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QUALIFIER EXAM
Quantum Mechanics
Fall 2012

This is an open book exam. You may refer to Lectures on Quantum Mechanics by Baym & Quantum Mechanics by Sakurai. There are blank pages available to provide sufficient workspace. To receive credit you must show all of your work. Make an effort to present clear and organized solutions. Little or no credit will be given for obscure mathematics.

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Quantum Mechanics Qualifying Exam—Fall 2012

Note: None of the problems below require extensive or sophisticated calculations to solve.

1. The behavior of a particle moving in one dimension is governed by a Hamiltonian

$$H = -iv\frac{d}{dx} + V(x), \tag{1}$$

where v is a velocity and $V(x)$ corresponds to a step potential with

$$V(x) = \begin{cases} 0, & x < 0 \\ V_0 & x \geq 0 \end{cases}. \tag{2}$$

For concreteness let's take $v > 0$ and $V_0 > 0$.

(a) [15 points] Recall that Schrodinger's equation $i\hbar\frac{d}{dt}\psi = H\psi$ implies a continuity equation $\frac{d}{dt}P + \frac{d}{dx}J = 0$, where $P = |\psi|^2$ is the probability density and J is the corresponding probability current density. Using H above derive an expression for J. (Note that you needn't solve Schrodinger's equation; just give J in terms of a general solution ψ.)

(b) [20 points] Consider a particle incident from the left with energy E. Deduce the probability that the particle gets reflected at $x = 0$ and explain your answer.

2. Two spin-1/2 degrees of freedom separated by a distance a along the $\hat{\mathbf{z}}$ direction couple through dipolar interactions:

$$H = \frac{1}{a^3}[\vec{\mu}_1 \cdot \vec{\mu}_2 - 3(\vec{\mu}_1 \cdot \hat{\mathbf{z}})(\vec{\mu}_2 \cdot \hat{\mathbf{z}})]. \tag{3}$$

Here $\vec{\mu}_j = \gamma \mathbf{S}_j$ is the magnetic moment for spin $j = 1, 2$ (γ is a constant and $\mathbf{S}_j$ is a spin-1/2 operator).

(a) [20 points] Find the energies and corresponding eigenstates of H. Express the latter in terms of $\mathbf{S}^2, S_z$ eigenstates, where $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$.

(b) [15 points] Suppose now that the system is prepared into some minimum-energy eigenstate of H. A measurement of S_{1x} is then performed and yields a value $\hbar/2$. Calculate the probability that a subsequent measurement of S_{2z} will yield a value $-\hbar/2$.

3. [15 points] Consider a particle in a one-dimensional harmonic oscillator potential, with

Hamiltonian

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2. \tag{4}$$

Let $|n\rangle$ denote the eigenstates of H with energy $E_n = \hbar\omega(n+1/2)$. At time $t = 0$ the particle is initialized into the ground state $|0\rangle$. One then turns on a time-dependent anharmonic potential $\delta H = f(t)X^6$ where $f(t)$ adiabatically increases with time from 0 to some finite (and not necessarily small) value. Find the probability that the system will eventually be found in the state $|3\rangle$.

4. [15 points] Let H be a time-independent Hamiltonian that commutes with two unitary operators $\mathcal{O}_1$ and $\mathcal{O}_2$. Assume that these operators satisfy

$$\mathcal{O}_1\mathcal{O}_2 = e^{i\pi/N}\mathcal{O}_2\mathcal{O}_1 \tag{5}$$

for some integer $N > 0$. Argue that the ground state must be degenerate and deduce the minimal ground-state degeneracy required by Eq. (5) as a function of N.

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QUALIFIER EXAM
Classical Mechanics
Fall 2012

This is an open book exam. You may refer to *Mechanics* by Landau and Lifshitz. Write directly on this examination and not in a Blue Book. Ever other page is left blank to provide sufficient workspace. To receive credit you must show all of your work. Make an effort to present clear and organized solutions. Little or no credit will be given for obscure mathematics.

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Classical Mechanics PhD Qualifying Exam, Fall 2012

Question:	1	2	3	4	Total
Points:	15	15	10	10	50
Score:					

1. (15 points) A hoop of mass m and radius R rolls without slipping down an inclined plane of mass M, which makes an angle α with the horizontal plane.

   (a) Find the moment of inertia of the hoop.

   (b) Write down the Lagrangian and Euler-Lagrange equations for this system if the plane can slide without friction along a horizontal surface.

   (c) Find the integrals of motion.

2. (15 points) Consider an oscillator of mass m and frequency ω under the influence of time-varying force $F(t)$. Consider also a generating function of canonical transformations:

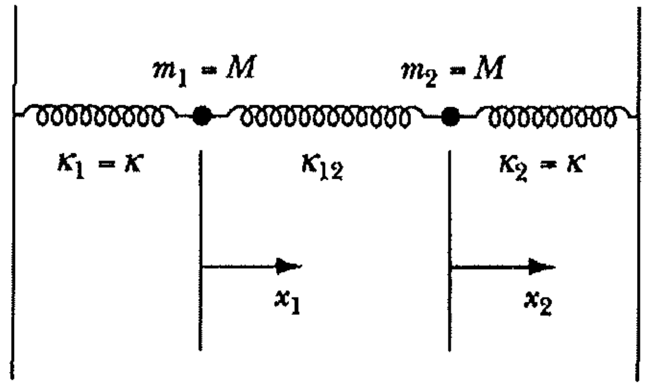
$$F(q, Q, t) = \frac{1}{2} m\omega \left[q - \frac{F(t)}{m\omega^2} \right]^2 \cot Q \,.$$

(a) Write down Lagrangian and Hamiltonean for the system.

(b) Using the generating function above, find new canonical variables P and Q as well as a new Hamiltonean $H(P, Q)$.

(c) Write equations of motion in new variables.

3. (10 points) Consider a system of two coupled oscillators shown in the Figure below



(a) Write down the Lagrangian and equations of motion

(b) Find the eigenfrequencies and normal coordinates.

4. (10 points) A non-relativistic deuteron with speed v collides elastically with a neutron at rest. Assume that $m_d = 2m_n$

   (a) If the deuteron scattered through an angle ψ in the LAB frame, what are the final speeds of both particles.

   (b) What is the LAB scattering angle of the neutron?

   (c) What is the maximal possible scattering angle of the deuteron?

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Electricity and Magnetism Qualifying Exam
Fall 2012

This is an open book exam. You may refer to Jackson but to no other materials. You may utilize any intermediate result in Jackson that is appropriate. There is table of possibly useful mathematical information on page 2 of the exam.

Do all seven (7) problems. In this exam we follow the unit conventions of Jackson's 3^{rd} edition. That is, for the non-relativistic problems (Numbers 1-5) we use SI units, and for the relativistic problems (Numbers 6-7) we use Gaussian units. To receive credit you must show all of your work. You do not need a calculator. It is preferable to leave answers in exact form, e.g. $\sqrt{2}-1$.

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Total: ______________________________

Possibly Useful Mathematical Information

$$\int_0^{\pi} \cos^m \theta \, d\theta = \begin{cases} \pi & m = 0 \\ 0 & m = 1 \\ \pi/2 & m = 2 \\ 0 & m = 3 \\ 3\pi/8 & m = 4 \\ 0 & m = 5 \end{cases}$$

$$\int_0^{\pi} \cos^m \theta \sin\theta \, d\theta = \begin{cases} 2 & m = 0 \\ 0 & m = 1 \\ 2/3 & m = 2 \\ 0 & m = 3 \\ 2/5 & m = 4 \\ 0 & m = 5 \end{cases}$$

$$\int_0^{\pi} \cos^m \theta \sin^2\theta \, d\theta = \begin{cases} \pi/2 & m = 0 \\ 0 & m = 1 \\ \pi/8 & m = 2 \\ 0 & m = 3 \\ \pi/16 & m = 4 \\ 0 & m = 5 \end{cases}$$

1. **[10 points]** Two concentric conducting cylindrical shells, with radii $\rho = a$ and $\rho = 2a$, are centered on the $\rho = 0$ axis with vacuum in between. The inner shell is maintained at potential V and the outer shell is maintained at potential $2V$. What is the potential at a point with radial coordinate $\rho = \frac{3}{2}a$?

2. **[15 points]** A uniform electric field of magnitude E_0 points in the $+x$ direction and fills all of space. Into this field is placed a solid conducting cylinder of radius a, centered on the z-axis and extending to infinity in both directions. Find the surface charge density induced on the cylinder as a function of the azimuthal angle ϕ. As usual $\phi = 0$ corresponds to the positive x-axis and $\phi = \frac{\pi}{2}$ corresponds to the positive y-axis.

3. **[15 points]** A hollow sphere of radius a is centered on the origin. The surface of the sphere is maintained at potential:

$$\Phi(a, \theta, \phi) = V_0 \sin\theta \cos\theta$$

...independent of ϕ.

(a) Find the leading non-vanishing term in the potential at the point (r, θ, ϕ) where $r \gg a$.

(b) Find the leading non-vanishing term in the potential at the point (r, θ, ϕ) where $r \ll a$.

(c) Find the net charge on the sphere.

(d) Find the dipole moment of the sphere.

4. **[15 points]** A uniform magnetic field of magnitude B points along the $+z$-axis and fills all of space. A magnetic dipole of dipole moment m is located at the origin; its direction is fixed along the $+z$-axis. A second magnetic dipole of the same dipole moment m is located at $x = a$ along the x-axis, but is free to rotate in the x-z plane.

   (a) Show that the second dipole is in equilibrium when it also points along the $+z$-axis. Is the equilibrium necessarily stable? If so, prove it. If not, find a condition on B that guarantees stability.

   (b) Repeat Part (a), but this time assume that the rotating dipole is located at $z = a$ along the z-axis.

5. **[15 points]** Two equal and opposite charges move near the origin. The charge $+q$ moves such that its coordinates at time t are:

$$x = a\cos\omega t, \qquad y = 0, \qquad z = 0$$

where ω is a constant angular frequency. The charge $-q$ moves such that its coordinates are:

$$x = 0, \qquad y = a\sin\omega t, \qquad z = 0$$

(a) Find the time-averaged power per unit solid angle $<dP/d\Omega>$ received by a distant observer along the $+z$-axis.

(b) Suppose now that *both* charges are $+q$. One positive charge moves as such that its coordinates are (as before):

$$x = a\cos\omega t, \qquad y = 0, \qquad z = 0$$

but the other positive charge moves such that its coordinates are now:

$$x = 0, \qquad y = a\cos\omega t, \qquad z = 0$$

Find the time-averaged power per unit solid angle received by a distant observer in the direction $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + \hat{\mathbf{y}})$.

6. **[15 points]** A charge q moves along the z-axis under the action of an unspecified force such that:

$$z(t) = \frac{4}{5}ct + \alpha t^2$$

where α is a positive constant, c is the speed of light, t is the time, and $|t|$ is not too far from zero. Note that the particle velocity is relativistic.

Express your answers below in terms of q, α, r, c only.

(a) Show that the retarded time associated with the distant point P with spherical coordinates $(r,\ \pi/3,\ 0)$ and observation time $t = r/c$ is $t_{ret} = 0$.

(b) Find the Cartesian components E_x, E_y, E_z of the electric *radiation* field produced by the particle as seen at the above point P for observation time $t = r/c$. Neglect the velocity field.

7. **[15 points]** The frame S′ moves as usual through the laboratory along the x-axis, such that the origins of S′ and the laboratory frame S coincide at time $t = t' = 0$. A circular ring of radius a is at rest in S′, centered on the origin of S′, and carries a total charge Q uniformly distributed about the ring (as seen from S′). The orientation of the ring is different in Parts (a) and (b) below. The speed of the frame S′ is $\frac{12}{13}c$ in both parts.

   (a) Suppose the ring lies in the $x' = 0$ plane in S′. Find the electric and magnetic fields observed at the origin in the laboratory frame for a negative time t for which the above equation for $z(t)$ holds. At such times the ring will be to the left of the origin in the lab frame.

   (b) Suppose the ring lies in the $z' = 0$ plane in S′. Find the electric and magnetic fields observed at the point with Cartesian coordinates $(0,\ a,\ 0)$ in the laboratory frame at the instant when the center of the ring crosses the laboratory origin.

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QUALIFIER EXAM
Statistical Mechanics
Fall 2012

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Statistical Mechanics Qualifying –Fall, 2012

During the exam you may consult **one** open book.

I. 10 points] Following Debye's reasoning show that the low temperature behavior of the specific heat at constant area of a <u>two</u> dimensional solid is of the form

$$C_{\text{area}} = aT^n ,$$

and determine n

II. [10 points] The energy-momentum relation of some spin-0 particles in three dimensions is

$$E = cp^n ,$$

c and n are constants. For what range of values of n will a Bose-Einstein condensation occur?

III. [10 points] Two species of spin-$\frac{1}{2}$ particles, A and B, have energy momenum relations

$$\begin{aligned} E_A(p) &= \epsilon_0 + \delta + \frac{p^2}{2m} , \\ E_B(p) &= \epsilon_0 - \delta + \frac{p^2}{2m} . \end{aligned}$$

The reaction $A \leftrightarrow B$ is allowed. At $T = 0$ the total number of particles is N; what is the nuber of type A particles? Consider δ to be small.

IV. [20 points] For a system in a canonical ensemble at a temperature T show that

$$< [E- < E >]^3 >= k_B^2 \left\{ T^4 \frac{\partial C_v}{\partial T} + 2T^3 C_v \right\} ,$$

where $< \cdots >$ denotes an average in this ensemble and C_v is the specific heat at fixed volume.

[Hint: Look at $\frac{\partial^3 \ln Z_{\text{can}}}{\partial \beta^3}$.]

For a large number of particles, N, in this system, what is the dependence of $\frac{<[E-<E>]^3]}{<E>^3}$ on N?