Classical Mechanics Qualifying Exam

1. Let a particle of mass m orbit in a central force field with potential $V(r)$. Consider only circular orbits.

(a) If the particle moves with speed v in a circular orbit, find the radius of that orbit in terms of m, v, and V or any of its derivatives.

(b) Show that if there is a *stable* circular orbit at a radius b, then the following inequality must hold:

$$V''(b)/V'(b) > -\alpha/b$$

where α is a real constant that should be determined.

(c) Find the values of the radius where there are stable circular orbits for the potential function

$$V(r) = -\frac{k}{r}e^{-r/a}$$

2. A proton beam is incident on a target of stationary protons. (The protons are in the form of hydrogen atoms as parts of molecules, but we ignore the electrons and consider the protons isolated and stationary.)

(a) What is the minimum kinetic energy of the incoming protons for them to produce proton-antiproton pairs? (Two protons colliding turn into 3 protons and an antiproton.) Express your answer in terms of the rest mass of a proton.

(b) What is the momentum of the incoming proton?

Statistical Mechanics (2022 Spring)

Problem 1 (40 points) For a two-level system with a total number of N>>1 non-interacting identical particles, the one-particle ground state energy is 0 and the exited state energy is Δ.

(i) (20 points) In the micro-canonical ensemble, derive the energy $E(T)$ as a function of temperature T, and also obtain the equilibrium entropy function $S(T)$

(ii) (20 points) In the canonical ensemble, calculate the N-particle partition function, the population in the excited state at temperature T, and obtain the free energy, energy, and entropy, all as the functions of T. See if the energy and entropy are obtained the same as in (i).

Problem 2 (40 points) A quantum particle of mass m is confined within a one-dimension box of length L for free motion. Calculate the avearge force the particle exterts on the two ends of the box (or 'walls') in thermal equilibrium, and analyze the temperature dependence of the force, in particular, on both the low-temperature and high-temperature limits [Hint: One may use the series $\Theta(\xi) = \sum_{n=1}^{\infty} \exp(-\xi n^2)$ as a known function of ξ].

Problem 3 (20 points) For a system of N (>>1) indistinguishable and independent fermions in thermal contact with a bath of temperature T, each particle has two non-degenerate energy levels separated by an energy gap Δ: calculate the chemical potential of the system.

1. A spin-$\frac{1}{2}$ particle is in the $+\hbar/2$ eigenstate of S_z, i.e., $|+\rangle$. It is then evolved in a magnetic field such that the Hamiltonian is $H = \omega S_x$.
   (a) Find the probability of finding the system in the $+\hbar/2$ eigenstate of S_y after a time T. At what times T is this probability maximized?
   (b) Find the expectation values $\langle S_x \rangle$, $\langle S_y \rangle$, $\langle S_z \rangle$ as a function of time and make the connection to spin precession.

2. A particle in a simple harmonic oscillator potential $V = m\omega^2 x^2/2$ is in the ground state ($|0\rangle$) at $t < 0$. At t = 0, a perturbation $H' = hm\omega^2 x^2 e^{-t/\tau}$ is switched on. Assume that h is small enough for perturbation theory to be valid. Calculate the probability for the system to transition to the $n = 2$ excited state ($|2\rangle$) after a sufficiently long time $t \gg \tau$ to the lowest non-vanishing order in h. Is transition to any other excited state possible at this order in h?

3. Consider the Hamiltonian of a two-level system given by $H = H_0 + H'$ with $H_0 = \epsilon_0 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $H' = \epsilon_0 \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$. Assume $\alpha \ll 1$ so that H' can be considered a perturbation.
   (a) Find the eigenvalues and eigenvectors of the unperturbed Hamiltonian H_0. Calculate the corrections to the energy eigenvalues and eigenvectors due to H' to the lowest non-vanishing order in α.
   (b) Solve for the exact eigenvalues of the full Hamiltonian, H. Expand the eigenvalues to the lowest non-vanising order in α and compare with the perturbation results obtained in (a).

EM Qual Fall 22

1. An insulating sphere of radius R is centered at the origin, and has its axis along the +z axis, and it carries a charge $\rho = \rho_0(1+\cos\theta)^3$. It is placed in an external field oriented along the z-axis; $\vec{E} = E_0\hat{z}$

(a) Find the charge and dipole moment of the sphere.

(b) Find the force on the sphere.

(c) The sphere is rotated in the x-z plane by half a rotation so that its axis now points along the -z axis. Find the work done.

2. One loop of radius a is in the x-y plane, is centered at the origin and carries a clockwise current I. Another loop of the same radius a is parallel to the first loop and carries the same current, but is centered at $(r\sin\theta, 0, r\cos\theta)$, where $r \gg a$. Find (a) the force on the second loop to leading order in (a/r) (b) the torque on the second loop to leading order in (a/r).

3. An antenna of length L has its ends at the origin and at $(0, 0, L)$. It carries a current $I = z(z-L)e^{i\omega t}$. Find the radiation pattern to lowest order in the frequency.