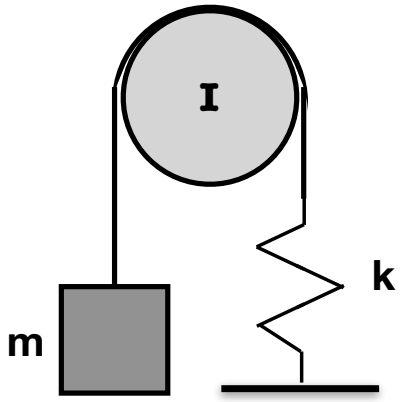
Classical Mechanics Qualifying Exam

1. Define three inertial frames: $\mathcal{S}_0$ is the lab frame; $\mathcal{S}_1$ is moving at a speed $4c/5$ in the positive x-direction relative to $\mathcal{S}_0$; and $\mathcal{S}_2$ is moving at a speed $12c/13$ in the positive y-direction relative to $\mathcal{S}_1$. (Note these speeds make the associated γ's come out as simple fractions! No need for a calculator.) All three frames have the same orientation and their origins coincide at $t = 0$. An object moves at speed $c/3$ in the positive z direction in frame $\mathcal{S}_2$. (a) What is its velocity in frame $\mathcal{S}_1$? (x, y, and z components) (b) What is its velocity in frame $\mathcal{S}_0$?

2. As shown in the figure, a mass m is suspended by a rope which goes over a cylinder with radius R, mass M, and moment of inertia $I = \frac{1}{2}MR^2$. An axle through the center of the cylinder keeps its center stationary but it can rotate. The rope has a high coefficient of static friction so the cylinder and the rope do not slip. The other end of the rope is connected to a massless spring with constant k. The rope is massless. Let y measure the position of the mass (y going up) and let $y = 0$ correspond to the spring being unstretched. (a) Find the equilibrium position y_0. Is it a stable or unstable point? (b) Write down the Lagrangian for the system and the Euler-Lagrange equations of motion. (c) Find the conjugate momentum and the Hamiltonian. (d) Write down the Hamilton equations. (e) Solve for the motion given the initial conditions: at $t = 0$ the system is at $y = 0$ and $y'(0) = 0$. You can use either the E-L or the Hamiltonian equations. (f) What is the frequency with which y oscillates?



Statistical Mechanics (2023 Fall)

Problem 1 (30 points) For a system consisting of *N* oscillators with frequency ν, considering the quantum number of the *i-th* oscillator denoted n_i and the corresponding energy level $\varepsilon_i = \left(n_i + \frac{1}{2}\right) h\nu$, then the total energy of the system is written as $E = (M + \frac{1}{2}N)h\nu$ for $M \equiv n_1 + n_2 + \cdots n_N$.

(i) (10 points) Find the number of ways allocating the *N* oscillators for a total energy of *E* with a certain number of *M*, i.e., the thermodynamic weight W_M;

(ii) (10 points) Calculate the entropy of the system for very large values of *N* and *M* (i.e., using the Stirling's approximation $\ln X! \sim XlnX - X$ for a large number $X \gg 1$);

(iii) (10 points) Derive the temperature *T* of the system and then write the energy as the function of the temperature *E(T)*.

Problem 2 (35 points) Consider an idea gas system consisting of *N* monoatomic molecules and calculate thermodynamic functions following procedures below:

(i) (10 points) Calculate the partition function in the canonical ensemble *Z(N,V,T)* (note that $\int_{-\infty}^{+\infty} e^{-x^2} = \pi$) and the Helmholtz free eerngy;

(ii) (10 points) Calculate the partition function in the constant temperature-pressure ensemble *Y(N,P,T)* (note that $\int_0^{+\infty} e^{-ax} x^n dx = n!/a^{n+1}$) and the Gibbs free energy, taking thermodynamic limit N>>1;

(iii) (15 points) Prove the state equation $PV = Nk_BT$ and derive the entropy *S* and chemical potential μ in in either of the above ensemble at the thermodynamics limit.

Problem 3 (35 points) An electron moving in a uniform magnetic field H (along z) has its orbital motion projected in the xy-plane as circular motion. The circular motions can be regarded as for a quantized harmonic oscillator, so that the energy level of the electron can be written as: $E = \frac{eH\hbar}{mc}\left(l + \frac{1}{2}\right) + \frac{p_z^2}{2m}$, with $l = 0,1,2 \ldots$ and $\frac{p_z^2}{2m}$ the kinetic energy of the electron translation motion along z.

(i) (10 points) Assume that the electron is confirmed in a box of length *L* or volume L^3. In near zero magnetic field, calculate the degeneracy around the energy level l (hint: for $2\mu_B Hl < \frac{p_x^2+p_y^2}{2m} < 2\mu_B H(l+1), with\ \mu_B \equiv \frac{e\hbar}{2mc}$ the Bohr magneton);

(ii) (15 points) Assume that the temperature is high and the electron is treated by the Boltzmann statistics, calculate the partition function Z_l for one electron, using the energy *E* formula and degeneracy calculated from (i);

(iii) (10 points) For *N* electrons, one can then show the magnetic moment as $M = -N\mu_B L(\frac{\mu_B H}{k_B T})$, with $L(x) = \coth x - 1/x$, based on calculation from (ii).

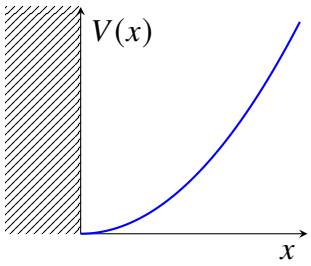
Comprehensive Exam
in
Quantum Mechanics

Fall 2023

- This exam has 5 problems and 90 points.
- You have 180 minutes.
- The points are indicated in the page margin.
- You can use either *Introduction to Quantum Mechanics* by Griffiths (any edition) **or** *Modern Quantum Mechanics* by Sakurai (any edition).
- Please write clearly. Answers that are hard to read will not be graded.
- Points will be taken off for wrong and/or irrelevant statements. That is, if you write things just so that the grader can find points, this will backfire and you will end up losing points.

1. Consider the potential

$$V(x) = \begin{cases} \infty\,, & x < 0 \\ \dfrac{1}{2} m\,\omega^2\,x^2\,, & x \geq 0 \end{cases}$$



where $m, \omega > 0$.

2 (a) Write down the time-independent Schrödinger equation for the problem.

6 (b) What are the boundary conditions for the solution $\psi(x)$ of the time-independent Schrödinger equation?

6 (c) Find the solutions of the Schrödinger equation.

Hint

The solutions of the harmonic oscillators are given by

$$\psi_n^{\text{H.O.}}(x) = \frac{1}{\sqrt{2^n\, n!\, \sqrt{\pi}\, x_0}}\, H_n\left(\frac{x}{x_0}\right) \exp\left\{-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right\}$$

with $x_0 = \sqrt{\dfrac{\hbar}{m\omega}}$ *and*

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2}(2x)^{n-4} - \dots$$

4 (d) Find the energy eigenvalues of the solutions.

8 (e) Compute the expectation value $\langle \boldsymbol{x} \rangle$ in the ground state.

Hint

Integrals of the form $\displaystyle\int_0^\infty \mathrm{d}x\, x^n\, \mathrm{e}^{-\alpha x^2}$ *can be obtained by differentiating*

$$I_0(\alpha) = \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-\alpha x^2} \quad \textit{or} \quad I_1(\alpha) = \int_0^\infty \mathrm{d}x\, x\, \mathrm{e}^{-\alpha x^2}\ ,$$

respectively, w.r.t. α.

2. $\boldsymbol{A}$ and $\boldsymbol{B}$ are Hermitean operators with nondegenerate spectrum.

3 (a) Write down the inverse of $\boldsymbol{A}\,\boldsymbol{B}$.

6 (b) Show that $\boldsymbol{A}$ and $\boldsymbol{B}$ can be diagonalized simultaneously if $[\boldsymbol{A}, \boldsymbol{B}] = 0$.

6 (c) Show that the eigenvalues of $\boldsymbol{A}$ are real.

6 (d) Assume that $\boldsymbol{A}$ commutes with two components of the operator of angular momentum, $\vec{\boldsymbol{L}}$. Show that this implies that $\boldsymbol{A}$ commutes with all three components.

5 (e) What properties must α have such that $\exp(\alpha \boldsymbol{A})$ is unitary? Prove your answer.

12 **3.** Consider the one-dimensional δ-potential

$$V(x) = -F\,\delta(x)$$

with $F > 0$. A plane wave is coming in from the left. Compute the reflection coefficient

$$R = \frac{|j_{\text{reflected}}|}{|j_{\text{incoming}}|} ,$$

where $j_{\text{reflected}}$ and j_{incoming} denote the probability currents of the reflected and incoming waves, respectively.

Hint

Use the condition

$$\lim_{\varepsilon\to 0}\left\{ \left.\frac{\partial\psi(x,t)}{\partial x}\right|_{x=\varepsilon} - \left.\frac{\partial\psi(x,t)}{\partial x}\right|_{x=-\varepsilon} \right\} = -\frac{2m}{\hbar^2} F\psi(0,t) .$$

4. An object with moment of inertia I is described by the Hamilton operator

$$\boldsymbol{H}_0 = \frac{1}{2I}\vec{\boldsymbol{L}}^2 .$$

$\vec{\boldsymbol{L}}$ denotes the operator of angular momentum.

4 (a) Which values can the energy assume? What are the degeneracies of the corresponding energy eigenstates?

10 (b) Assume now that the object has, in addition, a magnetic dipole moment $\vec{\mu}$. In an external magnetic field $\vec{B}$ this leads to an interaction term

$$\boldsymbol{H}_1 = -\vec{\mu}\cdot\vec{B} = -\mu B\cos\theta .$$

Treat $\boldsymbol{H}_1$ as a perturbation. Compute the first nontrivial correction to the ground state energy.

Hint

The two "lowest" spherical harmonics $Y_{\ell m}(\theta,\phi)$ are given by $Y_{00} = \sqrt{\frac{1}{4\pi}}$ and $Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$. You can write $\boldsymbol{H}_1$ in terms of these spherical har-

monics and use their orthogonality relations.

5. Consider a theory in which the Coulomb potential is modified for small distances,

$$V(\vec{r}) = \begin{cases} -\dfrac{e^2}{4\pi}\dfrac{R}{r^2} & \text{for } r \le R\ , \\ -\dfrac{e^2}{4\pi}\dfrac{1}{r} & \text{for } r > R\ , \quad \text{where } r = |\vec{r}|\ . \end{cases}$$

4 (a) Decompose the Hamilton operator describing an electron into two pieces, $\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}_1$, such that $\boldsymbol{H}_0$ is the Hamilton operator for the usual Coulomb law and $\boldsymbol{H}_1$ is a perturbation causes by the modification.

8 (b) Compute the change of the energy caused by the modification in first order perturbation theory.

Hint

Use the ground state wave function $\psi_0(r, \vartheta, \varphi) = \dfrac{1}{\sqrt{a_0^3 \pi}} \exp\left(-\dfrac{r}{a_0}\right)$.

EM Qual Fall 2023

1. A hollow spherical shell has inner radius a and outer radius b. It carries a charge density ρ. Find the field everywhere if

(a) the charge density is uniform i.e. $\rho = \rho_0$.

(b) the charge density is $\rho = \rho_1 \cos\theta$.

In each case, find the total charge and dipole moment.

2. A magnetic field fills the left half of the universe i.e. the magnetic field has magnitude B in the z-direction for $x < 0$ and zero otherwise. A CIRCULAR conducting loop (radius a, resistance R) is oriented in the $x-y$ plane, and moves to the $-x$ axis at a speed v. It begins to enter the field region at $t = 0$. Find the current in the loop as a function of time for all times.

3. An antenna of length L has its ends at the origin and at $(0, 0, L)$. It carries a current $I = \sin(\omega t)\hat{z}$. Find the radiation pattern (i.e. $\frac{dP}{d\Omega}$) to lowest order in the frequency.