A field theory approach to plasma self-organization

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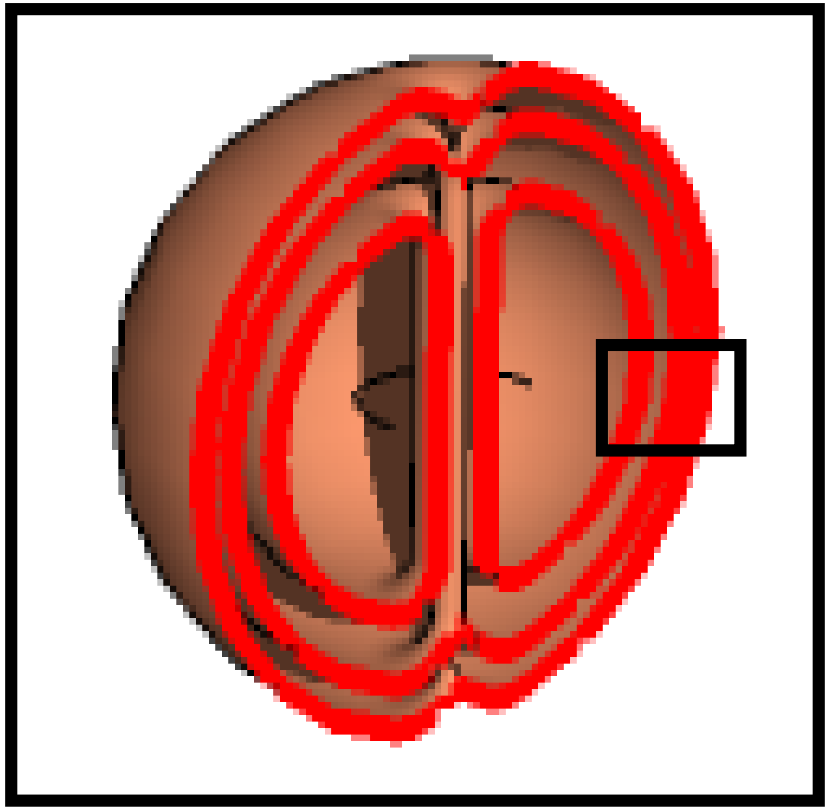
CT configuration from self-organization

Spontaneous formation of new configuration subj. to constraint, init. conditions, to reduce or maximize energy.



e.g.	Magnetostatic single fluid [Taylor, Woltjer, etc.]; Flowing two-fluid [Steinhauer, Ishida, etc.] Flowing neutral fluid [Moffat, etc.]
Strength	No need for detailed dynamics; Can predict final state from initial & boundary conditions.
Q:	Open configurations ? pressure, density , temperature gradients ? L-H transition ? kinetic regime ? magnetic dynamo ? dynamic collimation and stability ? relativity ?
A:	(partial) Can transform plasma physics to a generalized form of Maxwell’s equations, valid in all regimes. Maybe simpler to solve full system in this frame of reference ? In physics, often, a suitable transformation simplifies analysis.

Demonstrate

The equations of motion can be expressed as Generalized Maxwell Equations

Wanted the simplest, most general expression for plasmas.

Helicity evolution valid in kinetic regimes, relativistic regimes too

Wanted to see if helicity conservation, plasma self-organization, relaxation valid in/across regimes beyond fluid regime.

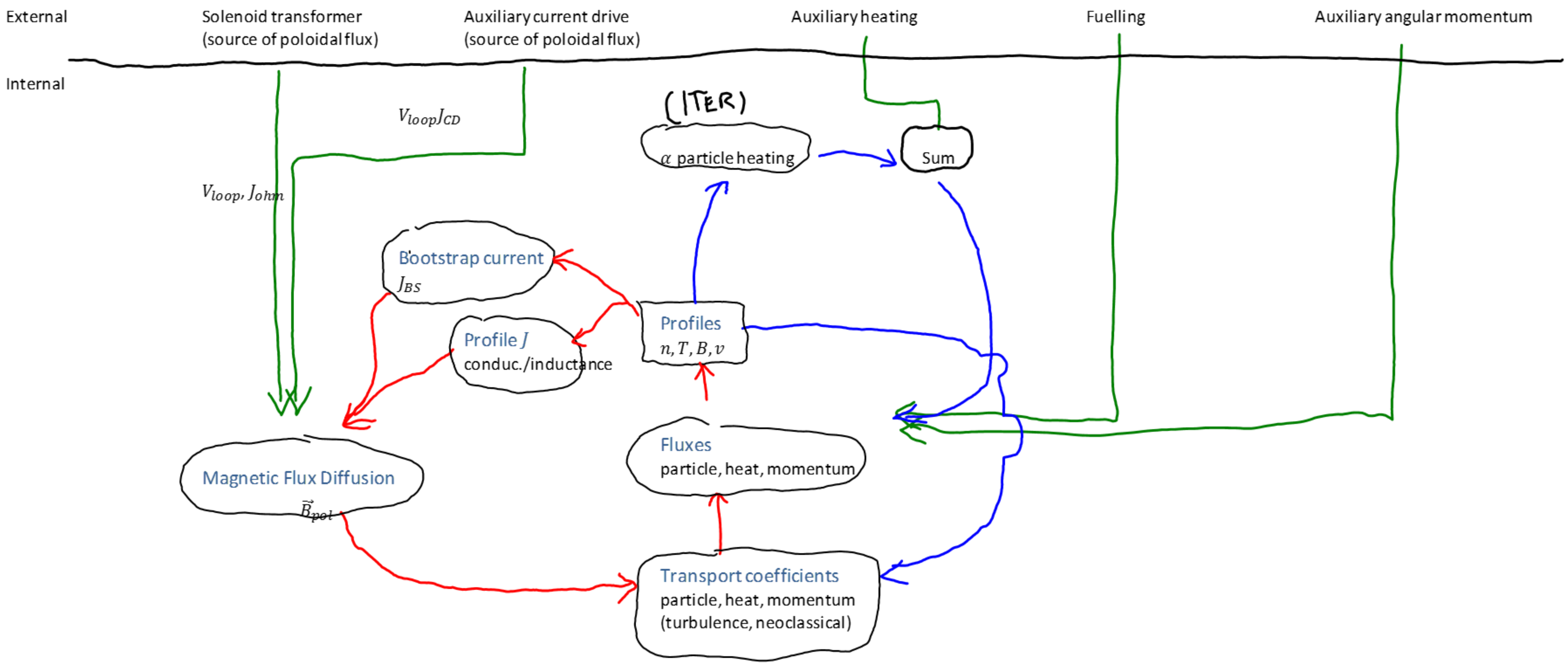
Criterion for helicity conservation vs energy conservation

Wanted to have a general criterion for when/where helicity is conserved

Helicity conversion from one form to another

Conversion of magnetic helicity to flows and vice-versa

Ingredients for magnetic confinement fusion concepts



ITER Physics Basis, 2007, Ch. 2 p.S41

Define relative canonical helicity $K_{\sigma rel} = \int \vec{P}_{\sigma -} \cdot \vec{\Omega}_{\sigma +} \, dV$

eqs. of motion

$$\boxed{\frac{dK_{erel}}{dt} = -\frac{dK_{irel}}{dt} = -2\int \vec{R}\cdot\vec{\Omega}\, dV + 2\int_{S_{sep}} (h_i - h_e)\, \vec{\Omega}\cdot d\vec{S} + \cdots} \qquad (1)$$

$$q_\sigma \phi + \frac{1}{2} m_\sigma u_\sigma^2 + \frac{\mathcal{P}_\sigma}{n_\sigma}$$

enthalpy

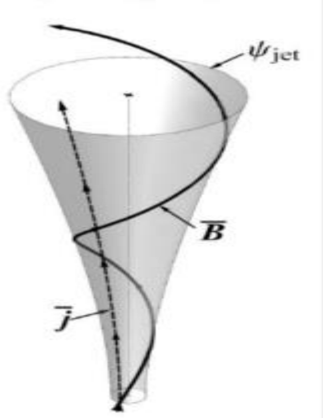
e.g.

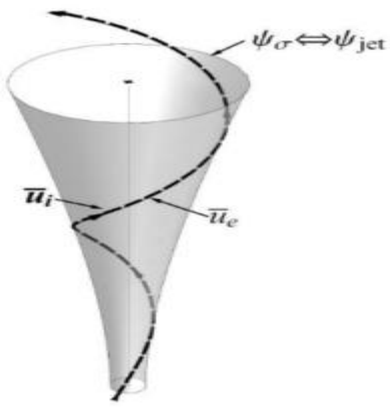
Can. momentum	$\vec{P}_\sigma = m_\sigma \vec{u}_\sigma + q_\sigma \vec{A}$
Can. vorticity	$\vec{\Omega}_\sigma = \nabla \times \vec{P}_\sigma$
Relative	$\vec{X}_\pm = \vec{X} \pm \vec{X}_{\mathrm{ref}}$
Also becomes	$K_\sigma = m_\sigma^2 K_{kin\sigma} + m_\sigma q_\sigma K_{crs\sigma} + q_\sigma^2 K_{mag}$

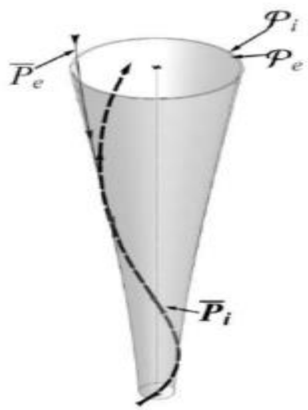
$K_{kin\sigma}$: $\int \vec{u}_\sigma \cdot \vec{\omega}_\sigma \, dV$ kinetic helicity

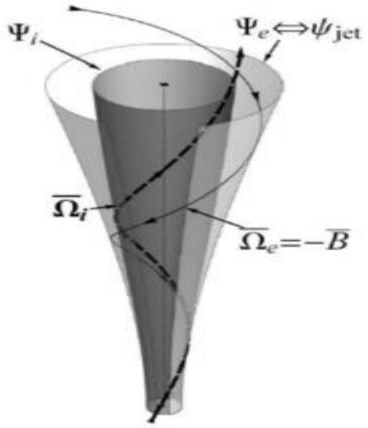
$K_{crs\sigma}$: $2\int \vec{u}_\sigma \cdot \vec{B} \, dV$ cross helicity

K_{mag}: $\int \vec{A} \cdot \vec{B} \, dV$ magnetic helicity









magnetic flux tube

canonical flux tube

S.You, Phys. Plasmas **19**, 092107 (2012); S. You, Plasma Phys. Control. Fusion, **56**, 064007 (2014)

Define relative canonical helicity $K_{\sigma rel} = \int \vec{P}_{\sigma-} \cdot \vec{\Omega}_{\sigma+} \, dV$

eqs. of motion

e.g.

Can. momentum $\quad \vec{P}_\sigma = m_\sigma \vec{u}_\sigma + q_\sigma \vec{A}$

Can. vorticity $\quad \vec{\Omega}_\sigma = \nabla \times \vec{P}_\sigma$

Relati $\quad \vec{X}_\pm = \vec{X} \pm \vec{X}_{\mathrm{ref}}$

Also b $\quad K_\sigma = m_\sigma^2 K_{kin\sigma} + m_\sigma q_\sigma K_{crs\sigma} + q_\sigma^2 K_{mag}$

Note:
$\nabla \cdot \vec{\Omega} = 0$

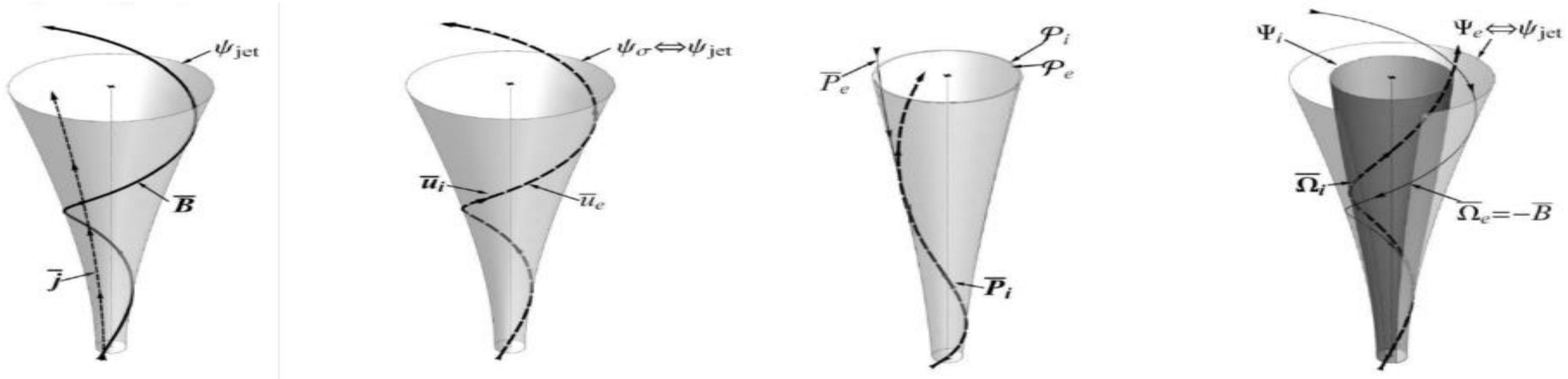
$$\boxed{\frac{dK_{erel}}{dt} = -\frac{dK_{irel}}{dt} = -2\int \vec{R}\cdot\vec{\Omega}\, dV + 2\int_{S_{sep}} (h_i - h_e)\,\vec{\Omega}\cdot d\vec{S} + \cdots} \qquad (1)$$

$\int \vec{u}_\sigma \cdot \vec{\omega}_\sigma \, dV$ kinetic helicity

$2\int \vec{u}_\sigma \cdot \vec{B} \, dV$ cross helicity

$\int \vec{A} \cdot \vec{B} \, dV$ magnetic helicity

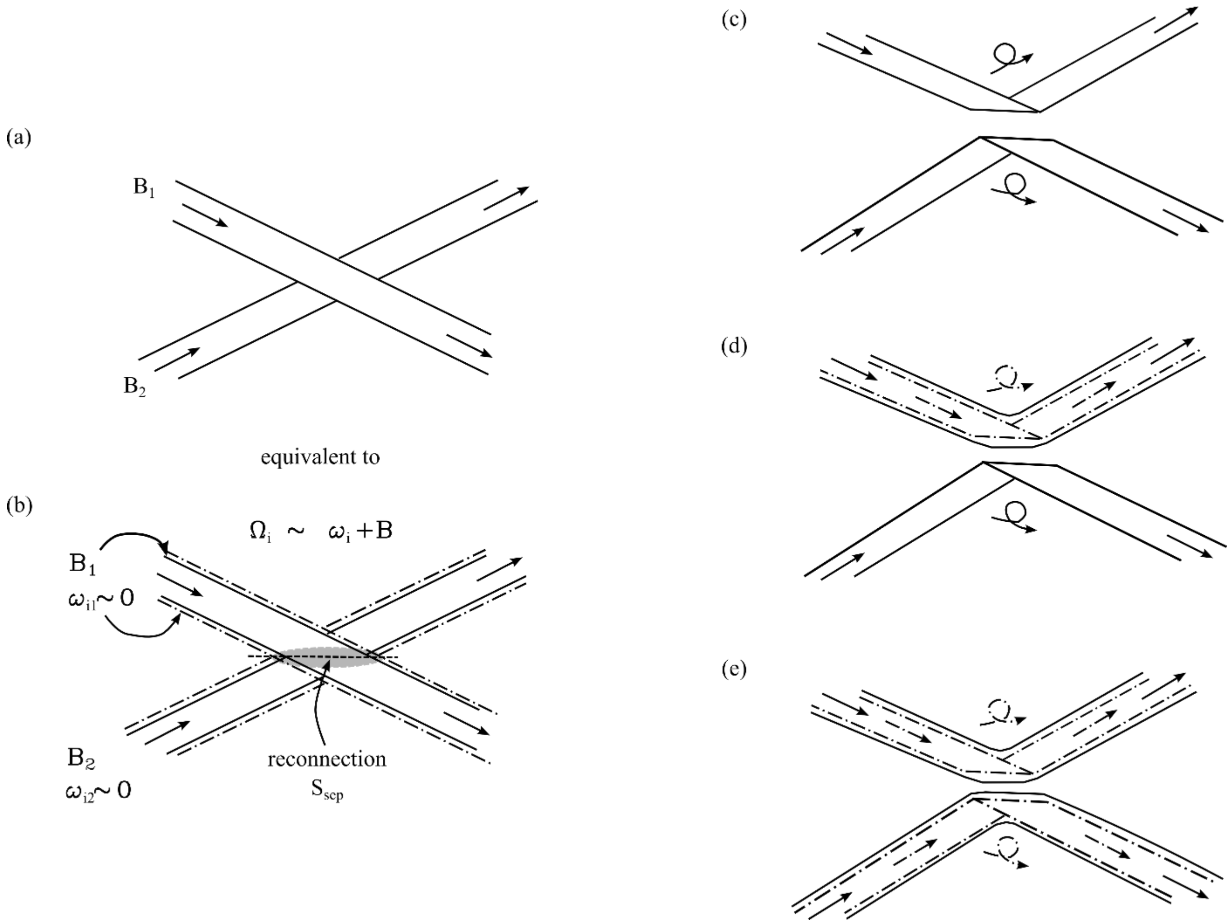
$q_\sigma \phi + \frac{1}{2} m_\sigma u_\sigma^2 + \frac{\mathcal{P}_\sigma}{n_\sigma}$ enthalpy



magnetic flux tube $\qquad$ canonical flux tube

S.You, Phys. Plasmas **19**, 092107 (2012); S. You, Plasma Phys. Control. Fusion, **56**, 064007 (2014)

Reconnection of two flux tubes with three possible results



Two magnetic flux tubes
Half twist each

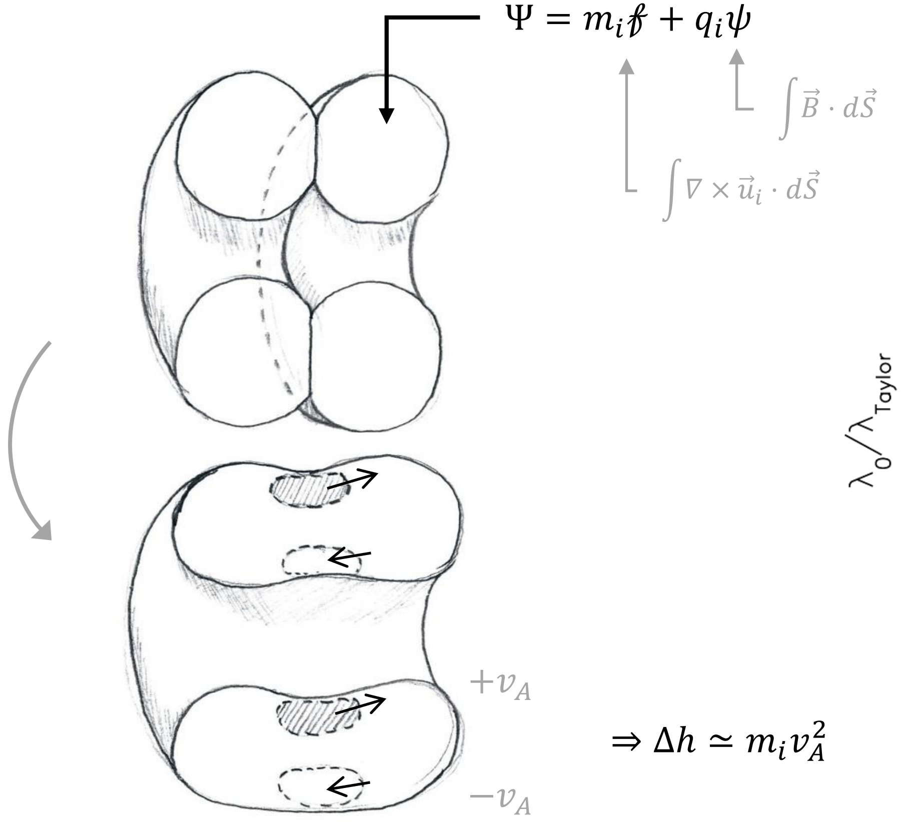
One magnetic flux tube, half twist
One vorticity flow, half twist

Two flow vorticity flux tubes.
Half twist each

S.You, Plasmas Phys. Contol. Fusion, **56,** 064007 (2014)

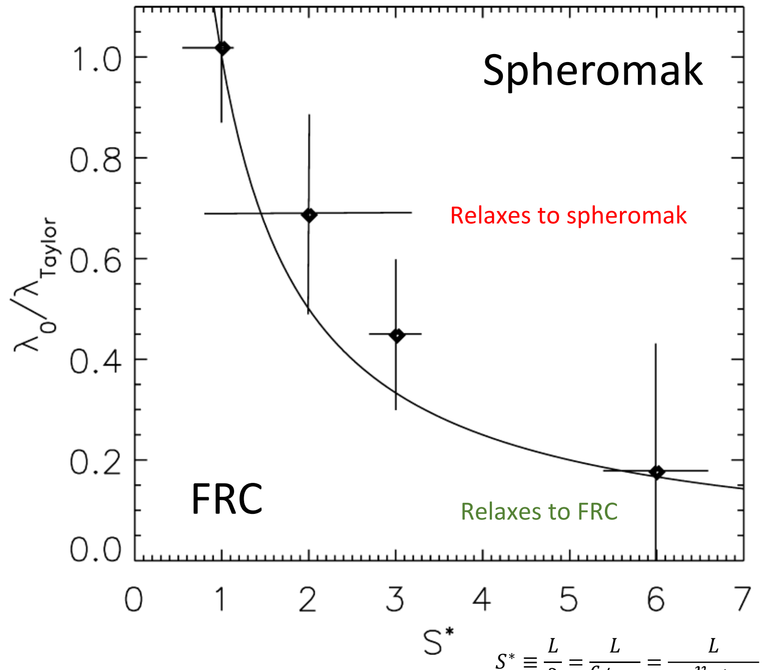
The model explains why bifurcation in CT formation depends on 1/S*.

Counter-helicity merging



$\Rightarrow \Delta h \simeq m_i v_A^2$

$\dot{K}_i = m_i \, \Delta h \oint + q_i \, \Delta h \, \psi$



$S^* \equiv \frac{L}{\delta_i} = \frac{L}{c/\omega_{pi}} = \frac{L}{\rho_i \, {}^{v_A}/_{v_{thi}}}$

Ratio of canon. helicity injection into vortex tube or magnetic flux tube is $\frac{K_i^{\oint}}{K_i^{\psi}} \sim \frac{1}{S^*}$

S.You, Phys. Plasmas **19**, 092107 (2012); E. Kawamori et al, Nucl. Fusion **45**, 843 (2005)

Maxwell's Eqs.

$\Box A_\mu = -\mu_0 j_\mu$

Particle $\{q; m\}$

$\vec{x}(t)$

$\vec{v}(t)$

Electrodynamics

$$\frac{d(\gamma m\vec{v})}{dt} = \vec{F}$$

Kinetic

$f_\sigma(\vec{x}, \vec{v}, t)$

Stat. Mech.

$$\frac{df_\sigma}{dt} = C$$

Multi-fluid

$n_\sigma(\vec{x}, t)$

$\vec{u}_\sigma(\vec{x}, t)$

$\mathcal{P}_\sigma(\vec{x}, t)$

Fluid eqs. of motion *x 2*

MHD

$n(\vec{x}, t)$

$\vec{U}(\vec{x}, t)$

$\mathcal{P}(\vec{x}, t)$

Single fluid eq. of motion

+

Ohm's law

$\vec{E} + \vec{U} \times \vec{B} = \ \dots$

System configuration

$\vec{B}(\vec{x}), \vec{u}(\vec{x})$

(quasi static evolution)

Multi-fluid

$$\begin{Bmatrix} \vec{P}_{\sigma fluid} \equiv \rho_\sigma \vec{u}_\sigma + \rho_{c\sigma}\vec{A} \\ h_{\sigma fluid} \equiv \rho c^2 + \frac{1}{2}\rho_\sigma u_\sigma^2 + \rho_c\phi + \rho\phi_g + \mathcal{P}_\sigma \\ \vec{R}_{\sigma fluid} \equiv \vec{R}' + \vec{P}_{\sigma fluid}\nabla\cdot\vec{u}_\sigma + \mathcal{L}_{\sigma fluid}\frac{\nabla n}{n} \end{Bmatrix}$$

Canonical Helicity p.o.v

$K_\sigma = K_{kin} + K_{cross} + K_{mag}$

$$\frac{dK_\sigma}{dt} = \cdots$$

$\int \dots dV$

Eqs. of motion (canonical form)

$$\Leftrightarrow \frac{\partial\vec{P}}{\partial t} \underbrace{- \vec{v}\times\vec{\Omega}}_{\textit{no-work}} \underbrace{= -\nabla h}_{\textit{conserv.}} \underbrace{- \vec{R}}_{\textit{non-cons.}}$$

$$\Leftrightarrow \boxed{\vec{\Sigma} + \vec{v}\times\vec{\Omega} = \vec{R}}$$

Canonical "electric field" (force-field)

$$\vec{\Sigma} \equiv -\nabla h - \frac{\partial\vec{P}}{\partial t}$$

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Fluid eqs. of motion x 2

MHD

$n(\vec{x}, t)$

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Single fluid eq. of motion + Ohm's law

$\vec{E} + \vec{U} \times \vec{B} = \ \ldots$

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Note:

$$\nabla \times \vec{\Sigma} = -\frac{\partial\vec{\Omega}}{\partial t}$$

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Kinetic

$$\begin{Bmatrix} \vec{P}_{\sigma kin} \equiv f\vec{P}_{part} = fm_\sigma\vec{v} + fq_\sigma\vec{A} \\ h_{\sigma kin} \equiv fh_{part} = \cdots \\ \vec{R}_{\sigma kin} \equiv -\vec{P}_{part}C + \nabla_v\left(f\vec{a}\vec{P}_{part}\right) + \mathcal{L}_{part}\nabla f \end{Bmatrix}$$

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$$\Leftrightarrow \boxed{\vec{\Sigma} + \vec{v}\times\vec{\Omega} = \vec{R}}$$

Canonical "electric field" (force-field)

$$\vec{\Sigma} \equiv -\nabla h - \frac{\partial\vec{P}}{\partial t}$$

(2)

Is there a more fundamental p.o.v. ?

Where does the canonical form of the equation of motion come from ?
The first two canonical Maxwell's eqs give us a clue:

Canonical Gauss' law (no monopole)	$\nabla \cdot \vec{\Omega} = 0$
Canonical Faraday's law	$\nabla \times \vec{\Sigma} = -\frac{\partial \vec{\Omega}}{\partial t}$
Canonical Gauss' law	$\nabla \cdot \vec{\Sigma} = ?$
Canonical Ampere's law	$\nabla \times \vec{\Omega} = ?$

Is there a more fundamental p.o.v. ?

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The first two canonical Maxwell's eqs give us a clue:

Canonical Gauss' law (no monopole)

Canonical

Lagrangian $\mathcal{L}$ → Euler-Lagrange eq. →

$$\nabla \cdot \vec{\Omega} = 0$$

$$\nabla \times \vec{\Sigma} = -\frac{\partial \vec{\Omega}}{\partial t}$$

$$\nabla \cdot \vec{\Sigma} = ?$$

$$\nabla \times \vec{\Omega} = ?$$

Canonical Gauss' law

Canonical Ampere's law

Define a new Lagrangian

$$\mathcal{L}_\sigma \equiv \underbrace{\mathbb{J}^\mu \mathbb{P}_\mu}_{} - \underbrace{\frac{1}{4\mu_\sigma} \mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}}_{} + \mathcal{L}_{EinsteinHilbert}$$

Coupling interaction between "sources" $\mathbb{J}^\mu$ and "field" $\mathbb{P}_\mu$

Coupling interaction between "field" $\mathbb{P}_\mu$ components (its derivatives)

Canon. four-current

$$\mathbb{J}^\mu \equiv \mathbb{q}\frac{\partial x^\mu}{\partial t} = \begin{bmatrix} \mathbb{q}c \\ \vec{\mathbb{J}} = \mathbb{q}\vec{v} \end{bmatrix}$$

Canon. four-field

$$\mathbb{P}_\mu \equiv \left[\frac{h}{c}, \vec{P}\right]$$

Canon. charge density (switch)

$$\mathbb{q} \equiv \epsilon_\sigma \nabla \cdot \left[\vec{R}_\sigma - \vec{v}_\sigma \times \vec{\Omega}_\sigma\right]$$

Canon. field tensor

$$\mathbb{F}_{\mu\nu} \equiv \partial_\mu \mathbb{P}_\nu - \partial_\nu \mathbb{P}_\mu$$

Particle

$$\vec{P}_{\boldsymbol{par}} \equiv \boldsymbol{\gamma m\vec{v} + q\vec{A}}$$
$$h_{par} \equiv \gamma mc^2 + q\phi$$

Kinetic

$$\vec{P}_{\boldsymbol{kin}} \equiv \boldsymbol{f}\,\vec{\boldsymbol{P}}_{\boldsymbol{par}}$$
$$h_{kin} \equiv f\ h_{par}$$

Multi-fluid

$$\vec{P}_{\sigma fluid} \equiv \int \vec{P}_{\boldsymbol{\sigma kin}}\, d\vec{v}_\sigma$$
$$h_{\sigma fluid} \equiv \int h_{\sigma kin}\, d\vec{v}_\sigma$$

S.You, Phys. Plasmas **23,** 072108 (2016)

In flat space …

$$\mathcal{L} \equiv \underbrace{\mathbb{q}\,\vec{v}\cdot\vec{P} - \mathbb{q}\,h}_{\text{Balance between generalized kinetic energy and potential energy (enthalpy).}} + \underbrace{\frac{1}{2}\epsilon\Sigma^2 - \frac{\Omega^2}{2\mu}}_{\text{Balance between generalized “electrical” energy (canon. forces) and “magnetic” energy (canon. vorticity)}}$$

Particle

$$\vec{P}_{\boldsymbol{par}} \equiv \boldsymbol{\gamma m \vec{v}} + \boldsymbol{q\vec{A}}$$
$$h_{par} \equiv \gamma mc^2 + q\phi$$

Kinetic

$$\vec{P}_{\boldsymbol{kin}} \equiv \boldsymbol{f}\,\vec{\boldsymbol{P}}_{\boldsymbol{par}}$$
$$h_{kin} \equiv f\ h_{par}$$

Multi-fluid

$$\vec{P}_{\sigma fluid} \equiv \int \vec{P}_{\boldsymbol{\sigma kin}}\, d\vec{v}_\sigma$$
$$h_{\sigma fluid} \equiv \int h_{\sigma kin}\, d\vec{v}_\sigma$$

$$\mathcal{L} \equiv \mathbb{q}\,\vec{v}\cdot\vec{P} - \mathbb{q}\,h + \frac{1}{2}\epsilon\Sigma^2 - \frac{\Omega^2}{2\mu}$$

e.g. particle in e.m. field

$$\mathcal{L}_{part} = -\frac{1}{\gamma}mc^2 + q\vec{v}\cdot\vec{A} - q\phi$$

$$\vec{\Sigma} = -\nabla h - \frac{\partial \vec{P}}{\partial t}$$
$$= -q_\sigma \nabla \phi - q_\sigma \frac{\partial \vec{A}}{\partial t} + \ \dots$$
$$= q_\sigma \ \vec{E} + \cdots$$

Particle	$\vec{\Sigma} = q_\sigma \vec{E} + \vec{\Sigma}'$	$\vec{\Omega} = q_\sigma \vec{B} + \vec{\Omega}'$
Kinetic	$\vec{\Sigma} = f_\sigma q_\sigma \vec{E} + \Sigma'$	$\vec{\Omega} = f_\sigma q_\sigma \vec{B} + \vec{\Omega}'$
Fluid	$\vec{\Sigma} = \rho_\sigma \vec{E} + \vec{\Sigma}'$	$\vec{\Omega} = \rho_\sigma \vec{B} + \vec{\Omega}'$

$$\mathcal{L} \ \equiv \ \mathbb{q}\, \vec{v} \cdot \vec{P} \ - \ \mathbb{q}\, h \ + \underbrace{\frac{1}{2} \epsilon \Sigma^2 - \frac{\Omega^2}{2\mu}}_{\textbf{Field}}$$

$$\underbrace{\frac{1}{2}\epsilon_0 E^2 - \frac{B^2}{2\mu_0}}_{\mathcal{L}_{Maxwell}} \ + \ \underbrace{2q\left[\frac{1}{2}\epsilon \vec{E} \cdot \vec{\Sigma}' - \frac{\vec{B} \cdot \vec{\Omega}'}{\mu}\right]}_{\mathcal{L}_{e.m.-m}} \ + \ \underbrace{\frac{1}{2}\epsilon \Sigma'^2 - \frac{\Omega'^2}{2\mu}}_{\mathcal{L}_{m-m}}$$

Particle $\vec{\Sigma} = q_\sigma \vec{E} + \vec{\Sigma}'$ $\vec{\Omega} = q_\sigma \vec{B} + \vec{\Omega}'$

Kinetic $\vec{\Sigma} = f_\sigma q_\sigma \vec{E} + \Sigma'$ $\vec{\Omega} = f_\sigma q_\sigma \vec{B} + \vec{\Omega}'$

Fluid $\vec{\Sigma} = \rho_\sigma \vec{E} + \vec{\Sigma}'$ $\vec{\Omega} = \rho_\sigma \vec{B} + \vec{\Omega}'$

$$\mathcal{L} \equiv \mathfrak{q}\,\vec{v}\cdot\vec{P} - \mathfrak{q}\,h + \underbrace{\frac{1}{2}\epsilon\Sigma^2 - \frac{\Omega^2}{2\mu}}_{\text{Field}}$$

$$\underbrace{\frac{1}{2}\epsilon_0 E^2 - \frac{B^2}{2\mu_0}}_{\mathcal{L}_{Maxwell}} + \underbrace{2q\left[\frac{1}{2}\epsilon\vec{E}\cdot\vec{\Sigma}' - \frac{\vec{B}\cdot\vec{\Omega}'}{\mu}\right]}_{\mathcal{L}_{e.m.-m}} + \underbrace{\frac{1}{2}\epsilon\Sigma'^2 - \frac{\Omega'^2}{2\mu}}_{\mathcal{L}_{m-m}}$$

Just a reformulation of existing Lagrangians.

New Lagrangians that represent the collective coupling between kinetic distributions, & e.m. fields, between kinetic distributions themselves, and relativistic distributions

Eq. of motion

Insert $\mathcal{L}$ into Euler-Lagrange equation for space coordinates $\vec{x}$:

$$\Rightarrow \boxed{\mathbb{q}\left[\vec{\Sigma} + \vec{\mathrm{v}} \times \vec{\Omega}\right] = \underbrace{-\nabla\left[\frac{1}{2}\epsilon\Sigma^2 - \frac{\Omega^2}{2\mu}\right]}_{\equiv \vec{R}}}$$

Dissipative forces:

Incomplete conversion between canon. “electrical”/"kinetic“ energy and canon. “magnetic”/"potential“ energy (canon. vorticity).

Single particle:

$$\vec{R}^{par} \simeq 0 \text{ (classical)}$$
$$\vec{R}^{par} \neq 0 \text{ (relativistic)}$$

Kinetic:

$$\vec{R}_\sigma^{kin} \equiv \left(\vec{v}_\sigma \cdot \vec{P}_\sigma^{par} - h_\sigma^{par}\right) \nabla f_\sigma - \vec{P}_\sigma^{par} \frac{df_\sigma}{dt}$$

Fluid:

$$\vec{R}_\sigma^{flu} \equiv \vec{R}_{\sigma\alpha} + \vec{R}_{\sigma\sigma} + \vec{R}_{\sigma c} + \vec{R}_{\sigma n}$$

Canon. Maxwell equations

Insert $\mathcal{L}$ into Euler-Lagrange equation for four-field $\mathbb{P}_\mu$ (with Lorenz gauge):

$$\Leftrightarrow \quad \boxed{D_\mu \mathbb{F}^{\mu\nu} = -\,\mu_\sigma\,\mathbb{J}^\nu} \qquad\qquad \Leftrightarrow \quad \partial_\mu \partial^\mu \mathbb{P}^\nu = -\mu_\sigma \mathbb{J}^\nu$$

$$\Leftrightarrow \begin{cases} \nabla \cdot \vec{\Sigma} = \dfrac{\mathbb{q}}{\epsilon} \\ \nabla \times \vec{\Omega} = \mu\,\mathbb{q}\,\vec{v} \; + \; \mu\epsilon \dfrac{\partial \vec{\Sigma}}{\partial t} \end{cases} \qquad\qquad \Leftrightarrow \begin{cases} \boxdot\, h = -\dfrac{\mathbb{q}}{\epsilon} \\ \boxdot\, \vec{P} = -\mu\,\mathbb{q}\,\vec{v} \end{cases}$$

(also gives a canonical Poynting flux equation for $\vec{\Sigma} \times \vec{\Omega}$)

d’Alembert wave operator

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eq. of motion !

$$\Leftrightarrow \begin{cases} \boxed{\nabla \cdot \vec{\Sigma} = \dfrac{\mathbb{q}}{\epsilon}} \\ \nabla \times \vec{\Omega} = \mu\,\mathbb{q}\,\vec{v} + \mu\epsilon \dfrac{\partial \vec{\Sigma}}{\partial t} \end{cases} \qquad\qquad \Leftrightarrow \begin{cases} \boxdot\, h = -\dfrac{\mathbb{q}}{\epsilon} \\ \boxdot\, \vec{P} = -\mu\,\mathbb{q}\,\vec{v} \end{cases}$$

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d'Alembert wave operator

Hamiltonian, energy evolution

Perform Legendre transformation of $\mathcal{L}$

$$\Rightarrow \boxed{\mathcal{H} = \underbrace{\frac{1}{2}\epsilon\Sigma^2 + \frac{\Omega^2}{2\mu}}_{} + \underbrace{\text{q}\, h}_{}}$$

Sum of energy density of the plasma field and the source enthalpy (if any)

Equivalent derivation of eq. of motion $\text{q}\left[\vec{\Sigma}_\sigma + \vec{v}_\sigma \times \vec{\Omega}_\sigma\right] = -\vec{R}_\sigma$, by inserting $\mathcal{H}$ into Hamilton's third equation:

$$\frac{\partial \mathcal{H}}{\partial q_i} = -\dot{p}_i \Leftrightarrow \frac{\partial \vec{P}}{\partial t} + (\vec{v} \cdot \nabla)\vec{P} = -\nabla\mathcal{H}$$

Using the Poynting theorem for $\vec{\Sigma}, \vec{\Omega}$ in the Hamilton equation $\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$ gives energy evolution equation

$$\Rightarrow \frac{dH_\sigma}{dt} = \int \text{q} \frac{\partial h_\sigma}{dt} dV - \int \text{q}\, \vec{\Sigma}_\sigma \cdot \vec{v}_\sigma \, dV - \int \frac{\vec{\Sigma}_\sigma \times \vec{\Omega}_\sigma}{\mu_\sigma} \cdot d\vec{S} + \int \mathcal{H}_\sigma \vec{u}_\sigma \cdot d\vec{S} \qquad (3)$$

Therefore

Can express eq. of motion as generalized Maxwell equations.
Canonical helicity transport is valid at all regimes.
Helicity vs energy conservation depends on density scale length.

Particle

$$\vec{P}_{par} \equiv \gamma m\vec{v} + q\vec{A}$$
$$h_{par} \equiv \gamma mc^2 + q\phi$$

Kinetic

$$\vec{P}_{kin} \equiv f\,\vec{P}_{par}$$
$$h_{kin} \equiv f\,h_{par}$$

Multi-fluid

$$\vec{P}_{\sigma fluid} \equiv \int \vec{P}_{\sigma kin}\, d\vec{v}_\sigma$$
$$h_{\sigma fluid} \equiv \int h_{\sigma kin}\, d\vec{v}_\sigma$$

$$\mathcal{L}_\sigma \equiv \mathbb{J}^\nu \mathbb{P}_\nu - \frac{1}{4\mu_\sigma}\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}$$

Eqs. of motion (canonical form)

$$\boxed{\mathbb{q}\left(\vec{\Sigma} + \vec{v} \times \vec{\Omega}\right) = \vec{R}} \Leftrightarrow \boxed{D_\mu \mathbb{F}^{\mu\nu} = -\mu_\sigma \mathbb{J}^\nu}$$

$\int \dots dV$

Canonical Helicity p.o.v

$$K_\sigma = K_{kin} + K_{cross} + K_{mag}$$
$$\frac{dK_\sigma}{dt} = \cdots$$
$$\mathbb{K} \equiv K_i + K_e = \text{constant}$$

System configuration
$\vec{B}(\vec{x}), \vec{u}(\vec{x})$
(quasi static evolution)

Particle

$$\vec{R}_{part} \simeq 0$$

Kinetic

$$\vec{R}_{\sigma kin} = \left(\vec{v}_\sigma \cdot \vec{P}_{\sigma par} - h_{\sigma par}\right)\nabla f - \vec{P}_{\sigma par}\frac{df_\sigma}{dt}$$

Multi-fluid

$$\vec{R}_{\sigma fluid} = \vec{R}_{\sigma\alpha} + \vec{R}_{\sigma\sigma} + \vec{R}_{\sigma c} + \vec{R}_{\sigma n}$$

Change of (species) helicity vs change in energy ?

Take the ratio of evolution equations (Eq. 1)/(Eq. 3).

e.g. isolated, fixed, non-conservative forces

$$\frac{\Delta K_\sigma / K_{\sigma 0}}{\Delta H_\sigma / H_{\sigma 0}} = 2\frac{\int \vec{R}_\sigma \cdot \vec{\Omega}_\sigma \, dV}{\int \vec{R}_\sigma \cdot \vec{v}_\sigma \, dV}\frac{H_{\sigma 0}}{K_{\sigma 0}} \simeq 2\frac{\Omega_\sigma}{v_\sigma}\frac{H_{\sigma 0}}{K_{\sigma 0}} \sim \frac{\rho_\sigma}{L_s}\frac{H_{\sigma 0}}{K_{\sigma 0}}$$

$1/L_s \equiv 2(1/L_{circ} + 1/r_L)$

Density scale length:

shallow ⇒ species helicity changes little

steep ⇒ species helicity changes a lot

MST experiment:

$H_{e0} \sim W_{mag} \sim 50$ kJ

$\Omega_e \sim \rho_{ce} B \sim 0.2$ kg m^{-3} s^{-1}

$n_e \sim 10^{19}$ m^{-3}

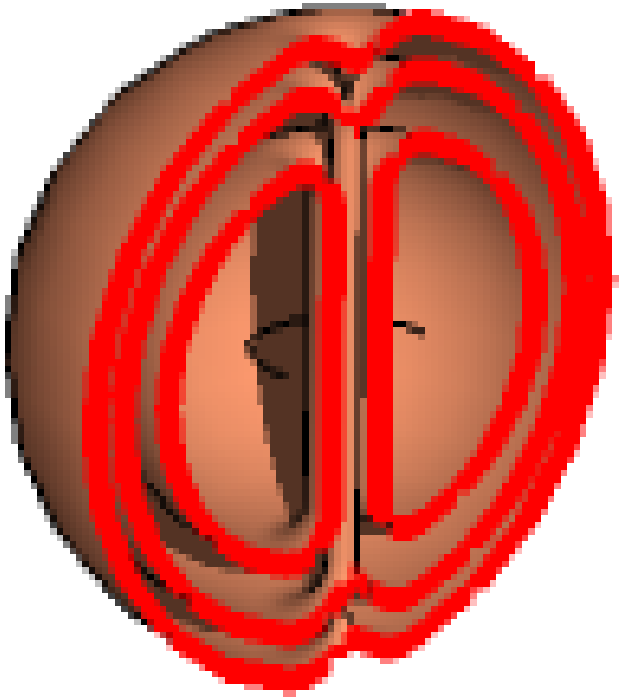
$B_{tor} \sim 0.12$ T

$T_e \sim 300$ eV

$K_{e0} \sim 23$ mWb^2

=> magnetic energy is dissipated 33 times more than magnetic helicity in an isolated, purely dissipative, magnetically dominated system.

Helicity conversion at the edge of plasmas where density gradients are steep



Density scale length:

shallow ⇒ species helicity changes little

steep ⇒ species helicity changes a lot

L-H transition as helicity-constrained relaxation ? use kinetic form of canon. Maxwell's eq. ? in spheromak ?

First, examine helicity transport in cyl. geom.

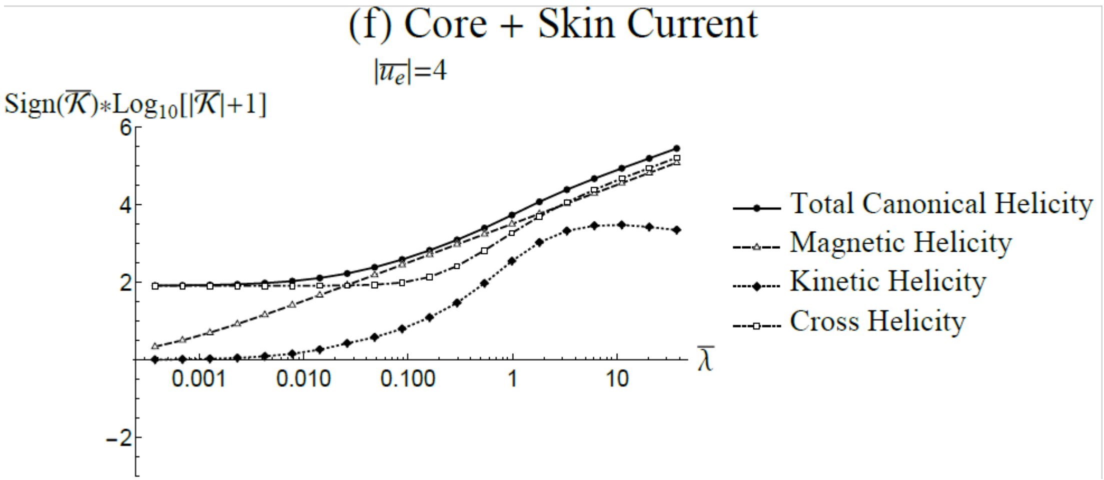
Generalization of $\nabla n \cdot \vec{B} = 0 \Rightarrow \quad \nabla h_\sigma \cdot \vec{\Omega}_\sigma = 0$

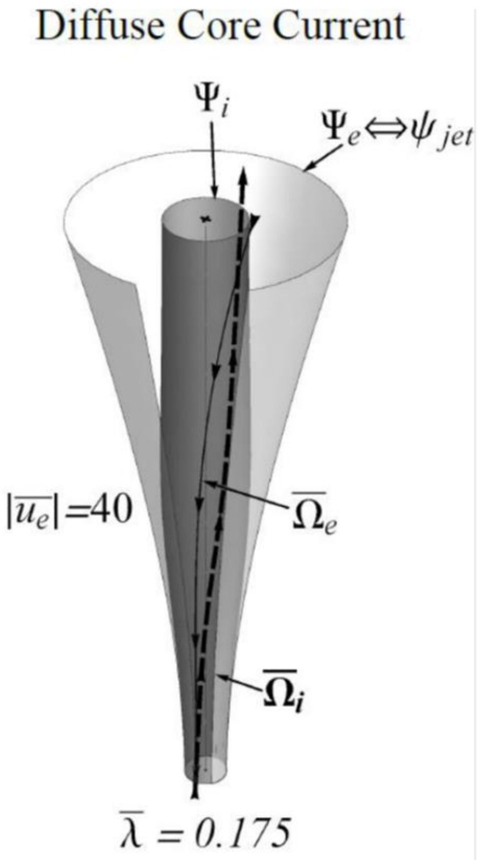
Dot steady-state $\vec{\Sigma}_\sigma + \vec{v}_\sigma \times \vec{\Omega}_\sigma \simeq 0$ with $\vec{\Omega}_\sigma$

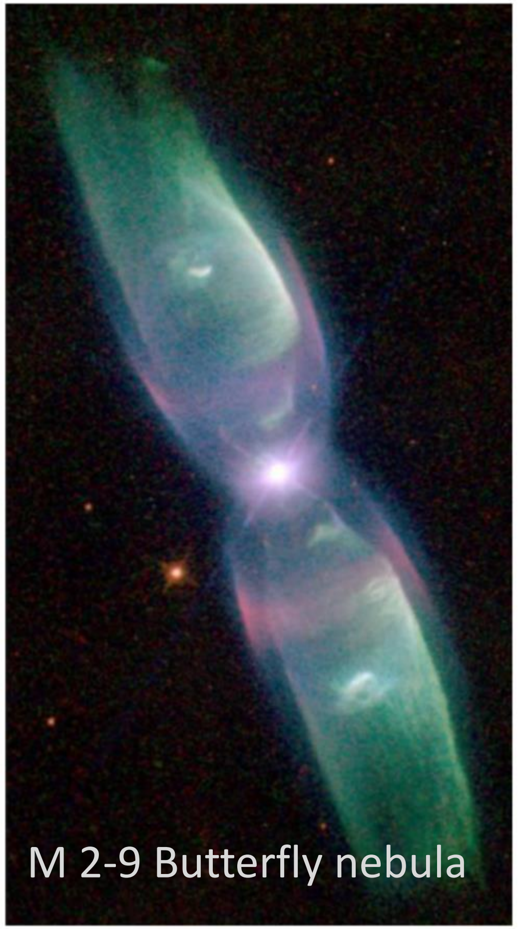
Generalized induction equation for $\vec{\Omega}_\sigma$

Take curl of $\vec{\Sigma}_\sigma + \vec{v}_\sigma \times \vec{\Omega}_\sigma \simeq 0$

So plasma motion ⇔ canonical flux tube motion in many regimes beyond magnetostatic.

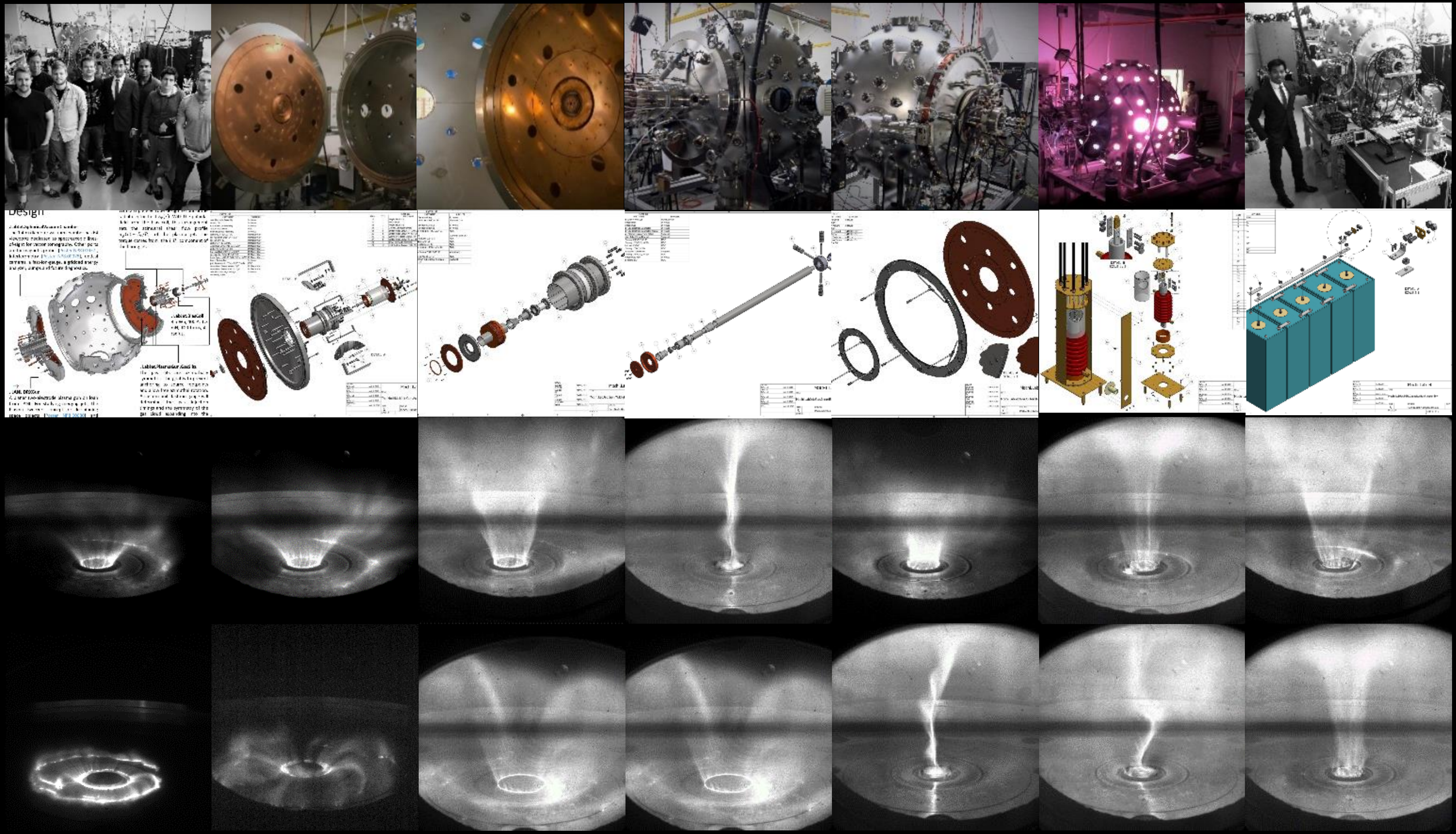






Mochi experiment just became operational

Designed to study interaction between flows and magnetic fields



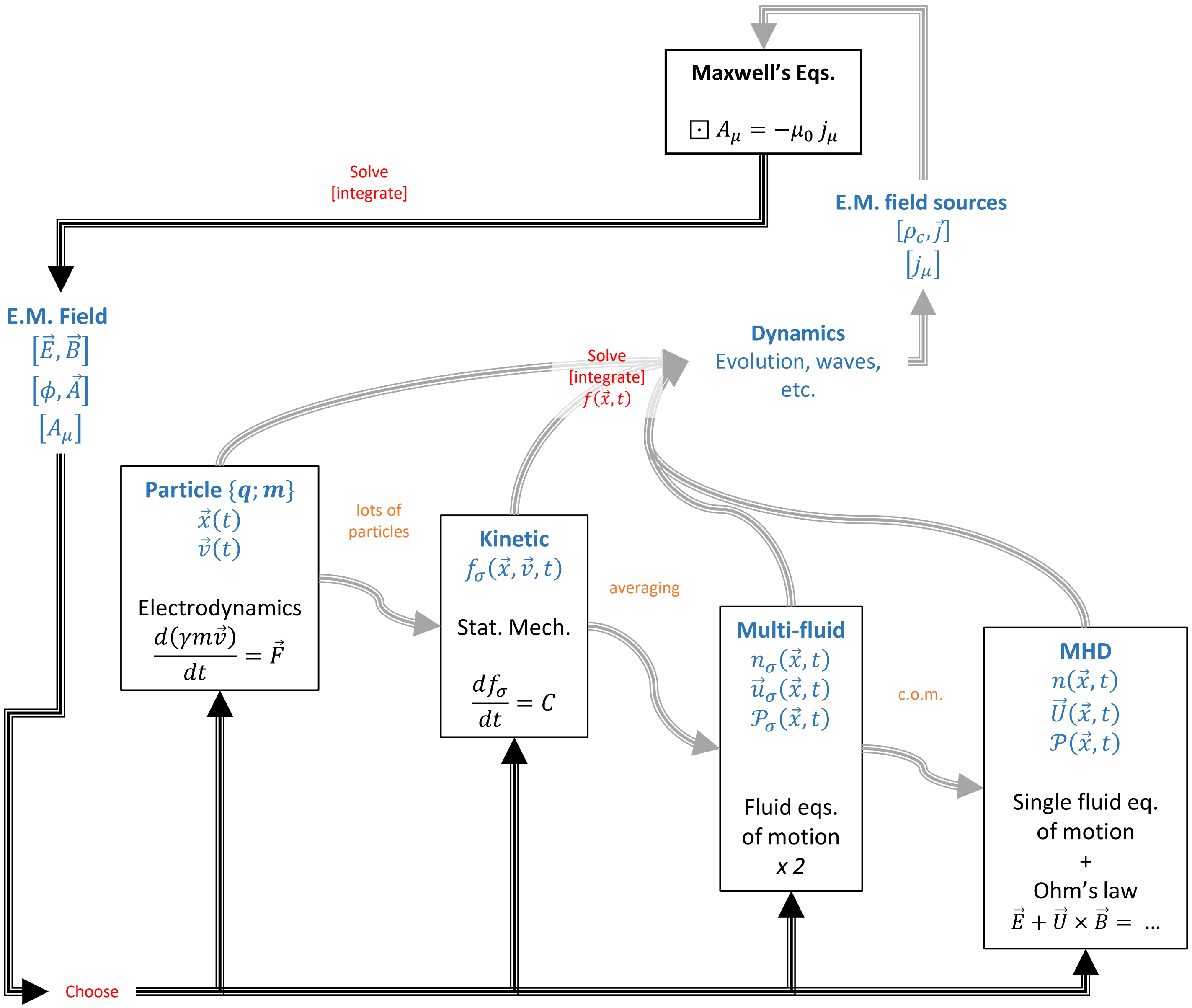
End

Energy equation

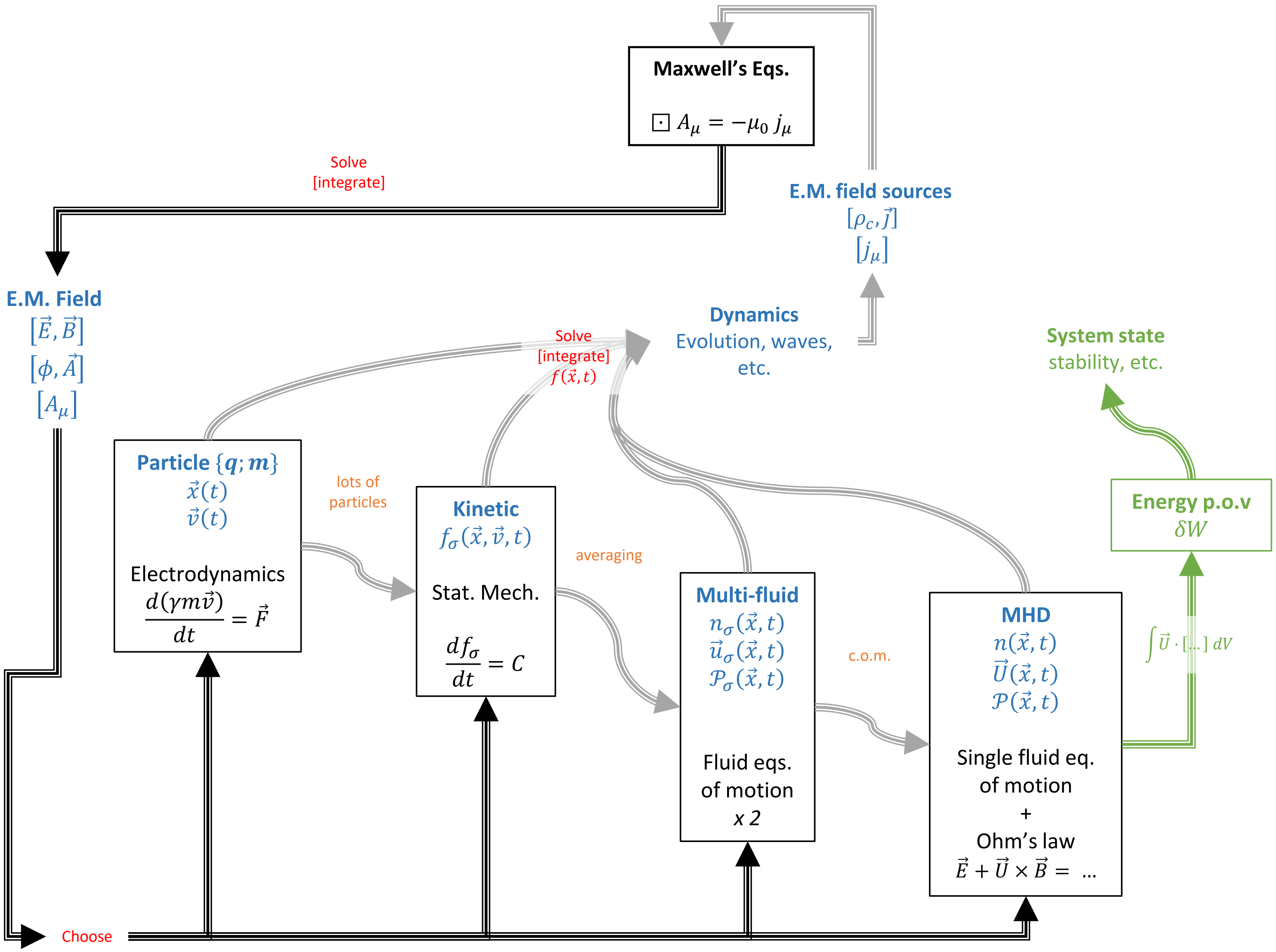
Insert $\mathcal{L}$ into Euler-Lagrange equation for time t:

$$\Rightarrow \quad \boxed{\mathbb{q}\,\vec{v}\cdot\frac{\partial\vec{P}}{\partial t} = -\frac{\partial}{\partial t}\left[\frac{1}{2}\epsilon\mathbb{E}^2 - \frac{\Omega^2}{2\mu}\right]}$$

If the "plasma field" has a time dependence, energy changes.



Maxwell's Eqs.
$\boxdot\, A_\mu = -\mu_0\, j_\mu$
Solve
[integrate]
E.M. field sources
$[\rho_c, \vec{j}]$
$[j_\mu]$
E.M. Field
$[\vec{E}, \vec{B}]$
$[\phi, \vec{A}]$
$[A_\mu]$
Dynamics
Evolution, waves, etc.
Solve
[integrate]
$f(\vec{x}, t)$
Particle $\{q; m\}$
$\vec{x}(t)$
$\vec{v}(t)$
Electrodynamics
$\frac{d(\gamma m \vec{v})}{dt} = \vec{F}$
lots of particles
Kinetic
$f_\sigma(\vec{x}, \vec{v}, t)$
Stat. Mech.
$\frac{df_\sigma}{dt} = C$
averaging
Multi-fluid
$n_\sigma(\vec{x}, t)$
$\vec{u}_\sigma(\vec{x}, t)$
$\mathcal{P}_\sigma(\vec{x}, t)$
Fluid eqs. of motion
x 2
c.o.m.
MHD
$n(\vec{x}, t)$
$\vec{U}(\vec{x}, t)$
$\mathcal{P}(\vec{x}, t)$
Single fluid eq. of motion
+
Ohm's law
$\vec{E} + \vec{U} \times \vec{B} = \ \ldots$
Choose



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Dynamics
Evolution, waves, etc.
Solve
[integrate]
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System state
stability, etc.
Particle $\{\boldsymbol{q}; \boldsymbol{m}\}$
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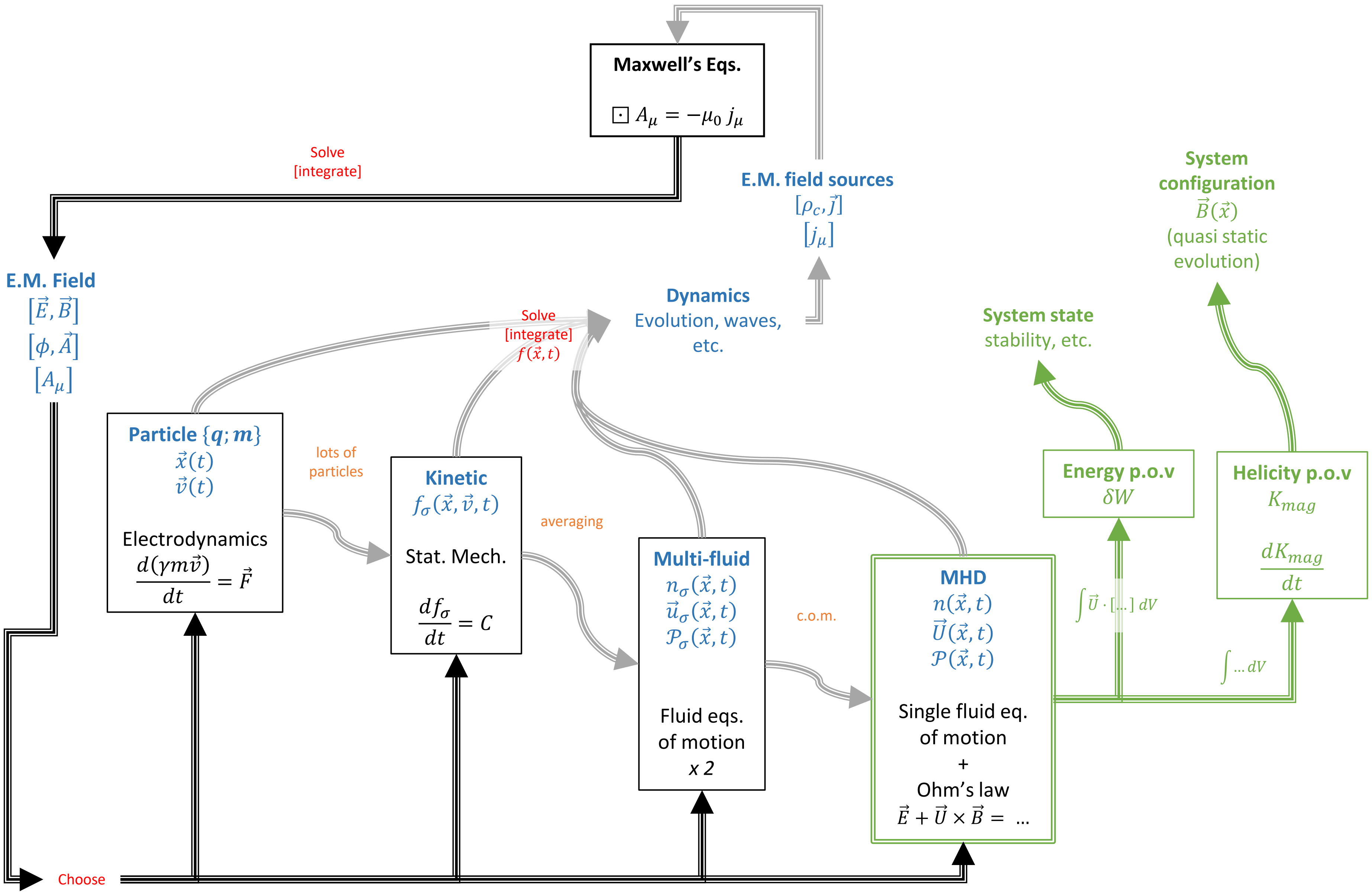
Energy p.o.v
δW
$\int \vec{U} \cdot [\ldots]\, dV$
Choose



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$\vec{B}(\vec{x})$
(quasi static
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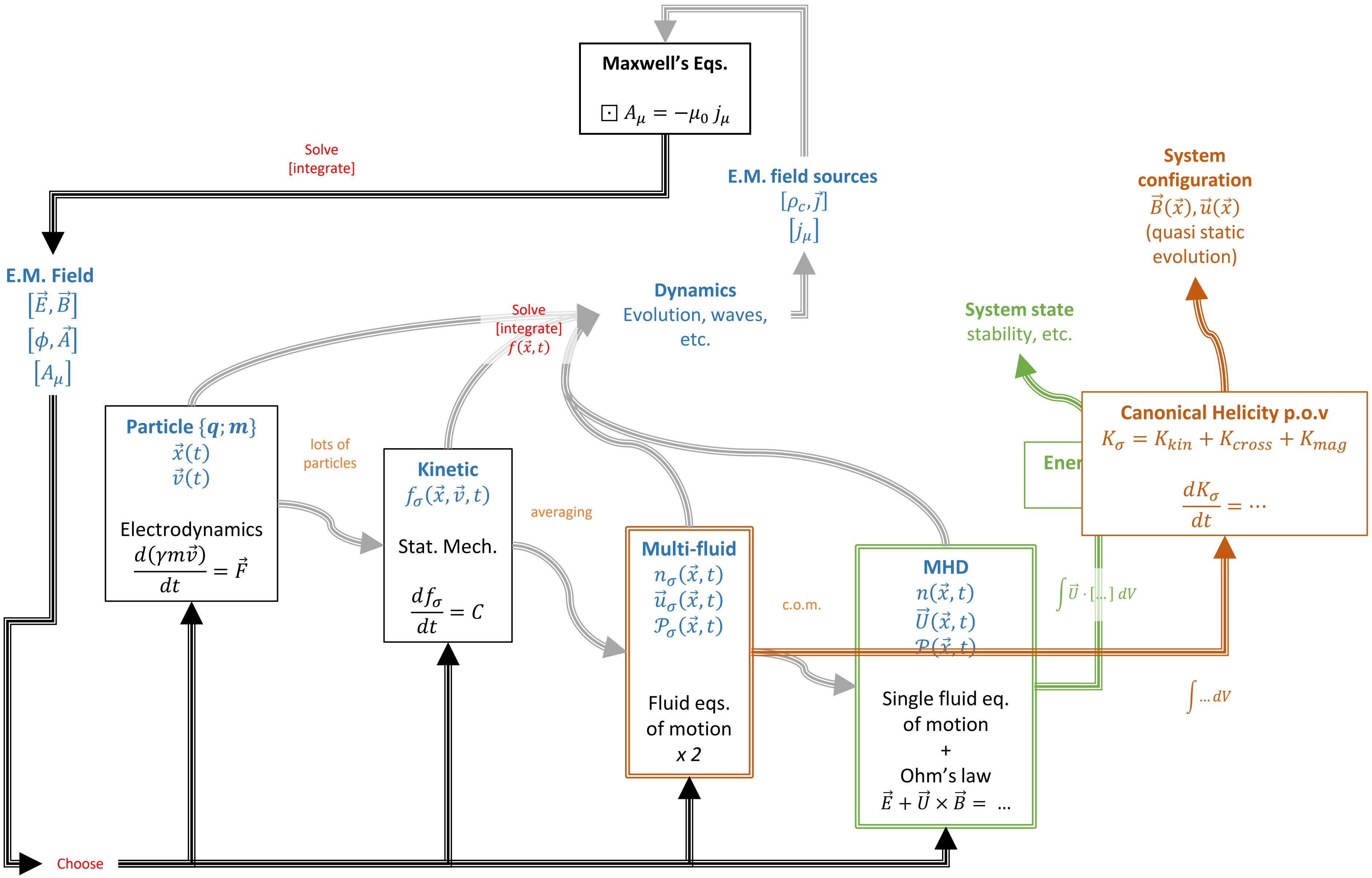
Energy p.o.v
δW
$\int \vec{U} \cdot [...] \, dV$
Helicity p.o.v
K_{mag}
$\frac{dK_{mag}}{dt}$
$\int ... dV$
Choose



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$\frac{dK_\sigma}{dt} = \cdots$
$\int \vec{U} \cdot [\ldots]\, dV$
$\int \ldots dV$
Choose