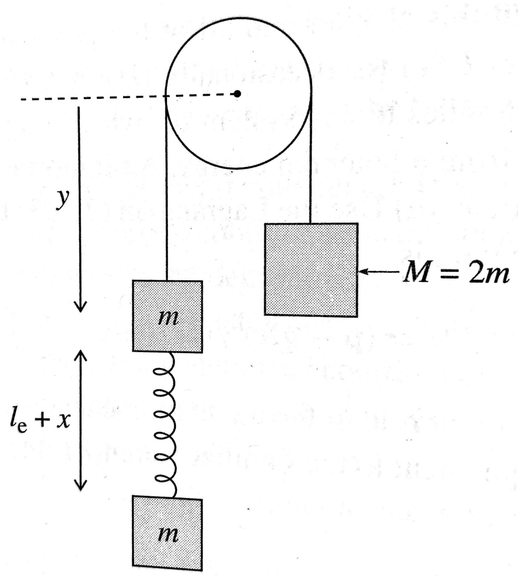
Classical Mechanics Qualifying Exam

1. A particle a traveling along the positive x axis of frame $\mathcal{S}$ with speed $0.5c$ decays into two identical particles, $a \to b+b$, both of which continue to travel on the x axis. (a) Given that $m_a = 2.5m_b$, find the speed of either b particle in the rest frame of particle a. (b) By making the necessary transformation on the result of part a, find the velocities of the two b particles in the original frame $\mathcal{S}$.

2. As shown in the figure, two weights on the left have equal masses m and are connected by a massless spring with constant k. The pulley is frictionless and massless, and the mass on the right is $M = 2m$. Let the unloaded length of the spring be l_0 and its equilibrium length, with all masses stationary but the spring supporting the lower mass m, be l_e. The coordinate x is the extension of the spring from l_e, so the length of the spring is $l_e + x$. (a) Find the potential energy of the system, which you should find does not depend on y. (b) Write down the Lagrangian for the system and the Euler-Lagrange equations of motion. (c) Find the conjugate momenta and the Hamiltonian. (d) Write down the four Hamilton equations. What is special about p_y? (e) Solve for the motion given the initial conditions: you hold M fixed, with the whole system in equilibrium and $y = y_0$, and then you pull down the bottom m by an extra distance x_0, hold it briefly to set $\dot{x} = 0$, and release everything. (f) Find the frequency with which x oscillates.



Statistical Mechanics (2020 Spring)

1. (30 points) Derive the equation of state of the ideal classical gas (with indistinguishable particles) from the grand canonical distribution, with fixed temperature T and chemical potential μ, but variable number N of the particles.

2. (30 points) Each of two similar particles, not interacting directly, may be in any of two quantum states, with single-particle energies ε equal to 0 and Δ. Write down the statistical sum Z of the system, and use it to calculate its average total energy E at temperature T, for the cases when the particles are:

   (i) distinguishable;
   (ii) indistinguishable fermions;
   (iii) indistinguishable bosons.

   Analyze and interpret the temperature dependence of $\langle E \rangle$ for each case, assuming that $\Delta > 0$.

3. (40 points) Magnetic susceptibility of a spinful Boltzmann gas. Consider a classical ideal gas of N spin-1/2 atoms moving in a container of volume V. In the presence of a weak external magnetic field H of the energy of the n-th such atom may be taken to be

$$E(\vec{p}_{n,}\sigma_n) = \frac{\vec{p}_n^{\,2}}{2m} - \gamma H \sigma_n$$

   Here $\sigma_n = \pm 1$ describes the two possible spin orientations of the atom and $\vec{p}_n$ is the momentum of the atom. γ is a positive constant.

   (i) Calculate the change in free energy due to the magnetic field.
   (ii) Calculate the average magnetization per atom $\langle M \rangle = \frac{1}{N}\gamma\langle \sum_n \sigma_n \rangle$.
   (iii) Calculate the variance in the magnetization $\langle M^2 \rangle - \langle M \rangle^2$.
   (iv) Use the above to calculate the magnetic susceptibility $\chi = \frac{d\langle M \rangle}{dH}$.
   How is the susceptibility related to the variance?

The notation used below is consistent with Sakurai's textbook.

1. For a spin-$\frac{1}{2}$ particle in an $\ell = 1$ state, write down all the kets $|j, m\rangle$ labeled by the total angular momentum quantum numbers j and m in terms of the kets $|m_\ell, m_s\rangle$. All symbols have the usual meaning: $\vec{J} = \vec{L} + \vec{S}$, $J^2|j, m\rangle = j(j+1)\hbar^2|j, m\rangle$, $J_z|j, m\rangle = m\hbar|j, m\rangle$, $L_z|m_\ell, m_s\rangle = m_\ell\hbar|m_\ell, m_s\rangle$, and $S_z|m_\ell, m_s\rangle = m_s\hbar|m_\ell, m_s\rangle$. [10]

2. Consider a system described by the Hamiltonian, $H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} + kx$. Assume that kx is a perturbation on top of the Harmonic Oscillator (HO) Hamiltonian.

   (a) Describe a physical situation where a perturbation like kx is generated? [2]

   (b) Calculate the energy of the first excited state to the lowest order in perturbation theory. [8]

   (c) Calculate the first excited state ket to the lowest order in perturbation theory in terms of the unperturbed HO energy eigenkets. [6]

   (d) Derive a condition to be satisfied by k for perturbation theory to be valid. [4]

3. Consider a system of two spin-$\frac{1}{2}$ particles described by the Hamiltonian $H = \omega_0(S_{1z} + S_{2z}) + (\omega/\hbar)(S_{1+}S_{2+} + S_{1-}S_{2-})$, where $S_\pm = S_x \pm iS_y$ are the spin raising and lowering operators. (The first piece of H can arise from the interaction energy of the spins with a uniform B-field, $-\vec{\mu} \cdot \vec{B}$, and the second piece is an anisotropic exchange interaction between the spins.)

   (a) Find the exact energy eigenvalues. Find the corresponding eigenkets in terms of the kets $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$. [16]

   (b) Assuming $\omega^2 \ll \omega_0^2$, find the energy eigenvalues to the lowest order in perturbation theory. Do not use the exact solution for this part. [8]

   (c) Expand your exact solution assuming $\omega^2 \ll \omega_0^2$ and compare to the answer from the perturbation method. [4]

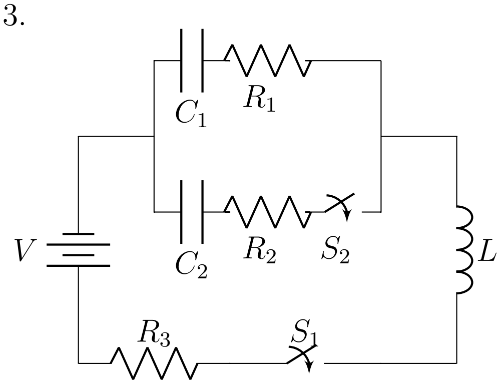
4. A one-dimensional Haromic Oscillator (Hamiltonian $H_0 = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$) is in its *first excited state* at $t < 0$. At $t > 0$, it is subject to an additional *perturbative* time-dependent potential $V(t) = kx\exp(-t/\tau)$ where τ is a finite time constant. As $t \to \infty$, find the probability of finding the system in the ground and second excited states to the lowest order in perturbation theory. [12]

Spring 2020 EM Qual

1. An insulating sphere of radius a centered at the origin has a volume charge density $\rho = kr^2$.
(a) Find the E-field and potential everywhere.
(b) A charge q is placed at $(a, b, 0)$. What are the magnitude and direction of the force on it?
(c) The charge of part (b) is moved to $(-a, b, 0)$. How much work is done?

2. A magnetic field fills the left half of the universe i.e. the magnetic field has magnitude B in the z-direction for $x < 0$ and zero otherwise. A CIRCULAR conducting loop (radius a , resistance R) is oriented in the $x - y$ plane, and moves to the $-x$ axis at a speed v. It begins to enter the field region at $t = 0$. Find the current in the loop as a function of time for all times.

3.



Initially S_2 is closed, S_1 is open and the capacitors are uncharged. At $t = 0$, switch S_1 is closed. After a long time, switch S_1 is opened again.
(a) What is the final charge on each capacitor?
(b) Capacitor C_2 is now filled with a dielectric of dielectric constant ϵ so that its capacitance increases by a factor ϵ. What is the final charge on C_2?