Classical Mechanics Qualifying Exam

1. A thin flat rectangular plate with uniform mass density has length $2a$ in the x direction and width a in the y direction. Its total mass is M. (a) Find its moment of inertia tensor about its center of mass. (b) An axle is connected to it so that it coincides with a diagonal of the plate. The axle rotates about its axis, taking the plate with it, with constant angular velocity. Find the magnitude and direction of the torque (about the center of mass) that the axle exerts on the plate. (Hint: Euler's equations are $\lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3 = \Gamma_1$, and cyclic permutations.)

2. An X-particle at rest decays into particles A, B, and C:

$$X \to A + B + C$$

The mass of the X-particle is $12m$, the mass of the A-particle is $6m$, and the mass of the B paricle is $4m$. The C particle is massless and travels at the speed of light.

(a) If after the decay, the B-particle is at rest, find the energy E_C of the C-particle and the Lorentz factor γ_A of the A-particle.

(a) If instead, after the decay, the A-particle is at rest, find the energy E_C of the C-particle and the Lorentz factor γ_B of the B-particle.

EM Qual Fall 22

1. A sphere has outer radius a and inner radius b. It carries a charge distribution $\rho = \rho_0(\cos\theta)^2$.

(a) Find the charge and dipole moment.

(b) Find the exact potential outside the sphere.

(d) The charge on the sphere is rearranged so that the charge distribution becomes spherically symmetric. What is the change in energy?

2. There is a magnetic field in the region $0 < x < a$, with constant value $\vec{B} = B_0\hat{z}$ in this region, and zero outside the region. A screen is placed at $x = 2a$.

A charged particle of charge q starts at $(-2a, 0, 0)$ and travels along the x-axis at speed v_0 toward the magnetic field.

(a) Where does it hit the screen?

(b) At what velocity will it never hit the screen? Explain.

3. A loop of radius a is in the x-y plane, and is centered at the origin. A magnetic field has a form $\vec{B} = B_0\hat{\imath}\sin(\omega t) + B_1\hat{k}\sin(2\omega t)$. Find the potential across the wire.

1. A photon state vector with right- or left-circular polarization can be written as $|R\rangle = (|x\rangle + i|y\rangle)/\sqrt{2}$ and $|L\rangle = (|x\rangle - i|y\rangle)/\sqrt{2}$ in terms of the states polarized in the x-direction $|x\rangle$ and y-direction (perpendicular to x-direction) $|y\rangle$.

   (a) Show explicitly that $|R\rangle$ and $|L\rangle$ are the eigenvectors of the rotation operator $\mathcal{R}(\theta)$ in the x-y plane (i.e., rotation about the z-axis). $\mathcal{R}(\theta)$ can be represented as the square matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ in the $|x\rangle$-$|y\rangle$ basis. [10]

   (b) Consider three sequential polaroids with the photon prepared in the $|x\rangle$ state striking the polaroid oriented at 30 degrees with respect to x first, then a polaroid oriented at 60 degrees with respect to x and and finally a polaroid oriented at 90 degrees (i.e., along y). Find the probability of the photon getting through all three polaroids. Compare this to the probability you would obtain if the first and second polaroids (oriented at 30 and 60 degrees) were not present. Explain why the two probabilities differ. [15]

2. A particle of mass m is in an infinite one-dimensional square well potential: $V = 0$ for $|x| < L/2$ and infinite for $|x| \geq L/2$. The particle is prepared in a state that is a superposition of the ground ($|1\rangle$) and first excited ($|2\rangle$) states such that $|\alpha\rangle = (3|1\rangle + i|2\rangle)/\sqrt{10}$.

   (a) If the energy of the particle is measured, what are the possibile values? What is the probability of obtaining each energy value? What is the expectation value for the energy of the particle? [15]

   (b) What is the expectation value for the momentum of the particle? [10]

3. The Hamiltonian of a rigid rotator in a magnetic field can be written in the following way: $H = \alpha L^2 + \beta L_z + \gamma L_y$. Assume that the γL_y term is a perturbation to the Hamiltonian $H^{(0)} = \alpha L^2 + \beta L_z$. Find the energy eigenvalues correct to the lowest order in γ. (Note that this problem can be solved exactly, but use perturbation theory here.) [25]

4. Consider a particle of mass $m = 10$ MeV/c^2 and velocity $v = 0.01c$ being scattered by a spherically-symmetric finite square well potential given by $V(r) = 1$ MeV for $r \leq 10$ fm and $V(r) = 0$ for $r > 1$ fm.

   (a) Calculate the differential cross section ($d\sigma/d\Omega$) for scattering at angles of 60 and 90 degrees using the Born approximation. [20]

   (b) Explain why $d\sigma/d\Omega$ is almost independent of the angle of scattering. [5]

   (Some useful constants: $\hbar = 6.58 \times 10^{-22}$ MeV $\cdot$ s, $\hbar c = 1.97 \times 10^{-13}$ MeV $\cdot$ m, 1 MeV $=$ 1.6022×10^{-13} J, 1 MeV/c^2 $= 1.78 \times 10^{-30}$ kg, 1 fm $= 10^{-15}$m, $c = 3 \times 10^8$ m/s.)

Statistical Mechanics (2022 Fall)

1. (30 points) An ideal gas in a box of volume V containing N identical mass points:

   (i) Calculate the number of states or phase integral $\Omega\,(E,V,N)$ for the system at an energy E, knowing that the volume of a d-dimensional sphere is $\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2}+1)$

   (ii) Calculate the entropy of the of the system as $S(E,N,V) = k_B \log \Omega(E,V,N)$, and one can use the Stirling's approximation $logN! \approx NlogN - N$ for large N;

   (iii) Derive the equation of state, e.g., using $\frac{1}{T} = (\frac{\partial S}{\partial E})_{V,N}$ or $\frac{p}{T} = (\frac{\partial S}{\partial V})_{E,N}$

2. (40 points) In a magnetic field H, a particle of spin ½ and magnetic moment μ *or* -μ along the direction of the magnetic field thus has its energy split into -μH *and* +μH, respectively. For a system of N such distinguishable particles in the magnetic field of H and at a temperature of T:

   (i) Write down the partition function of the one-particle and N-particle system, in the canonical ensemble;

   (ii) Derive the Helmholtz free energy and entropy of the N-particle system;

   (iii) Calculate the internal energy and the specific heat of the above system;

   (iv) Derive the total magnetic moment M of the above system.

3. (30 points) An ideal Fermi gas with a volume V and $N \gg 1$ non-interacting, indistinguishable, and ultra-relativistic particles:

   (i) Derive a general expression for the energy E of the system;

   (ii) Calculate the chemical potential μ in the degenerate limit $T \rightarrow 0$;

   (iii) At the degenerate limit, derive the pressure-volume relation.