Classical Mechanics Qualifying Exam

1. (a) A uniform cubical block of mass m and sides $2a$ is balanced on top of a rough sphere of radius b. Show that the potential energy function when the block wobbles can be written as

$$V = mg[C_1 \cos\theta + C_2\theta \sin\theta]$$

where θ is the angle between the point of contact and the vertical, measured from the center of the sphere. Here C_1 and C_2 depend on a and b, and you should determine what they are. (b) Find the condition on a and b that determines whether the oscillation is stable or unstable. (c) For the stable oscillation case, determine the frequency. (Hint: the moment of inertia of the cube is $(2/3)ma^2$.)

2. A particle of unknown mass M decays into two particles, with masses $m_a = 0.5GeV/c^2$ and $m_b = 1.0GeV/c^2$. Their momenta are measured to be $\vec{p}_a = 2GeV/c\ \hat{y}$ and $\vec{p}_b = 1.5GeV/c\ \hat{x}$. Find the mass M and its speed.

Statistical Mechanics Qualifying exam

This qualifying exam is open book: you may use Kardar's *Statistical Physics of Particles* or Pathria's *Statistical Mechanics*. PDF versions of the books listed above are allowed on laptops and tablets as long as their Wi-Fi and cellular connections are turned off.

10:30 am – 12:00 pm Wednesday, April 2, 2025

1.

Consider a one-dimensional gas identical non-interacting, spin-less fermions in equilibrium at temperature T. The particles are confined in a harmonic potential and thus have an energy spectrum $\varepsilon_n = \hbar\omega(n + \frac{1}{2})$, where n is a non-negative integer.

(a) Find the Fermi energy ε_F of the gas at $T = 0$ if there are N particles.

(b) Using the sum

$$\sum_{n=0}^{N-1} n = \frac{N(N-1)}{2} ,$$

calculate the exact total energy per particle, $\frac{E}{N}$, at $T = 0$.

Then, compute the average single-particle density of states $g(\varepsilon) = \frac{dN_s}{d\varepsilon}$ in the harmonic trap. Hint: the states in a harmonic oscillator are equally spaced by $\hbar\omega$ and the number of states N_s below ε is $\varepsilon/(\hbar\omega)$. Use $g(\varepsilon)$ to calculate the *average* total energy per particle. Compare with your result for $\frac{E}{N}$.

(c) Determine the grand potential $\Omega(\mu, T) = -k_B T \ln \mathcal{Z}$, where k_B is the Boltzmann constant, $\mathcal{Z}$ is the grand canonical partition function of the gas and μ is the chemical potential. (Do not attempt to evaluate the infinite sum.)

(d) Using your result for $\Omega(\mu, T)$, find the average total number of particles $\overline{N}(\mu, T)$. (Do not attempt to evaluate the infinite sum.)

(e) Compute $\overline{N}(\mu, T)$ in the high temperature limit ($\beta \to 0$ and $e^{\beta\mu} \to 1$) by evaluating the infinite sum explicitly. What does this limit correspond to?

2.

Consider a paramagnetic material made of magnetic atoms with total angular momentum quantum number J that can have $2J + 1$ different z components: $J^z = -J, -J + 1, \ldots, J - 1$, and J. The z component of the magnetic moment then takes the values of μJ^z, where μ is a constant related to the Landé factor and the Bohr magneton. Let's apply an external magnetic field B in the z direction.

(a) Using the identity for the sum of a finite geometric series:

$$1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x} ,$$

and that neglecting interactions between dipoles the magnetic energy is $-\mu J^z B$, show that the partition function can be written as

$$Z = \frac{\sinh[\beta\mu B\left(J+\frac{1}{2}\right)]}{\sinh\left(\frac{\beta\mu B}{2}\right)} \ ,$$

(b) Compute the free energy (F) of a material of N such particles.

(c) Compute the total magnetization. (You can do this directly from the definition of the average of M or via $M = -\frac{\partial F}{\partial B}$, where F is the free energy.)

(d) Show that the magnetization has the expected behavior as $T \to 0$.

(e) Show that the magnetization follows Curie's law (i.e. it is proportional to $1/T$) in the limit $T \to \infty$. (Hint: $\coth(x) \approx \frac{1}{x} + \frac{x}{3}$ when $x \ll 1$.)

3.

Consider non-interacting electrons in two dimensions with a linear dispersion relation (like graphene) $\varepsilon_{\pm}(\mathbf{k}) = \pm\hbar v k$, where $k = |\mathbf{k}|$. Positive energy states (with ε_+) correspond to the conduction band and negative energy states (with ε_-) to the valence band. The allowed wave vectors correspond to periodic boundary conditions over a rectangular are $A = L_x L_y$.

At zero temperature, the valence band is completely filled and the conduction band is empty. At finite T, we can have excitations above this ground state, which are occupied states (electrons) in the conduction band and empty states (holes) in the valence band.

(a) Find the single-particle density of states $g(\varepsilon)$ as a function of the energy ε in terms of $\hbar$, v, and A.

(b) Assuming that the chemical potential is 0 at any temperature, and using the Fermi-Dirac distribution, prove that the probability of finding an electron at energy ε is equal to the probability of finding a hole at energy $-\varepsilon$.

(c) Still assuming $\mu = 0$, compute the total energy, $U(T) - U(0)$, of the excitations above the $T = 0$ ground state expressed in terms of A, v, $\hbar$, and k_B . Note that since $U(0)$ is subtracted, in the valence band you only need to count the energy associated with holes. (Hint: $U(T) - U(0)$ will be the sum of the average energy of the holes in the valence band and the electrons in the conduction band; $\int_{-\infty}^{0}(-\varepsilon)n_h(\varepsilon)g(\varepsilon)d\varepsilon + \int_0^\infty \varepsilon n_e(\varepsilon)g(\varepsilon)d\varepsilon$) A useful integral is

$$\frac{1}{n!}\int_0^\infty \frac{x^n}{e^x+1} = (1-2^{-n})\zeta(n+1) \ , \qquad (1)$$

where $\zeta(n+1)$ is the Riemann zeta function (no need to compute it!).

(d) Use your result for the internal energy and compute the heat capacity $C(T) \propto T^\alpha$. What is the exponent α.

QM Qalification, 2025S

1. 20% A particle in one-dimensional motion has a Hamiltonian $H = \frac{p^2}{2m} + V(x)$. $\{\psi_n\}$ are eigenfunctions of H, namely, $H\psi_n = E_n\psi_n$. Prove
   a. $\langle\psi_n|p|\psi_{n'}\rangle = \alpha_{nn'}\langle\psi_n|x|\psi_{n'}\rangle$ and find $\alpha_{nn'}$
   b. $\langle\psi_n|p^2|\psi_n\rangle = \frac{m^2}{\hbar^2}\sum_{n'}(E_n - E_{n'})^2|\langle\psi_n|x|\psi_{n'}\rangle|^2$

2. 30% A particle with mass *m* moves in a one-dimensional potential V(x). Assume that its normalized eigenfunction is $\psi(x) = (\frac{\gamma^2}{\pi})^{\frac{1}{4}}e^{-\gamma^2x^2/2}$ for a state with an eigenenergy of $E = \hbar^2\gamma^2/2m$, find
   (a) $<x^2>$
   (b) $<p^2>$
   (c) V(x)
   (d) The probability density of the particle in the momentum space, i.e., P(p).

3. 30% A spin-½ particle is placed in a magnetic field B=$\frac{B_0}{\sqrt{2}}(\hat{\imath} + \hat{k})$. In bases $\chi_+ = \begin{pmatrix}1\\0\end{pmatrix}$ $and\ \chi_- = \begin{pmatrix}0\\1\end{pmatrix}$, find
   (a) the Hamiltonian matrix and its eigenvalues and eigenvectors
   (b) the probability of getting each eigenvalue if we measure the energy at t=0 with $\psi(0) = \begin{pmatrix}0\\1\end{pmatrix}$
   (c) $\psi(t)$ from this initial state and $<S_x>(t)$.

4. 20% Using the raising or lowering operators for the orbital angular momentum, get angular wave functions of the 1s, 2s, 2p states of a hydrogen atom (Show your steps as the answers are well known).

Name:____________________ **ID#_______________**

Electrodynamics Comprehensive Exam
Spring 2025

You may use any intermediate results in the textbook. No electronic devices (calculator, computer, cell phone etc) are allowed.

For Administrative use only:

#1:_____________________

#2:_____________________

#3:_____________________

#4:_____________________

#5:_____________________

Total:___________________

1. **Mean value theorem** (20pts)-- Prove that for charge-free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of any sphere centered on that point.
2. **Spherical shell** (20pts)-- Thin spherical shell of radius R has a surface charge density of $\sigma = k\cos 2\theta$ in spherical coordinates, where k is a constant. The spaces inside and outside the shell are vacuum. Calculate potential in all spaces.
3. **Magnetic dipole** (20pts)-- What is the magnetic dipole moment due to the motion of a charged particle with a mass M and a charge q in a uniform magnetic field $\mathbf{B} = B\mathbf{e}_z$? The velocity of the particle is $\mathbf{v} = v_{||}\mathbf{e}_z + v_{\perp}\mathbf{e}_{\perp}$, where $\mathbf{e}_{\perp}$ represents a unit vector on the x-y plane. What is angular momentum $\mathbf{L}$ and ratio of absolute values of magnetic moment to angular momentum?
4. **Coaxial cable** (20pts)-- An infinitely long coaxial cable has an inner shell with a radius a, and an outer shell with a radius b. The material between the two shells has a conductivity σ and permittivity ε. Initially at t=0, the inner shell has a positive charge line density q_0 and the outer shell a negative charge line density $-q_0$. Find the electric and magnetic fields between the two shells as a function of time t.
5. **Rotor radiation** (20pts)-- Two equal and opposite charges (q and $-q$, respectively) are attached to the ends of a rod of length d in the x-y plane. The rod rotates counterclockwise in the x-y plane with an angular frequency ω. Find the total radiation power.