Classical Mechanics Qualifying Exam

1. An electron moving in the positive x direction has kinetic energy equal to its rest mass. It strikes a stationary positron and the two annihilate and produce two photons. Photon 1 travels in the original direction of the electron, photon 2 in the opposite direction. Find the energies E_1 and E_2 of the two photons.

2. Consider a uniform rigid rod of length L and mass M that is being used as a simple pendulum. The rod is free to swing about the pivot at its top in the vertical plane. A gold mass m is embedded in the rod at its midpoint. Ignore air resistance and friction at the pivot for this problem. (a) Determine the potential energy of the system as a function of the angle θ between the rod and the vertical. (b) Calculate the moment of inertia of the pendulum (rod plus embedded gold mass) about the pivot. (c) Using the moment of inertia found in part (b), derive an expression for the rotational kinetic energy of the pendulum at an angle θ from the vertical. Construct the Lagrangian of the system. (d) Assuming small oscillations, find an expression for the period of the pendulum's oscillation. Check to make sure the correct result is obtained in the limit $m \to \infty$.

Statistical Mechanics (2024 Spring)

1. Consider a paramagnetic material consisting of N non-interacting spin-1 particles. Suppose that these spins reside in (and are in thermal equilibrium with) a crystal lattice at temperature T. We apply a magnetic field of magnitude B in the z-direction.

   a) Write the partition function of the system in terms of B, $k_B T$ and μ_B.
   b) What is the internal energy of the N spins?
   c) Calculate the spin contribution to the entropy of the crystal.

2. Consider equilibrium between a solid and a vapor made up of monatomic molecules with mass m. It is assumed that the energy ϕ is required for transforming an atom from the solid to vapor. For simplicity, take the Einstein model for the solid, i.e., assuming that a three-dimensional harmonic oscillator with a frequency ω represents each atom independently. Evaluate the chemical potentials of vapor and solid in terms of T, m, ϕ and ω and give the vapor pressure as a function of temperature.

3. Consider an ideal Bose gas confined to a region of area A in *two* dimensions. Express the number of particles in the excited states, N_e, and the number of particles in the ground state, N_0, in terms of z, T, and A, and show that the system does not exhibit Bose–Einstein condensation unless $T \sim \frac{h^2}{mkl^2}\frac{1}{\ln N}$ → 0 K for large N. Here, $l = \sqrt{A/N}$ is the mean interparticle distance in the system.

Quantum Qualifying Exam 2024

1. A particle is in a square well potential; the potential is

$$V = \begin{cases} \infty & x \leq 0 \\ 0 & 0 < x < L \\ \infty & x \geq L \end{cases} \tag{1}$$

At time $t = 0$, the particle is in the left half of the box; the wavefunction of the particle is

$$\psi(x) = \begin{cases} N \sin(2\pi x/L) & 0 < x < L/2 \\ 0 & L/2 < x < L \end{cases} \tag{2}$$

where N is a normalization constant.

Find an expression for the probability that the particle is at $x = 3L/4$ at time $t = t_0$. **You do not need to perform the sums or integrals that may occur.**

2. Consider a hydrogen atom in the limit where the proton mass is infinite, and the proton is at the origin. The atom is perturbed by an extra Hamiltonian for the electron

$$H = \epsilon(x^2 + y^2) = \epsilon(r^2 - z^2) \tag{3}$$

Treat ϵ as a small parameter. Find the ground state energy to first order in ϵ. Perform all integrals and sums that appear.

3. Two spin 1/2 particles are coupled by a Hamiltonian

$$H_0 = 2\lambda \vec{S}_1 . \vec{S}_2 = \lambda[(\vec{S}_1 + \vec{S}_2)^2 - \vec{S}_1^2 - \vec{S}_2^2] \tag{4}$$

a. What are the eigenstates and eigenvalues of the Hamiltonian H_0?

b. The system is further perturbed by an additional Hamiltonian

$$H_1 = \begin{cases} \epsilon S_{1x} t & 0 < t < 1 \\ 0 & \text{else} \end{cases} \tag{5}$$

Treat ϵ as a small parameter. Here S_{1x} is the x-component of the spin S_1.

At $t = 0$, the system is in the state where the z-component of S_1 is in the eigenstate $S_{1z} = 1/2$ and the z-component of S_2 is in the eigenstate $S_{2z} = -1/2$. The system evolves according to the total Hamiltonian $H_0 + H_1$. At $t = 1$ the z-components of both spins are measured.

What is the probability that at $t = 1$, the z-component of S_1 is $S_{1z} = 1/2$? Provide this answer to first order in ϵ.

Name:_____ ID#_____

Electrodynamics Comprehensive Exam
Spring 2024

You may use any intermediate results in the textbook. No electronic devices (calculator, computer, cell phone etc) are allowed.

For Administrative use only:

#1:_____

#2:_____

#3:_____

#4:_____

#5:_____

Total:_____

1. **Poynting vector** (15pts) A current *I* flows in a straight cylindrical wire with a radius *R*, a length *L*, and a conductivity σ. Using Poynting vector, calculate the total energy flux into the wire through the surface at the radius *R* and compare this total energy flux to the total Joule heating inside the wire.

2. **Proton orbit in magnetic fields** (25pts) A proton with mass M and charge e has an initial velocity v perpendicular to a uniform magnetic field B. (a) What is frequency ω for the cyclotron motion and what is magnetic flux Φ enclosed by the cyclotron orbit? (b) Now let the magnetic field amplitude increases slowly, i.e., $\frac{dB}{dt} \equiv \dot{B} \ll \omega B$. (Hint: assume a circular motion with a fixed radius when calculating *EMF* along proton orbit). Calculate $\dot{v}$ and $\dot{\Phi}$ as a function of $\dot{B}$. Is Φ conserved?

3. **Magnetized cylinder** (20pts). A solid, infinitely long, magnetized cylinder runs parallel to and is centered on the z-axis. It has a radius R and contains a magnetization $\mathbf{M} = \frac{rM_0}{R}\mathbf{e}_r$ in cylindrical coordinates (r,θ,z). M_0 is a constant. There is no free current and no magnetic fields far away from the cylinder. Find **B** and **H** inside and outside the cylinder. (Hint: $\mathbf{J}_\mathrm{b} = \nabla \times \mathbf{M}, \mathbf{K}_\mathrm{b} = \mathbf{M} \times \hat{n}$)

4. **Topological insulator** (15pts) A class of materials called topological insulators has the following constitutive relations: $\mathbf{D} = \epsilon_0\mathbf{E} - \alpha_0\mathbf{B}$, $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} + \alpha_0\mathbf{E}$, where $\alpha_0$is a constant. Given magnetic field of a plane wave $\mathbf{B} = B_0 e^{i(kx-\omega t)}\mathbf{e}_z$, where B_0 is a constant, find electric field $\mathbf{E}(\mathrm{x},\mathrm{t})$. Find frequency ω in term of wave vector k. What is wave phase velocity? Hint: Maxwell's equations in matter without free charge or free current.

5. **Polarized sphere** (25 pts) A sphere of radius *R* carries a polarization $\mathbf{P} = k\mathbf{e}_z$, where *k* is a constant. There is no free charge and no external electric field. Using spherical coordinate (r,θ,ϕ), calculate (A) bound charge surface density σ_b at $r = R$ and volume density ρ_b; (B) monopole Q and total dipole **p** of the sphere; (C) potential V inside and outside the sphere assuming $V(r=\infty) = 0$. (hint: general solution of Laplace's equation)