and z.

1.6.6 (a) Show that a necessary and sufficient condition that $u(x, y, z)$ and $v(x, y, z)$ are related by some function $f(u, v) = 0$ is that $(\nabla u) \times (\nabla v) = 0$.
(b) If $u = u(x, y)$ and $v = v(x, y)$, show that the condition $(\nabla u) \times (\nabla v) = 0$ leads to the two-dimensional Jacobian

$$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0.$$

The functions u and v are assumed differentiable.

1.7 DIVERGENCE, $\nabla \cdot$

Differentiating a vector function is a simple extension of differentiating scalar quantities. Suppose $\mathbf{r}(t)$ describes the position of a satellite at some time t. Then, for differentiation with respect to time,

$$\begin{aligned} \frac{d\mathbf{r}(t)}{dt} &= \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \\ &= \mathbf{v}, \quad \text{linear velocity.} \end{aligned}$$

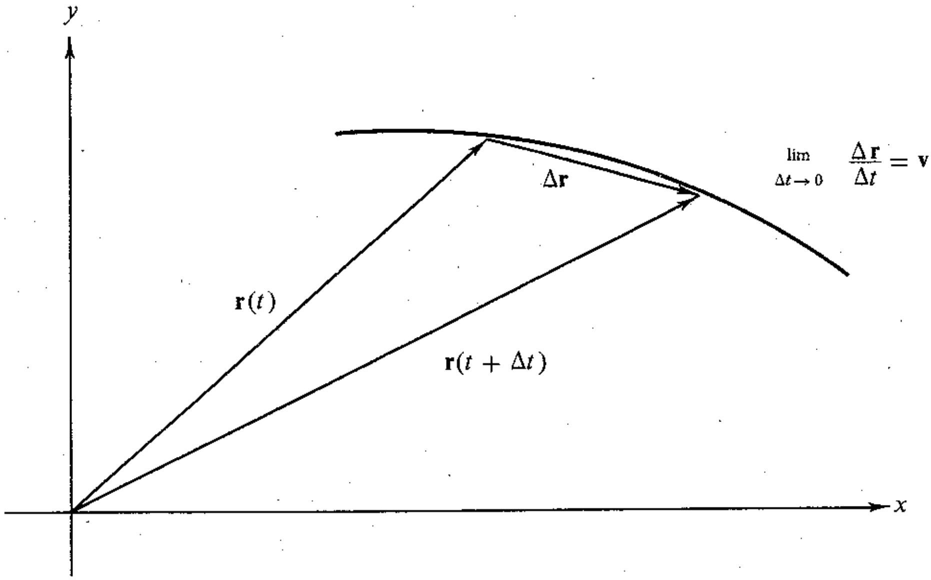

FIG. 1.18 Differentiation of a vector

Graphically, we again have the slope of a curve, orbit, or trajectory, as shown in Fig. 1.18.

If we resolve $\mathbf{r}(t)$ into its cartesian components, $d\mathbf{r}/dt$ always reduces directly to a vector sum of not more than three (for three-dimensional space) scalar derivatives. In other coordinate systems (Chapter 2) the situation is a little more complicated, for the unit vectors are no longer constant in direction. Differentiation with respect to the space coordinates is handled in the same way as differentiation with respect to time, as seen in the following paragraphs.

In Section 1.6 ∇ was defined as a vector operator. Now, paying careful attention to both its vector and its differential properties, we let it operate on a vector. First, as a vector we dot it into a second vector to obtain

$$\nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}, \tag{1.60}$$

known as the divergence of $\mathbf{V}$. This is a scalar, as discussed in Section 1.3.

EXAMPLE 1.7.1

Calculate $\nabla \cdot \mathbf{r}$.

$$\nabla \cdot \mathbf{r} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z},$$

or

$$\nabla \cdot \mathbf{r} = 3.$$

EXAMPLE 1.7.2

Generalizing Example 1.7.1,

$$\begin{aligned}\nabla\cdot\mathbf{r}f(r) &= \frac{\partial}{\partial x}[xf(r)] + \frac{\partial}{\partial y}[yf(r)] + \frac{\partial}{\partial z}[zf(r)] \\ &= 3f(r) + \frac{x^2}{r}\frac{df}{dr} + \frac{y^2}{r}\frac{df}{dr} + \frac{z^2}{r}\frac{df}{dr} \\ &= 3f(r) + r\frac{df}{dr}.\end{aligned}$$

The manipulation of the partial derivatives leading to the second equation in Example 1.7.2 is discussed in Example 1.6.1.

In particular, if $f(r) = r^{n-1}$,

$$\begin{aligned}\nabla\cdot\mathbf{r}r^{n-1} &= \nabla\cdot\mathbf{r}_0 r^n \\ &= 3r^{n-1} + (n-1)r^{n-1} \qquad (1.60a) \\ &= (n+2)r^{n-1}.\end{aligned}$$

This divergence vanishes for $n = -2$, an important fact in Section 1.14.

A PHYSICAL INTERPRETATION

To develop a feeling for the physical significance of the divergence, consider $\nabla\cdot(\rho\mathbf{v})$ with $\mathbf{v}(x, y, z)$, the velocity of a compressible fluid and $\rho(x, y, z)$, its density at point (x, y, z). If we consider a small volume $dx\,dy\,dz$ (Fig. 1.19), the fluid flowing into this volume per unit time (positive x-direction) through the face $EFGH$ is (rate of flow in)$_{EFGH} = \rho v_x|_{x=0}\,dy\,dz$. The components of the flow ρv_y and ρv_z tangential to this face contribute nothing to the flow through this face. The rate of flow out (still positive x-direction) through face $ABCD$ is $\rho v_x|_{x=dx}\,dy\,dz$. To compare these flows and to find the net flow out, we expand this last result in a Maclaurin series[1], Section 5.6. This yields

$$\begin{aligned}(\text{rate of flow out})_{ABCD} &= \rho v_x|_{x=dx}\,dy\,dz \\ &= \left[\rho v_x + \frac{\partial}{\partial x}(\rho v_x)\,dx\right]_{x=0} dy\,dz.\end{aligned}$$

Here the derivative term is a first correction term allowing for the possibility of nonuniform density or velocity or both[2]. The zero-order term $\rho v_x|_{x=0}$ (corresponding to uniform flow) cancels out.

---

[1] A Maclaurin expansion for a single variable is given by Eq. 5.88, Section 5.6. Here we have the increment x of Eq. 5.88 replaced by dx. We show a partial derivative with respect to x since ρv_x may also depend on y and z.

[2] Strictly speaking, ρv_x is averaged over face $EFGH$ and the expression $\rho v_x + (\partial/\partial x)(\rho v_x)\,dx$ is similarly averaged over face $ABCD$. Using an arbitrarily small differential volume, we find that the averages reduce to the values employed here.

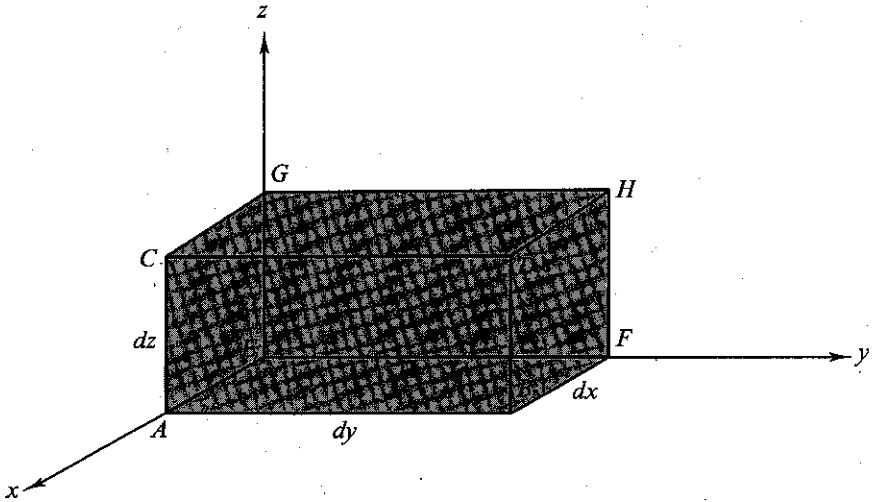

FIG. 1.19 Differential rectangular parallelepiped (in first or positive octant)

$$\text{Net rate of flow out}\,\Big|_x = \frac{\partial}{\partial x}(\rho v_x)\,dx\,dy\,dz.$$

Equivalently, we can arrive at this result by

$$\lim_{\Delta x \to 0} \frac{\rho v_x(\Delta x, 0, 0) - \rho v_x(0, 0, 0)}{\Delta x} \equiv \frac{\partial[\rho v_x(x, y, z)]}{\partial x}\bigg|_{0,0,0}$$

Now the x-axis is not entitled to any preferred treatment. The preceding result for the two faces perpendicular to the x-axis must hold for the two faces perpendicular to the y-axis, with x replaced by y and the corresponding changes for y and z: $y \to z$, $z \to x$. This is a cyclic permutation of the coordinates. A further cyclic permutation yields the result for the remaining two faces of our parallelepiped. Adding the net rate of flow out for all three pairs of surfaces of our volume element, we have

$$\begin{aligned}\text{net flow out}\quad&\\ \text{(per unit time)} &= \left[\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right] dx\,dy\,dz \qquad (1.61)\\ &= \nabla \cdot (\rho \mathbf{v})\,dx\,dy\,dz.\end{aligned}$$

Therefore the net flow of our compressible fluid out of the volume element $dx\,dy\,dz$ per unit volume per unit time is $\nabla \cdot (\rho\mathbf{v})$. Hence the name *divergence*. A direct application is in the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \qquad (1.62)$$

which simply states that a net flow out of the volume results in a decreased density inside the volume. Note that in Eq. 1.62 ρ is considered to be a possible function of time as well as of space: $\rho(x,y,z,t)$. The divergence appears in a wide variety of physical problems, ranging from a probability current density in quantum mechanics to neutron leakage in a nuclear reactor.

The combination $\nabla\cdot(f\mathbf{V})$, in which f is a scalar function and $\mathbf{V}$ a vector function, may be written

$$\begin{aligned}\nabla\cdot(f\mathbf{V}) &= \frac{\partial}{\partial x}(fV_x)+\frac{\partial}{\partial y}(fV_y)+\frac{\partial}{\partial z}(fV_z)\\ &= \frac{\partial f}{\partial x}V_x+f\frac{\partial V_x}{\partial x}+\frac{\partial f}{\partial y}V_y+f\frac{\partial V_y}{\partial y}+\frac{\partial f}{\partial z}V_z+f\frac{\partial V_z}{\partial z} \qquad (1.62a)\\ &= (\nabla f)\cdot\mathbf{V}+f\nabla\cdot\mathbf{V},\end{aligned}$$

which is just what we would expect for the derivative of a product. Notice that ∇ as a differential operator differentiates both f and $\mathbf{V}$; as a vector it is dotted into $\mathbf{V}$ (in each term).

If we have the special case of the divergence of a vector vanishing,

$$\nabla\cdot\mathbf{B}=0, \qquad (1.63)$$

the vector $\mathbf{B}$ is said to be solenoidal, the term coming from the example in which $\mathbf{B}$ is the magnetic induction and Eq. 1.63 appears as one of Maxwell's equations. When a vector is solenoidal it may be written as the curl of another vector known as the vector potential. In Section 1.13 we shall calculate such a vector potential.

EXERCISES

1.7.1 For a particle moving in a circular orbit $\mathbf{r}=\mathbf{i}r\cos\omega t+\mathbf{j}r\sin\omega t$,
(a) evaluate $\mathbf{r}\times\dot{\mathbf{r}}$.
(b) Show that $\ddot{\mathbf{r}}+\omega^2\mathbf{r}=0$.
The radius r and the angular velocity ω are constant.

ANS. (*a*) $\mathbf{k}\omega r^2$.

Note. $\dot{\mathbf{r}}=d\mathbf{r}/dt$, $\quad\ddot{\mathbf{r}}=d^2\mathbf{r}/dt^2$.

1.7.2 Vector $\mathbf{A}$ satisfies the vector transformation law, Eq. 1.15. Show directly that its time derivative $d\mathbf{A}/dt$ also satisfies Eq. 1.15 and is therefore a vector.

1.7.3 Show, by differentiating components, that
(a) $\dfrac{d}{dt}(\mathbf{A}\cdot\mathbf{B})=\dfrac{d\mathbf{A}}{dt}\cdot\mathbf{B}+\mathbf{A}\cdot\dfrac{d\mathbf{B}}{dt}$,
(b) $\dfrac{d}{dt}(\mathbf{A}\times\mathbf{B})=\dfrac{d\mathbf{A}}{dt}\times\mathbf{B}+\mathbf{A}\times\dfrac{d\mathbf{B}}{dt}$,
just like the derivative of the product of two algebraic functions.

1.7.4 In Chapter 2 it will be seen that the *unit* vectors in noncartesian coordinate systems are usually functions of the coordinate variables, $\mathbf{e}_i=\mathbf{e}(q_1,q_2,q_3)$ but $|\mathbf{e}_i|=1$. Show that either $\partial\mathbf{e}_i/\partial q_j=0$ or $\partial\mathbf{e}_i/\partial q_j$ is orthogonal to $\mathbf{e}_i$.

1.7.5 Prove $\nabla\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot\nabla\times\mathbf{a}-\mathbf{a}\cdot\nabla\times\mathbf{b}$.
Hint. Treat as a triple scalar product.

1.7.6 The electrostatic field of a point charge q is

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0} \cdot \frac{\mathbf{r}_0}{r^2}.$$

Calculate the divergence of $\mathbf{E}$. What happens at the origin?

1.8 CURL, $\nabla\times$

Another possible operation with the vector operator ∇ is to cross it into a vector. We obtain

$$\nabla \times \mathbf{V} = \mathbf{i}\left(\frac{\partial}{\partial y}V_z - \frac{\partial}{\partial z}V_y\right) + \mathbf{j}\left(\frac{\partial}{\partial z}V_x - \frac{\partial}{\partial x}V_z\right) + \mathbf{k}\left(\frac{\partial}{\partial x}V_y - \frac{\partial}{\partial y}V_x\right)$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}, \tag{1.64}$$

which is called the curl of $\mathbf{V}$. In expanding this determinant form or in any operation with ∇, we must consider the derivative nature of ∇. Specifically, $\mathbf{V} \times \nabla$ is defined only as an operator, another vector differential operator. It is certainly not equal, in general, to $-\nabla \times \mathbf{V}$.[1] In the case of Eq. 1.64 the determinant must be expanded *from the top down* so that we get the derivatives as shown in the middle portion of Eq. 1.64. If ∇ is crossed into the product of a scalar and a vector, we can show

$$\nabla \times (f\mathbf{V})|_x = \left[\frac{\partial}{\partial y}(fV_z) - \frac{\partial}{\partial z}(fV_y)\right]$$
$$= \left(f\frac{\partial V_z}{\partial y} + \frac{\partial f}{\partial y}V_z - f\frac{\partial V_y}{\partial z} - \frac{\partial f}{\partial z}V_y\right) \tag{1.65}$$
$$= f\nabla \times \mathbf{V}|_x - (\nabla f) \times \mathbf{V}|_x.$$

If we permute the coordinates $x \to y, y \to z, z \to x$ to pick up the y-component and then permute them a second time to pick up the z-component,

$$\nabla \times (f\mathbf{V}) = f\nabla \times \mathbf{V} + (\nabla f) \times \mathbf{V}, \tag{1.66}$$

which is the vector product analog of Eq. 1.62*a*. Again, as a differential operator ∇ differentiates both f and $\mathbf{V}$. As a vector it is crossed into $\mathbf{V}$ (in each term).

EXAMPLE 1.8.1

Calculate $\nabla \times \mathbf{r}f(r)$
By Eq. 1.66

[1] In this same spirit, if $\mathbf{A}$ is a differential operator, it is not necessarily true that $\mathbf{A} \times \mathbf{A} = 0$. Specifically, for the quantum mechanical angular momentum *operator*, $\mathbf{L} = -i(\mathbf{r} \times \nabla)$, we find that $\mathbf{L} \times \mathbf{L} = i\mathbf{L}$.

$$\nabla \times \mathbf{r}f(r) = f(r)\nabla \times \mathbf{r} + [\nabla f(r)] \times \mathbf{r}. \tag{1.67}$$

First,

$$\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0. \tag{1.68}$$

Second, using $\nabla f(r) = \mathbf{r}_0(df/dr)$ (Example 1.6.1), we obtain

$$\nabla \times \mathbf{r}f(r) = \frac{df}{dr}\mathbf{r}_0 \times \mathbf{r} = 0. \tag{1.69}$$

The vector product vanishes, since $\mathbf{r} = \mathbf{r}_0 r$ and $\mathbf{r}_0 \times \mathbf{r}_0 = 0$.

To develop a better feeling for the physical significance of the curl, we consider the circulation of fluid around a differential loop in the xy-plane, Fig. 1.20.

Although the circulation is technically given by a vector line integral $\int \mathbf{V} \cdot d\boldsymbol{\lambda}$ (Section 1.10), we can set up the equivalent scalar integrals here. Let us take the circulation to be

$$\begin{aligned} \text{circulation}_{1234} &= \int_1 V_x(x,y)\,d\lambda_x + \int_2 V_y(x,y)\,d\lambda_y \\ &\quad + \int_3 V_x(x,y)\,d\lambda_x + \int_4 V_y(x,y)\,d\lambda_y. \end{aligned} \tag{1.70}$$

The numbers 1, 2, 3, and 4 refer to the numbered line segments in Fig. 1.20. In the first integral $d\lambda_x = +dx$ but in the third integral $d\lambda_x = -dx$ because the third line segment is traversed in the negative x-direction. Similarly, $d\lambda_y = +dy$ for the second integral, $-dy$ for the fourth. Next, the integrands are referred to the point (x_0, y_0) with a Taylor expansion[2] taking into account the

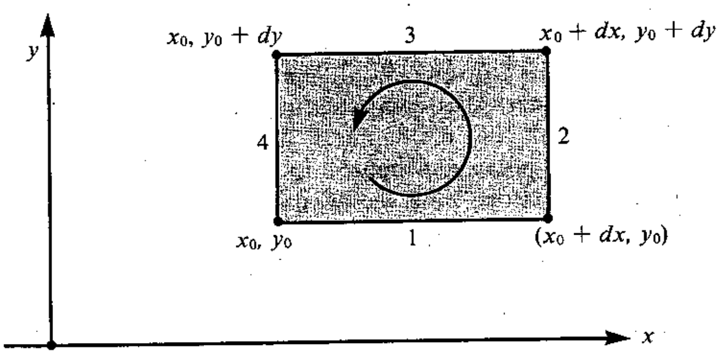

FIG. 1.20 Circulation around a differential loop

[2] $V_y(x_0 + dx, y_0) = V_y(x_0, y_0) + \left(\dfrac{\partial V_y}{\partial x}\right)_{x_0 y_0} dx + \cdots.$

The higher-order terms will drop out in the limit as $dx \to 0$. A correction term for the variation of V_y with y is canceled by the corresponding term in the fourth integral (see Section 5.6).

displacement of line segment 3 from 1 and 2 from 4. For our differential line segments this leads to

$$\begin{aligned}\text{circulation}_{1234} &= V_x(x_0, y_0)\,dx + \left[V_y(x_0, y_0) + \frac{\partial V_y}{\partial x}dx\right]dy \\ &\quad + \left[V_x(x_0, y_0) + \frac{\partial V_x}{\partial y}dy\right](-dx) + V_y(x_0, y_0)(-dy) \qquad (1.71) \\ &= \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)dx\,dy.\end{aligned}$$

Dividing by $dx\,dy$, we have

$$\text{circulation per unit area} = \nabla \times \mathbf{V}|_z. \qquad (1.72)$$

The circulation[3] about our differential area in the xy-plane is given by the z-component of $\nabla \times \mathbf{V}$. In principle, the curl, $\nabla \times \mathbf{V}$ at (x_0, y_0), could be determined by inserting a (differential) paddle wheel into the moving fluid at point (x_0, y_0). The rotation of the little paddle wheel would be a measure of the curl.

We shall use the result, Eq. 1.71, in Section 1.13 to derive Stokes's theorem.

Whenever the curl of a vector $\mathbf{V}$ vanishes,

$$\nabla \times \mathbf{V} = 0. \qquad (1.73)$$

$\mathbf{V}$ is labeled irrotational. The most important physical examples of irrotational vectors are the gravitational and electrostatic forces. In each case

$$\mathbf{V} = C\frac{\mathbf{r}_0}{r^2} = C\frac{\mathbf{r}}{r^3}, \qquad (1.74)$$

where C is a constant and $\mathbf{r}_0$ is the unit vector in the outward radial direction. For the gravitational case we have $C = -Gm_1m_2$, given by Newton's law of universal gravitation. If $C = q_1q_2/4\pi\varepsilon_0$, we have Coulomb's law of electrostatics (mks units). The force $\mathbf{V}$ given in Eq. 1.74 may be shown to be irrotational by direct expansion into cartesian components as we did in Example 1.8.1. Another approach is developed in Chapter 2, in which we express $\nabla\times$, the curl, in terms of spherical polar coordinates. In Section 1.13 we shall see that whenever a vector is irrotational, the vector may be written as the (negative) gradient of a scalar potential. In Section 1.15 we shall prove that a vector may be resolved into an irrotational part and a solenoidal part (subject to conditions at infinity). In terms of the electromagnetic field this corresponds to the resolution into an irrotational electric field and a solenoidal magnetic field.

For waves in an elastic medium, if the displacement $\mathbf{u}$ is irrotational, $\nabla \times \mathbf{u} = 0$, planes waves (or spherical waves at large distances) become longitudinal. If $\mathbf{u}$ is solenoidal, $\nabla \cdot \mathbf{u} = 0$, then the waves become transverse. A seismic disturbance will produce a displacement that may be resolved into a solenoidal

[3] In fluid dynamics $\nabla \times \mathbf{V}$ is called the "vorticity."

part and an irrotational part (compare Section 1.15). The irrotational part yields the longitudinal P (primary) earthquake waves. The solenoidal part gives rise to the slower transverse S (secondary) waves, Exercise 3.6.8.

Using the gradient, divergence, and curl, and of course the *BAC-CAB* rule, we may construct or verify a large number of useful vector identities. For verification, complete expansion into cartesian components is always a possibility. Sometimes if we use insight instead of routine shuffling of cartesian components, the verification process can be shortened drastically.

Remember that ∇ is a vector operator, a hybrid creature satisfying two sets of rules:

1. vector rules, and
2. partial differentiation rules—including differentiation of a product.

EXAMPLE 1.8.2. Gradient of a Dot Product

Verify that

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B}). \quad (1.75)$$

This particular example hinges on the recognition that $\nabla(\mathbf{A}\cdot\mathbf{B})$ is the type of term that appears in the *BAC-CAB* expansion of a triple vector product, Eq. 1.50. For instance,

$$\mathbf{A}\times(\nabla\times\mathbf{B}) = \nabla(\mathbf{A}\cdot\mathbf{B}) - (\mathbf{A}\cdot\nabla)\mathbf{B},$$

with the ∇ differentiating only $\mathbf{B}$, not $\mathbf{A}$. From the commutativity of factors in a scalar product we may interchange $\mathbf{A}$ and $\mathbf{B}$ and write

$$\mathbf{B}\times(\nabla\times\mathbf{A}) = \nabla(\mathbf{A}\cdot\mathbf{B}) - (\mathbf{B}\cdot\nabla)\mathbf{A},$$

now with ∇ differentiating only $\mathbf{A}$, not $\mathbf{B}$. Adding these two equations, we obtain ∇ differentiating the product $\mathbf{A}\cdot\mathbf{B}$ and the identity, Eq. (1.75).

This identity is used frequently in advanced electromagnetic theory. Exercise 1.8.15 is one simple illustration.

EXERCISES

1.8.1 Show, by rotating the coordinates, that the components of the curl of a vector transform as a vector.
Hint. The direction cosine identities of Eq. 1.41 are available as needed.

1.8.2 Show that $\mathbf{u}\times\mathbf{v}$ is solenoidal if $\mathbf{u}$ and $\mathbf{v}$ are each irrotational.

1.8.3 If $\mathbf{A}$ is irrotational, show that $\mathbf{A}\times\mathbf{r}$ is solenoidal.

1.8.4 A rigid body is rotating with constant angular velocity $\boldsymbol{\omega}$. Show that the linear velocity $\mathbf{v}$ is solenoidal.

1.8.5 A vector function $\mathbf{f}(x, y, z)$ is not irrotational but the product of $\mathbf{f}$ and a scalar function $g(x, y, z)$ is irrotational. Show that

$$\mathbf{f} \cdot \nabla \times \mathbf{f} = 0.$$

1.8.6 If (a) $\mathbf{V} = \mathbf{i}V_x(x, y) + \mathbf{j}V_y(x, y)$ and (b) $\nabla \times \mathbf{V} \neq 0$, prove that $\nabla \times \mathbf{V}$ is perpendicular to $\mathbf{V}$.

1.8.7 Classically, angular momentum is given by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{p}$ is the linear momentum. To go from classical mechanics to quantum mechanics, replace $\mathbf{p}$ by the operator $-i\nabla$ (Section 15.6). Show that the quantum mechanical angular momentum operator has cartesian components

$$L_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_y = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$L_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

(in units of $\hbar$).

1.8.8 Using the angular momentum operators previously given, show that they satisfy commutation relations of the form

$$[L_x, L_y] \equiv L_xL_y - L_yL_x = iL_z$$

and hence

$$\mathbf{L} \times \mathbf{L} = i\mathbf{L}.$$

These commutation relations will be taken later as the defining relations as an angular momentum operator—Exercise 4.2.15 and the following one and Section 12.7.

1.8.9 With the commutator bracket notation $[L_x, L_y] = L_xL_y - L_yL_x$, the angular momentum vector $\mathbf{L}$ satisfies $[L_x, L_y] = iL_z$, etc. and so on, or $\mathbf{L} \times \mathbf{L} = i\mathbf{L}$. Two other vectors $\mathbf{a}$ and $\mathbf{b}$ commute with each other and with $\mathbf{L}$, that is, $[\mathbf{a}, \mathbf{b}] = [\mathbf{a}, \mathbf{L}] = [\mathbf{b}, \mathbf{L}] = 0$. Show that

$$[\mathbf{a} \cdot \mathbf{L}, \mathbf{b} \cdot \mathbf{L}] = i(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{L}.$$

1.8.10 For $\mathbf{A} = \mathbf{i}A_x(x, y, z)$ and $\mathbf{B} = \mathbf{i}B_x(x, y, z)$ evaluate each term in the vector identity

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B})$$

and verify that the identity is satisfied.

1.8.11 Verify the vector identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} - \mathbf{B}(\nabla \cdot \mathbf{A}) + \mathbf{A}(\nabla \cdot \mathbf{B}).$$

1.8.12 As an alternative to the vector identity of Example 1.8.2 show that

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \times \nabla) \times \mathbf{B} + (\mathbf{B} \times \nabla) \times \mathbf{A} + \mathbf{A}(\nabla \cdot \mathbf{B}) + \mathbf{B}(\nabla \cdot \mathbf{A}).$$

1.8.13 Verify the identity

$$\mathbf{A} \times (\nabla \times \mathbf{A}) = \tfrac{1}{2}\nabla(A^2) - (\mathbf{A} \cdot \nabla)\mathbf{A}.$$

1.8.14 If $\mathbf{A}$ and $\mathbf{B}$ are constant vectors, show that

$$\nabla(\mathbf{A} \cdot \mathbf{B} \times \mathbf{r}) = \mathbf{A} \times \mathbf{B}.$$

1.8.15 A distribution of electric currents creates a constant magnetic moment $\mathbf{m}$. The force on $\mathbf{m}$ in an external magnetic induction $\mathbf{B}$ is given by

$$\mathbf{F} = \nabla \times (\mathbf{B} \times \mathbf{m}).$$

Show that

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}).$$

Note. Assuming no time dependence of the fields, Maxwell's equations yield $\nabla \times \mathbf{B} = 0$. Also $\nabla \cdot \mathbf{B} = 0$.

1.8.16 An electric dipole of moment $\mathbf{p}$ is located at the origin. The dipole creates an electric potential at $\mathbf{r}$ given by

$$\psi(\mathbf{r}) = \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi\varepsilon_0 r^3}.$$

Find the electric field, $\mathbf{E} = -\nabla\psi$ at $\mathbf{r}$.

1.8.17 The vector potential $\mathbf{A}$ of a magnetic dipole, dipole moment $\mathbf{m}$, is given by $\mathbf{A}(\mathbf{r}) = (\mu_0/4\pi)(\mathbf{m} \times \mathbf{r}/r^3)$. Show that the magnetic induction $\mathbf{B} = \nabla \times \mathbf{A}$ is given by

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{3\mathbf{r}_0(\mathbf{r}_0 \cdot \mathbf{m}) - \mathbf{m}}{r^3}.$$

Note. The limiting process leading to point dipoles is discussed in Section 12.1 for electric dipoles, Section 12.5 for magnetic dipoles.

1.8.18 The velocity of a two-dimensional flow of liquid is given by

$$\mathbf{V} = \mathbf{i}u(x, y) - \mathbf{j}v(x, y).$$

If the liquid is incompressible and the flow is irrotational show that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are the Cauchy–Riemann conditions of Section 6.2.

1.8.19 The evaluation in this section of the four integrals for the circulation omitted Taylor series terms such as $\partial V_x/\partial x$, $\partial V_y/\partial y$ and all second derivatives. Show that $\partial V_x/\partial x$, $\partial V_y/\partial y$ cancel out when the four integrals are added and that the second derivative terms drop out in the limit as $dx \to 0$, $dy \to 0$.
Hint. Calculate the circulation per unit area and then take the limit $dx \to 0$, $dy \to 0$.

1.9 SUCCESSIVE APPLICATIONS OF ∇

We have now defined gradient, divergence, and curl to obtain vector, scalar, and vector quantities, respectively. Letting ∇ operate on each of these quantities, we obtain

$$\text{(a) } \nabla \cdot \nabla\varphi \qquad \text{(b) } \nabla \times \nabla\varphi \qquad \text{(c) } \nabla\nabla \cdot \mathbf{V}$$
$$\text{(d) } \nabla \cdot \nabla \times \mathbf{V} \qquad \text{(e) } \nabla \times (\nabla \times \mathbf{V}),$$

all five expressions involving second derivatives and all five appearing in the second-order differential equations of mathematical physics, particularly in electromagnetic theory.

The first expression, $\nabla \cdot \nabla\varphi$, the divergence of the gradient, is named the Laplacian of φ. We have

$$\begin{aligned}\nabla \cdot \nabla\varphi &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(\mathbf{i}\frac{\partial\varphi}{\partial x} + \mathbf{j}\frac{\partial\varphi}{\partial y} + \mathbf{k}\frac{\partial\varphi}{\partial z}\right) \\ &= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}.\end{aligned} \tag{1.76a}$$

When φ is the electrostatic potential, we have

$$\nabla \cdot \nabla\varphi = 0. \tag{1.76b}$$

which is Laplace's equation of electrostatics. Often the combination $\nabla \cdot \nabla$ is written ∇^2.

EXAMPLE 1.9.1

Calculate $\nabla \cdot \nabla g(r)$.
Referring to Examples 1.6.1 and 1.7.2,

$$\begin{aligned}\nabla \cdot \nabla g(r) &= \nabla \cdot \mathbf{r}_0 \frac{dg}{dr} \\ &= \frac{2}{r}\frac{dg}{dr} + \frac{d^2 g}{dr^2},\end{aligned}$$

replacing $f(r)$ in Example 1.7.2 by $1/r \cdot dg/dr$. If $g(r) = r^n$, this reduces to

$$\nabla \cdot \nabla r^n = n(n+1)r^{n-2}.$$

This vanishes for $n = 0$ [$g(r) = \text{constant}$] and for $n = -1$; that is, $g(r) = 1/r$ is a solution of Laplace's equation, $\nabla^2 g(r) = 0$. This is for $r \neq 0$. At $r = 0$ a Dirac delta function is involved (see Eq. 1.173 and Section 8.7).

Expression (b) may be written

$$\nabla \times \nabla\varphi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial\varphi}{\partial x} & \dfrac{\partial\varphi}{\partial y} & \dfrac{\partial\varphi}{\partial z} \end{vmatrix}.$$

By expanding the determinant, we obtain

$$\begin{aligned}\nabla \times \nabla\varphi &= \mathbf{i}\left(\frac{\partial^2\varphi}{\partial y\,\partial z} - \frac{\partial^2\varphi}{\partial z\,\partial y}\right) + \mathbf{j}\left(\frac{\partial^2\varphi}{\partial z\,\partial x} - \frac{\partial^2\varphi}{\partial x\,\partial z}\right) + \mathbf{k}\left(\frac{\partial^2\varphi}{\partial x\,\partial y} - \frac{\partial^2\varphi}{\partial y\,\partial x}\right) \\ &= 0,\end{aligned} \tag{1.77}$$

assuming that the order of partial differentiation may be interchanged. This is true as long as these second partial derivatives of φ are continuous functions.

Then, from Eq. 1.77, the curl of a gradient is identically zero. All gradients, therefore, are irrotational. Note carefully that the zero in Eq. 1.77 comes as a mathematical identity, independent of any physics. The zero in Eq. 1.76*b* is a consequence of physics.

Expression (d) is a triple scalar product which may be written

$$\nabla \cdot \nabla \times \mathbf{V} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}. \tag{1.78}$$

Again, assuming continuity so that the order of differentiation is immaterial, we obtain

$$\nabla \cdot \nabla \times \mathbf{V} = 0. \tag{1.79}$$

The divergence of a curl vanishes or all curls are solenoidal. In Section 1.15 we shall see that vectors may be resolved into solenoidal and irrotational parts by Helmholtz's theorem.

The two remaining expressions satisfy a relation

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla\nabla \cdot \mathbf{V} - \nabla \cdot \nabla\mathbf{V}. \tag{1.80}$$

This follows immediately from Eq. 1.50, the *BAC-CAB* rule, which we rewrite so that **C** appears at the extreme right of each term. The term $\nabla \cdot \nabla\mathbf{V}$ was not included in our list, but it may be *defined* by Eq. 1.80. If **V** is expanded in cartesian coordinates so that the unit vectors are constant in direction as well as in magnitude, $\nabla \cdot \nabla\mathbf{V}$, a vector Laplacian, reduces to

$$\nabla \cdot \nabla\mathbf{V} = \mathbf{i}\nabla \cdot \nabla V_x + \mathbf{j}\nabla \cdot \nabla V_y + \mathbf{k}\nabla \cdot \nabla V_z,$$

a vector sum of ordinary scalar Laplacians. By expanding in cartesian coordinates, we may verify Eq. 1.80 as a vector identity.

EXAMPLE 1.9.2 Electromagnetic Wave Equation

One important application of this vector relation (Eq. 1.80) is in the derivation of the electromagnetic wave equation. In vacuum Maxwell's equations become

$$\nabla \cdot \mathbf{B} = 0, \tag{1.81a}$$

$$\nabla \cdot \mathbf{E} = 0, \tag{1.81b}$$

$$\nabla \times \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{1.81c}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.81d}$$

Here **E** is the electric field, **B** the magnetic induction, ε_0 the electric permittivity, and μ_0 the magnetic permeability (mks or SI units). Suppose we eliminate **B** from Eqs. 1.81*c* and 1.81*d*. We may do this by taking the curl of both sides of Eq. 1.81*d* and the time derivative of both sides of Eq. 1.81*c*. Since the space and time derivatives commute,

$$\frac{\partial}{\partial t}\nabla \times \mathbf{B} = \nabla \times \frac{\partial \mathbf{B}}{\partial t}, \tag{1.82}$$

and we obtain

$$\nabla \times (\nabla \times \mathbf{E}) = -\varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{1.83}$$

Application of Eqs. 1.80 and of 1.81*b* yields

$$\nabla \cdot \nabla \mathbf{E} = \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \tag{1.84}$$

the electromagnetic vector wave equation. Again, if **E** is expressed in cartesian coordinates, Eq. 1.84 separates into three scalar wave equations, each involving a scalar Laplacian.

EXERCISES

1.9.1 Verify Eq. 1.80

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla\nabla \cdot \mathbf{V} - \nabla \cdot \nabla \mathbf{V}$$

by direct expansion in cartesian coordinates.

1.9.2 Show that the identity

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla\nabla \cdot \mathbf{V} - \nabla \cdot \nabla \mathbf{V}$$

follows from the *BAC-CAB* rule for a triple vector product. Justify any alteration of the order of factors in the *BAC* and *CAB* terms.

1.9.3 Prove that $\nabla \times (\varphi \nabla \varphi) = 0$.

1.9.4 You are given that the curl of **F** equals the curl of **G**. Show that **F** and **G** may differ by (a) a constant and (b) a gradient of a scalar function.

1.9.5 The Navier–Stokes equation of hydrodynamics contains a nonlinear term $(\mathbf{v} \cdot \nabla)\mathbf{v}$. Show that the curl of this term may be written $-\nabla \times [\mathbf{v} \times (\nabla \times \mathbf{v})]$.

1.9.6 From the Navier–Stokes equation for the steady flow of an incompressible viscous fluid we have the term

$$\nabla \times [\mathbf{v} \times (\nabla \times \mathbf{v})]$$

where **v** is the fluid velocity. Show that this term vanishes for the special case

$$\mathbf{v} = \mathbf{i}v(y, z).$$

1.9.7 Prove that $(\nabla u) \times (\nabla v)$ is solenoidal where u and v are differentiable scalar functions.