Classical Mechanics Qualifying Exam

1. A pion \((mc^2=140 \text{ MeV})\) decays to a muon \((mc^2=106 \text{ MeV})\) and a neutrino \((mc^2=0)\) with a lifetime of \(26 \times 10^{-9} \text{ s}\).
   
   (a) In the rest frame of pion, what are the momenta and energies of the muon and neutrino?
   
   (b) In the lab frame, the pion has a momentum 500 MeV/c. What will be its average lifetime?
   
   (c) What are the maximum and minimum energies of the neutrino?

2. A triangular wedge of mass \(m\), and slope angle \(\alpha\) sits on a frictionless table. A solid sphere of radius \(R\), mass \(M\) \((I_{CM} = \frac{2}{5}MR^2)\) rolls down the wedge (there is friction between the sphere and the wedge). Find the angular acceleration of the sphere, by any means you like.
Qualify Exam: Statistical Mechanics

1. For a system with \( N \) identical molecules that have three internal motion energy levels: 0, \( \epsilon \), and 10\( \epsilon \), find the contributions of internal motions to the
   a. average internal energy, \( \langle E \rangle \),
   b. specific heat, \( C_v \).

2. Consider a system of \( N \) localized atoms, each of which has an intrinsic magnetic moment \( m \). The system is in equilibrium with a thermal reservoir at the temperature \( T \) and the interaction between atoms is negligible. When an external uniform magnetic field \( H \) is applied, the Hamiltonian of the system is
   \[
   -\mu H \sum_{i=1}^{N} \cos \theta_i
   \]
   where \( \theta \) is the angle between a magnetic moment and the magnetic field.
   a) Write out the partition function of the system.
   b) Show that the mean magnetic moment
   \[
   M_z = N < \mu \cos \theta > = N \mu \{ \coth \left( \frac{\mu H}{kT} \right) - \frac{kT}{\mu H} \}
   \]
   c) At high temperature, show the magnetic susceptibility \( \chi \) satisfies Curie’s law \( \chi \propto T^{-1} \).

(Hints 1: Langevin function \( L(x) = \coth x - \frac{1}{x} \approx \frac{x}{3} - \frac{x^3}{45} + \ldots \) at small \( x \); 2: the magnetic susceptibility \( \chi = \frac{\partial M_z}{\partial H} \))

3. For an ideal gas with \( N \) non-interacting relativistic particles \( \epsilon = pc \) in a box with a volume \( V \) and temperature \( T \),
   a. Calculate its entropy
   b. Find its internal energy and equation of state
   c. Compute its chemical potential as a function of \( T \) and \( P \)

4. Show that the total zero-point energy of a Debye solid is equal to \( 9Nk\Theta_D/8 \) (show you work from the Debye model).
1. Consider the Sommerfeld gas model for electrons in a solid, but in the case of a two-dimensional (2D) solid. (This can exist, for example, single layer graphene, which is of interest in solid state physics.) In this case,

\[ V(x, y) = \begin{cases} 
0, & \text{if } 0 < x < l_x, \ 0 < y < l_y; \\
\infty, & \text{otherwise}. 
\end{cases} \]

(a) Write down the normalized wave function for this case. [5]

(b) What are the allowed quantized energies? [5]

(c) What is the area (“volume”) of each state in \(k\)-space? [5]

(d) The energy for a given wavenumber is still \(E = \frac{\hbar^2 k^2}{2m}\), where \(k\) is the magnitude of the wave vector \(\mathbf{k}\). Calculate the total energy of the 2D gas (in its lowest energy state). [10]
2. Using a Gaussian trial wave function,

\[ \psi(x) = Ae^{-bx^2} \]

where \( A = (2b/\pi)^{1/4} \) and the variational principle, find the lowest upper bound on the ground state energy of a root potential, plotted below:

\[ V(x) = \alpha \sqrt{|x|}. \]

These integrals may be useful.

\[ \int_{-\infty}^{\infty} e^{-2bx^2} \, dx = \sqrt{\frac{\pi}{2b}}, \quad \int_{-\infty}^{\infty} x^2 e^{-2bx^2} \, dx = 2\sqrt{2\pi b}, \quad \int_{0}^{\infty} \sqrt{x} e^{-2bx^2} \, dx = \frac{\Gamma(3/4)}{2^{3/4}b^{3/4}}, \]

where \( \Gamma(z) \) is the gamma function, and \( \Gamma(3/4) \approx 1.23 \). [10]
3. Consider a quantum system with just three linearly independent states. Suppose the Hamiltonian, in its matrix form, is

\[
H = V_0 \begin{pmatrix}
3 & \epsilon & 0 \\
\epsilon & 1 & 0 \\
0 & 0 & 1 - \epsilon
\end{pmatrix}
\]

with \( V_0 \) as a constant term and \( \epsilon \ll 1 \) being a perturbative small parameter.

(a) Write down the eigenvectors and eigenvalues of the unperturbed Hamiltonian, with \( \epsilon = 0 \). Note that there will be two degenerate energies and one unique energy. [10]

(b) Solve for the exact eigenvalues of \( H \) (with \( \epsilon \) non-zero). [5]

(c) Write down the perturbation Hamiltonian matrix \( H' \). Remember, \( H = H^0 + H' \). [5]

(d) Use first and second-order nondegenerate perturbation theory to find the first \( E_1^1 \) and second order corrections due to \( H' \) found in the last part of this problem to the non-degenerate state found in part (a) of this problem. [5]

(e) Use degenerate perturbation theory to find the first-order correction to the two initially degenerate energy eigenstates found in part (a). [5]
4. An electron is in the spin state

\[ \chi = A \left( \frac{3i}{4} \right) \]

(a) Determine the normalization constant \( A \). [5]

(b) Find the expectation values of the spin along the axes, \( S_x \), \( S_y \), and \( S_z \). [5]

(c) Find the “uncertainties” \( \sigma_{S_x} \), \( \sigma_{S_y} \), and \( \sigma_{S_z} \), where these sigmas are standard deviations, not Pauli matrices. [5]

(d) Are these results consistent with all three uncertainty principles encompassed by the relation [5]

\[ \sigma_{S_i} \sigma_{S_i} \geq \frac{\hbar}{2} |\langle S_k \rangle| \]
1. A hollow spherical shell has inner radius $a$ and outer radius $b$. The shell is filled with a dielectric of dielectric constant $K$. A charge $Q$ is uniformly distributed through the dielectric. Find (a) the electric field (b) the energy of the system.

2. A plane capacitor has a plate area $A$ and a plate separation $h$. It is charged by a battery to a potential difference $V$. It is then disconnected from the battery, and without disturbing the charges, it is filled with a dielectric of dielectric constant $K$. Find (a) the final charge (b) the work done in introducing the dielectric (c) the final force on each plate.

3. A solenoid is aligned along the $z$-axis. It has $n$ turns per unit length, and carries a current $I$. It now moves in the $x$-direction at a relativistic speed $v$. Find the electric and magnetic fields in the solenoid.