Classical Mechanics Qualifying Exam

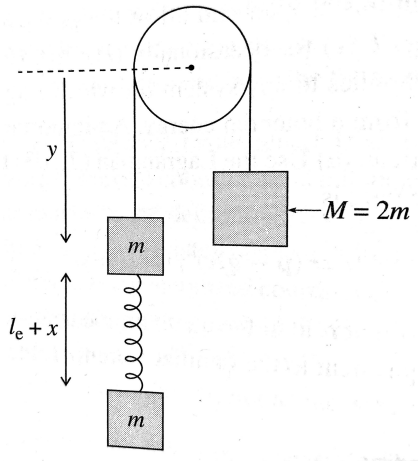
1.(a) Show that for any two particles a and b,

$$p_a \cdot p_b = m_a E_b = m_b E_a \tag{1}$$

where p_a and p_b are 4-vectors, E_b is the energy of b in a's rest frame, and vice versa for E_a.

(b) A particle d traveling along the positive x axis of frame $\mathcal{S}$ with speed $0.5c$ decays into two identical particles, $d \to g + g$, both of which continue to travel on the x axis. Given that $m_d = 2.5m_g$, find the speed of either g particle in the rest frame of particle d. (c) By making the necessary transformation on the result of part (b), find the velocities of the two g particles in the original frame $\mathcal{S}$.

2. As shown in the figure, two weights on the left have equal masses m and are connected by a massless spring with constant k. The pulley is frictionless and massless, and the mass on the right is $M = 2m$. Let the unloaded length of the spring be l_0 and its equilibrium length, with all masses stationary but the spring supporting the lower mass m, be l_e. The coordinate x is the extension of the spring from l_e, so the length of the spring is $l_e + x$. (a) Find the potential energy of the system, which you should find does not depend on y. (b) Write down the Lagrangian for the system and the Euler-Lagrange equations of motion. (c) Find the conjugate momenta and the Hamiltonian. (d) Write down the four Hamilton equations. What is special about p_y? (e) Solve for the motion given the initial conditions: with the whole system in equilibrium and $y = y_0$, you hold M fixed while simultaneously pulling down the bottom m by an extra distance x_0. With all three masses at rest you release both the bottom m and M. (f) Find the frequency with which x oscillates.



Statistical Mechanics (2023 Spring)

Problem 1 (35 points) For a classical ideal gas system consisting of N >>1 identical particles (with volume V and in contact with thermal bath of temperate T), the energy ε vs momentum p relation of the particle follows $\varepsilon = cp$, with c the constant.

(i) (10 points) Calculate the one-particle partition function and N-particle system partition function. Note: $\int_0^\infty e^{-ax}x^2dx = \frac{2}{a^3}$ (for $a > 0$)

(ii) (15 points) Calculate for the system the Helmholtz free energy, the energy, the entropy, as well as the pressure and the equation of the state. Note: one can use the Stirling formula $logN! = Nlog(\frac{N}{e})$.

(iii) (10 points) Derive the specific heat at a constant volume C_V and the specific heat at a constant pressure C_p

Problem 2 (30 points) Consider energy of an electron in a magnetic field H is given by $\mathcal{H} = -\mu_B\hat{\sigma}\cdot H$, with μ_B the Bohr magneton and $\hat{\sigma}$ the spin operator as the Pauli matrices. One can set the direction of the magnetic field H along the z-axis.

(i) (10 points) Write down and evaluate the density matrix in the diagonalized representation of $\hat{\sigma}_z$, i.e.,

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(ii) (10 points) Write down and evaluate the density matrix in the diagonalized representation of $\hat{\sigma}_x$, note that the matrix that transforms the operator is

$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ such that $U^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(iii) (10 points) Evaluate the average values measured for $\hat{\sigma}_z$ as $\langle\sigma_z\rangle$ and for $\hat{\sigma}_x$ as $\langle\sigma_x\rangle$ in both representations as above in (i) and (ii).

Problem 3 (35 points) For a system of N (>>1) non-interacting bosons confined in a spherically symmetric and very soft potential well of $U(r) = m\omega^2r^2/2$, which can be treated as 3D quantum harmonic oscillator (i.e., energy quantized as $\varepsilon_{n_x,\ n_y,\ n_z} = \hbar\omega\ (n_x + n_y + n_z$+3/2), with n_x, n_y, n_z non-negative integer, and $\hbar\omega$ very small; an additional spin degeneracy factor is denoted g), derive for the critical temperature T_c^ω ($k_BT_c^\omega \gg \hbar\omega$) at the Bose-Einstein condensation (BEC) condition of the system.

(i) (10 points) Consider the Bose-Einstein statistics for the population of bosons, write down the average value <N> ~ N at the thermodynamics limit

(ii) (10 points) As $n \equiv n_x + n_y + n_z$ can reach a large value of $n_{max} \equiv \frac{k_BT_c^\omega}{\hbar\omega}$ to be populated, compute N by integrating in the $\{n_x, n_y, n_z\}$ space, with an elementary volume $d\Omega \equiv dn_x\ dn_y\ dn_z = d\left(\frac{1}{3!}n^3\right) = \frac{1}{6}n^2dn$

(iii) (15 points) Derive the BEC critical temperature T_c^ω, using the given BE function $g_\nu(z)$

$$g_\nu(z) \equiv \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}dx}{z^{-1}e^x-1} = z + \frac{z^2}{2^\nu} + \frac{z^3}{3^\nu} + \cdots \text{ with } g_\nu(1) \equiv \xi(\nu) \text{ and } \Gamma(3)\xi(3) \approx 2.4$$

1. Consider a particle with spin quantum number $s = 1$ in a uniform magnetic field along the x-direction. Approximate the Hamiltonian as $H = \omega S_x$ (i.e., neglect spatial motion and other interactions). [20 points]

   (a) In the basis of the simultaneous eigenstates of S^2 and S_z, i.e., $|s, m_s\rangle$, show that the matrix for $S_x = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Show that the eigenvectors of S_x are $(|1,1\rangle + |1,-1\rangle - \sqrt{2}|1,0\rangle)/2$, $(|1,1\rangle + |1,-1\rangle + \sqrt{2}|1,0\rangle)/2$, $(|1,1\rangle - |1,-1\rangle)/\sqrt{2}$.

   (b) Let the particle be in the state $|s, m_s\rangle = |1,1\rangle$ at $t = 0$. What is the state at some time $t > 0$? What is the probability of finding the particle in the state $|1,-1\rangle$ at $t > 0$?

2. Consider an electron in the ground state of the Hydrogen atom. The normalized wavefunction for this state is $\psi(r) = e^{-r/a_0}/\sqrt{\pi a_0^3}$, where a_0 is the Bohr radius. [20 points]

   (a) What is the probability of finding the electron between $r = a_0$ and $r = a_0(1 + \delta)$ where $\delta \ll 1$.

   (b) Find the uncertainty in the momentum and position of the electron, and check explicitly that the uncertainty principle is satisfied. (Useful integral: $\int_0^\infty x^n e^{-2x} dx = n!/2^{n+1}$.)

3. A particle of charge q and mass m is in a harmonic oscillator potential $m\omega^2 x^2/2$. Assume that this system is perturbed by a weak electric field in the x-direction with magnitude $\mathcal{E}_0$ such that the change in potential energy is $-q\mathcal{E}_0 x$. To lowest non-vanishing order, find the corrections to the ground state wavefunction and the ground state energy. [20 points]

4. A particle of charge q and mass m is in a harmonic oscillator potential $m\omega^2 x^2/2$. Assume that this system is perturbed by a time-dependent weak electric field in the x-direction with magnitude $\mathcal{E}_0$ such that the change in potential energy with time is $-q\mathcal{E}_0 x e^{-t/\tau}$ for $t \geq 0$. To lowest order in perturbation theory, find the probabilities for the transition from the ground state to any excited state. [10 points]

EM Qualifying Exam Spring 2023

1. A thin insulating shell has radius R and is held at a potential $V = V_0 \cos^2\theta$.
(a) Find the potential outside the sphere.
(b) Deduce the charge and dipole moment of the configuration.
(c) Find the potential inside the sphere.

2. A magnetic field is such that for $0 < x < a$, the field is nonzero and has the value $\vec{B} = B_0\hat{k}$, while it is zero outside this region. A particle of mass m and charge q is initially travelling along the x-axis starting at negative values of x and travelling to the right at speed v_0. It reaches the region of magnetic field (i.e. $x = 0$) at time $t = 0$. Calculate the subsequent trajectory of the particle. What is the difference between large velocities and small velocities?

3. An antenna of length L is oriented along the x-axis from $x = 0$ to $x = L$. It carries a current $I = I_0 x\cos(\omega t)$.
(a) Find the charge density on the antenna. Be careful at the endpoints.
(b) Find the power radiated per unit angle for large wavelengths.
(c) Find the ratio of the power radiated along the x-axis to the power radiated along the y-axis.