Classical Mechanics Qualifying Exam

1. (a) Show that for any two particles a and b,

$$p_a \cdot p_b = m_a E_b = m_b E_a \tag{1}$$

where p_a and p_b are 4-vectors, E_b is the energy of b in a's rest frame, and vice versa for E_a.

(b) A meter stick is at rest in frame $\mathcal{S}_0$, which is traveling with speed $0.8c$ in the x direction, all axes aligned, relative to the lab frame $\mathcal{S}$. The stick makes an angle $\theta_0 = 60°$ with the x-axis in frame $\mathcal{S}_0$. What is its length and angle to x in the $\mathcal{S}$ frame?

2. A solid wooden uniform disk has mass M and radius R. The plane of the disk is confined to the $x - z$ plane, and the disk rolls without slipping in the x direction. We let x denote the horizontal location of the center of the disk. Embedded a distance r from the center of the disk is a small gold mass m which you can take as a point mass. At $x = 0$, m is straight below the center of the disk. (a) Find the potential energy of the object as a function of x. (b) Find the kinetic energy (including the rotational piece) as a function $\dot{x}$ and x (does it depend on x?) (c) Using conservation of energy, for a given total energy E find the velocity $\dot{x}$ as a function of position. (d) Write down the Euler-Lagrange equations for the system. (e) Near the equilibrium point $x = 0$, the system undergoes small oscillations. Find the frequency of the oscillations.

Statistical Mechanics (2020 Fall)

1. (30 points) Suppose the density of states (i.e., the number of states per energy unit) of the electrons is constant ρ for energy ε >0, while for $\varepsilon < 0$, ρ =0. The total number of electrons considered in this sample is *N*.

   (i) Calculate the Fermi potential μ_0 at temperature $T = 0°$ K;
   (ii) Show that the specific heat is proportional to *T* when the system is highly degenerate (Note: $\int_0^\infty \frac{z\,dz}{e^z+1} = \frac{\pi^2}{12}$, though it is not necessary to calculate it here);
   (iii) Derive the condition that the system becomes non-degenerate.

2. (30 points) *A* and *B* are two systems of identical composition, and are brought together and allowed to exchange both energy and particles, keeping respective volumes $\boldsymbol{V_A}$ and $\boldsymbol{V_B}$ constant, with the total energy $\boldsymbol{E} \equiv E_A + E_B$ and the total particle number $\boldsymbol{N} \equiv N_A + N_B$ remaining constant. Show that the minimum value of the quantity $\frac{dE_A}{dN_A}$ is given by $(\mu_A T_B - \mu_B T_A)/(T_B - T_A)$, where $\mu_{A,B}$, $T_{A,B}$ are the respective chemical potentials and temperatures of the *A* or *B* system during the exchange process.

3. (40 points) Use following ensemble distributions to calculate the average entropy, energy, and free energy of a classical 2D particle of mass *m*, with no internal degrees of freedom, free to move in an area *A*.

   (i) For a microcanonical ensemble, consider first for many (*N* >>1) of such distinct particles, then calculate the entropy, energy and free energy per particle.
   Note that for d-dimensional hypersphere, $V_d = \pi^{\frac{d}{2}}/\Gamma(\frac{d}{2}+1)$*;*
   The gramma function $\Gamma(n) = (n-1)!$ *for positive integer n;*
   Stirling's approximation $lnN! \approx NlnN - N$ *for very large N.*

   (ii) For a canonical ensemble at temperature *T* for the same particle system above, write down the partition function, and derive the free energy, energy, and entropy per particle.

Attempt all three problems.

1. Consider the following representations of the angular momentum operator:

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

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   (a) Find the eigenvalues of L_x and L_z. [10]
   (b) Find $|L_x;\ 1\rangle$, the normalized eigenstate of L_x corresponding to the eigenvalue of 1. Find the expectation values $\langle L_z\rangle$ and $\langle L_z^2\rangle$ in this eigenstate. [10]
   (c) Find $|L_z;\ 1\rangle$, the normalized eigenstate of L_z corresponding to the eigenvalue of 1. Find the expectation values $\langle L_x\rangle$ and $\langle L_x^2\rangle$ in this eigenstate. [10]
   (d) Find the probabilities of obtaining 0 and 1 when L_z is measured, if the particle is prepared in the state $|L_x;\ 1\rangle$. [10]
   (e) Find $\langle L_x^2 + L_y^2 + L_z^2\rangle$ in the $|L_z;\ 1\rangle$ state. [10]

2. Approximate the ammonia molecule NH_3 by a two-state system, corresponding to N being above or below the plane of the three H atoms. Each of the two states is approximately an energy eigenstate with the same energy E0. However, there is a small amplitude A for transition from the up state to the down state. Thus the Hamiltonian of the two-state system can be written as

$$H_0 = \begin{pmatrix} E_0 & V \\ V & E_0 \end{pmatrix}$$

for a real number V and $|V| \ll |E_0|$.

If the system is placed in a small electric field, then the two states can be distinguished. The new Hamiltonian is $H = H_0 + H'$ with

$$H' = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}$$

and $\delta_1 \neq \delta_2$.

   (a) Find the exact energy eigenvalues and eigenstates of H_0. [20]
   (b) Find the exact energy eigenvalues for the Hamiltonian $H = H_0 + H'$. [10]
   (c) Assuming $|\delta_1|, |\delta_2| \ll |V|$, find the energy eigenvalues to lowest order in $|\delta_1|$ and $|\delta_2|$ using perturbation theory. Compare to the appropriate Taylor expansion of the exact energy levels of H. [20]
   (d) Assuming $|\delta_1|, |\delta_2|, |\delta_1 - \delta_2| \gg |V|$, find the energy eigenvalues to lowest order in $|V|$ using perturbation theory. Compare to the appropriate Taylor expansion of the exact energy levels of H. [20]

3. Consider a particle in a one-dimensional Haromic Oscillator with Hamiltonian $H_0 = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$ at $t \leq 0$. Assume that the particle is in the ground state of H_0 at $t = 0$. At time $t = 0$ this system is perturbed by the addition of a linear potential, so that the Hamiltonian at $t > 0$ is $H = H_0 + kx$. What is the probability of finding the particle in the first excited state of H_0 at $t = \tau$ after the perturbation has been turned on? What is the probability of finding it in the second excited state at $t = \tau$? [30]

EM Qual Fall 2020

1. A hollow spherical shell has inner radius a and outer radius b. It carries a charge density ρ. Find the field everywhere if

(a) the charge density is uniform i.e. $\rho = \rho_0$.

(b) the charge density is $\rho = \rho_1 \cos\theta$.

In each case, find the total charge and dipole moment.

2. A linear antenna has length d. It carries a current such that exactly one wavelength of the current fits on the antenna, and furthermore, the current has a time dependence $e^{i\omega t}$. Find (a) the electric dipole

(b) the magnetic dipole

(c) the angular distribution of radiation at large wavelengths.

3. A circular loop of radius R is centered at the origin and is initially in the $x-y$ plane. Initially, therefore, the diameters are along the $x-$ and y-axis. The loop then rotates about the diameter along the y-axis at a frequency ω. At the same time, a uniform B-field is oriented along the x-direction and changes in time as $\vec{B} = B_0 cos(\omega t)\hat{\imath}$. Find the emf across the loop as a function of time.