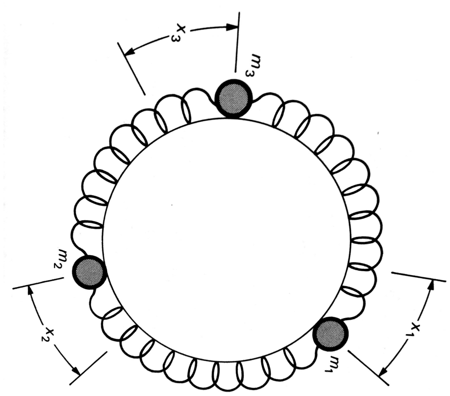
Classical Mechanics Qualifying Exam

1. Frame $\mathcal{S}$ and $\mathcal{S}'$ have the standard orientations, origins, etc. Frame $\mathcal{S}'$ moves along the positive x-axis at speed $V = \frac{5}{13}c$ as seen by $\mathcal{S}$. At $t = t' = 0$ a laser beam leaves the origin, traveling in the $x - y$ plane at some angle between the x and y axes, and strikes a bomb at some point in the plane. The bomb is at rest in $\mathcal{S}$. At the instant the beam strikes it, the bomb explodes. When it explodes, it has x coordinates $x = 24b$ in $\mathcal{S}$ and $x' = 11b$ in $\mathcal{S}'$, where b is a distance. (Answers for parts (a) and (b) can be given in terms of b and c. It is OK to just give the sine, cosine, or tangent of the angle for (c) and (d).)

(a) At what time does the bomb explode according to $\mathcal{S}$?
(b) At what time does the bomb explode according to $\mathcal{S}'$?
(c) At what angle does the laser beam leave the origin, according to $\mathcal{S}$?
(d) At what angle does the laser beam leave the origin, according to $\mathcal{S}'$?

2.(a) Three springs and three masses are arranged around a frictionless circle, as shown. The masses can only move along the circle. For the case of three equal masses m and three equally strong springs k, find the normal mode frequencies. Also, find the eigenvector for the lowest frequency normal mode.



Statistical Mechanics (2021 Fall)

1. (40 points) Consider a harmonic oscillator with angular frequency ω and mass m.
   (i) Calculate the partition function for this oscillator regarding it classically;
   (ii) Calculate the partition function regarding this oscillator quantum mechanically, and indicate the condition that the quantum partition function approaches to the classical one;
   (iii) Obtain the partition function and the Helmholtz free energy for an ensemble of N independent and distinguishable oscillators of the above type (quantum in general);
   (iv) Obtain the internal energy, heat capacity, and entropy as functions of temperature T for the above ensemble.

2. (30 points) At zero temperature the electrons are expected to occupy energy levels up to the Fermi potential ε_F. Consider that the electrons are under a magnetic field with strength H: the electron accordingly acquires an energy $\pm\mu_B H$, depending on whether its spin magnetic moment is parallel or anti-parallel to the field.
   (i) Calculate the numbers free electrons under the above magnetic field, with parallel and anti-parallel spin magnetic moments, N_+, and N_-, respectively (at T=0° K);
   (ii) Calculate the total magnetic moment M of the electrons given N_+ and N_- above;
   (iii) Obtain the spin paramagnetic susceptibility χ the above system of free electrons.

3. (30 points) For an ideal Bose gas confined in a constant volume V with particle mass m:
   (i) Calculate one-particle state density $D(\varepsilon)$ at energy ε for the ideal gas system;
   (ii) Calculate the internal energy E of the gas system at temperature T considering the Bose-Einstein statistics and $D(\varepsilon)$ obtained above, and show the expansion of E below when the system degeneracy is weak (i.e. chemical potential $\mu<0$ or $e^{-(\varepsilon-\mu)/k_BT} \ll 1$):

$$E = \frac{3}{2}k_BTV\left(\frac{2\pi m k_BT}{h^2}\right)^{\frac{3}{2}}\sum_{l=1}^{\infty}\frac{e^{l\mu/k_BT}}{l^{\frac{5}{2}}}$$

   (iii) Verify $PV = \frac{2}{3}E$ for the above system, with P the pressure.

Consider a slightly perturbed 1-d harmonic oscillator with potential energy $\frac{1}{2}m\omega^2x^2+kx$ and k, m, ω all real. Answer the following questions for this potential, with details of the calculations.

1. Find the exact energy levels and energy eigenstates for this potential. [20]
2. Use perturbation theory to find the energy levels to lowest order in k. Show that your answer is consistent with the exact energy levels obtained previously. [30]
3. The third order energy contribution can be written as

$$E_n^{(3)} = \sum_{k\neq n}\sum_{m\neq n} \frac{V_{nm}V_{mk}V_{kn}}{(E_n^{(0)}-E_m^{(0)})(E_n^{(0)}-E_k^{(0)})} - V_{nn}\sum_{m\neq n}\frac{|V_{nm}|^2}{(E_n^{(0)}-E_m^{(0)})^2} \,,$$

   where $V_{nm} = \langle n^{(0)}|kx|m^{(0)}\rangle$. Show that this term is zero. [20]
4. Use perturbation theory to find the energy eigenstates to lowest order in k. Show that your answer is consistent with the exact energy eigenstates obtained peviously. [30]

EM Qual Fall 21

1. A sphere of radius R is centered at the origin has a surface potential $V = V_0(1 + \cos\theta)^3$.

(a) Find the potential everywhere outside the sphere.

(b) Find the charge and dipole moment.

(c) A dipole $\vec{p} = p_1\hat{x} + p_2\hat{y}$ is placed far from the sphere at the point $(a, 0, b)$ where a, b are much larger than R. Find the force on the dipole to leading order in the inverse distance.

2. An infinite cylindrical shell has inner radius a, and outer radius b. A current flows along the axis of this cylinder with a current density $j = j_0 r^2$. Find the magnetic field everywhere.

3. A light wave has a maximum electric field of E_0, and a frequency ω. It is travelling along the vector (1,1,1) when it strikes and reflects off a plane mirror which occupies the y-z plane. Estimate the pressure on the mirror.