EM Qualifying Exam Fall 2015

1. A spherical shell is made of a conducting material, and has inner radius a and outer radius b. The sphere is centered at the origin. The shell therefore occupies the region $a < r < b$, with no material in the region $r < a$ or $r > b$. A point charge q is placed at the point $(0, 0, c)$ with $c < a$ i. e. the charge is in the interior part of the shell, but off-center.

Find

(a) the electric field everywhere

(b) the charge distribution on the inner surface of the conducting shell

(c) the charge distribution on the outer surface of the conducting shell.

An exact solution is preferable, but failing that, try and find a solution in the form of an infinite series.

2. A cube is made of a magnetized material. The cube has side length L. The sides are oriented along the coordinate axes with one corner at the origin, and three other corners are at $(0, 0, L), (0, L, 0), (L, 0, 0)$. For $0 < x \leq L/2$ the magnetism is $\vec{M} = M_0\hat{x}$ and for $L/2 < x \leq L$ the magnetism is $\vec{M} = -M_0\hat{x}$.

Find an expression for the magnetic field H everywhere.

3. A pendulum is made of a bob of mass M hanging from a string of length L. The bob also carries a charge q. The pendulum is displaced from the equilibrium position by a very small angle θ, and is allowed to oscillate about the minimum. Estimate how much energy is lost in each oscillation.

4. Two circular wires are oriented in the $x - y$ plane. One circular wire of radius R is centered at $(0, R, 0)$ while the second wire, also of radius R, is centered at $(0, -R, 0)$. They touch at the origin and the whole configuration looks like a figure 8. The first loop carries a current I flowing clockwise while the second loop carries a current I flowing anticlockwise. Find the electric and magnetic fields far away from the origin. Use spherical coordinates.

Clearly Print Name:______________________________

Signature:____________________________

CLASSICAL MECHANICS QUALIFYING EXAM

Fall 2015

You may consult only *Classical Mechanics* by Goldstein, Safko, and Poole. Do all four (4) problems. The exam is worth 50 points. Problem No. 4 is relativistic; the other three problems are non-relativistic. Write directly on this exam; do not use a blue book. Every other page has been left blank to provide extra work space. In order to receive credit you must show all of your work.

DO NOT OPEN THIS EXAM UNTIL YOU ARE TOLD TO DO SO

FOR ADMINISTRATIVE USE ONLY:

1 : ______________________________

2 : ______________________________

3 : ______________________________

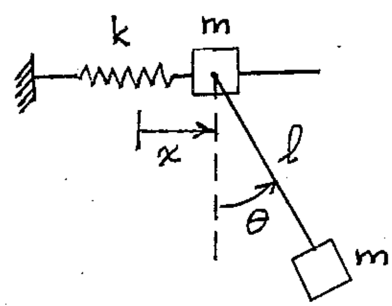
4 : ______________________________

Total: ______________________________

1. **[14 points]** A horizontal spring of spring constant k is attached to a wall on the left and to a small block of mass m on the right. The block is constrained to move along the horizontal (x-axis). A massless rigid rod of length ℓ is attached to the block and is free to swing below it so that the spring and rod always lie in the plane of the paper. A second small block of mass m is attached to the lower end of the rod. Let $x =$ the position of the upper block measured from the position where the spring is unstretched (and increasing away from the wall), and $\theta =$ the angle that the rod makes with the vertical (and increasing counter-clockwise). The acceleration of gravity is g. *Do not assume that the displacements from equilibrium are small.*

   a. Write the two Lagrange's equations for this system.

   b. Suppose the system is released from rest with the initial angle θ_0 between 0 and $\pi/2$. Is it possible to stretch the spring enough so that the lower block initially moves up rather than down, i.e. such that θ starts to move counter-clockwise? If it is possible, find the critical stretch x_0 of the spring. If it is impossible, prove this assertion.



2. **[12 points]** All given quantities in this problem are dimensionless. Numerical answers are expected.

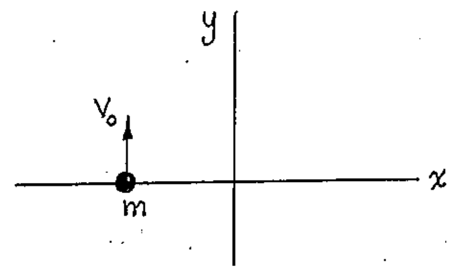
A particle of mass $m = 1$ moves in the x-y plane in response to an attractive potential $V = -k/r$, where $k = 1$. [This is the only potential energy in the problem.] The center of force is at the origin. At time $t = 0$ the particle has coordinates:

$$x = -1/7, \quad y = 0$$

and is moving with velocity $v_0 = \sqrt{10}$ in the $+y$-direction.

a. At what value of x does the particle cross the *positive* x-axis?

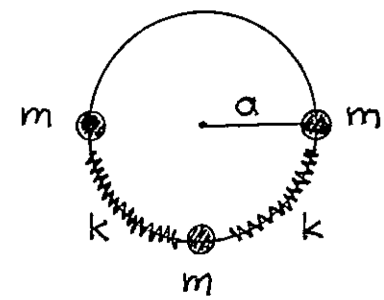
b. At what time does the particle cross the positive x-axis? [Of course it crosses multiple times. We want the time of the first crossing after $t = 0$.]



3. **[12 points]** A horizontal circular hoop has radius a. Since the loop is horizontal, gravity can be ignored. Two springs of spring constant k are wound around the loop so that the springs always conform to arcs of the circle. The springs connect three identical point masses m in the order: mass - spring - mass - spring - mass. When the springs are unstretched their lengths are each equal to 1/4 of the circumference so that two of the masses are diametrically opposite one another. In this problem we take the coordinates to be the angles θ_1, θ_2, θ_3 (subtended by the center of the circle) that the masses make with their unstretched positions and numbered so that θ_2 is the angle corresponding to the middle mass.

a. Find the (angular) frequencies of the three normal modes of this system.

b. Considering only the highest angular frequency, find the normal mode $\{\theta_1, \theta_2, \theta_3\}$, normalized so that $\theta_1 = 1$.



4. **[12 points]** Consider a one-dimensional reaction in which one relativistic A particle is incident on a B particle. The B particle is initially at rest. After the collision, there is still a B particle (of course in general no longer at rest), but there are now n distinct A particles, where n is an integer. The reaction may be written symbolically as:

$$A + B \to B + nA$$

The mass of the A particle is m and the mass of the B particle is $5m$. The Lorentz factor of the A particle before the collision is $\gamma_A = 11/2$.

a. What is the maximum number n_{max} of A particles that can be present after the collision?

b. When $\gamma_A = 11/2$ and there are n_{max} A particles after the collision as in Part (a), what is the Lorentz factor γ_B of the B particle after the collision?

Quantum Mechanics Ph.D. Qualifying Exam (Fall 2015)

I. Denote by E_0 the ground state energy of a quantum mechanical anharmonic oscillator, with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{k}{2}x^2 + \lambda x^4$$

Here the parameter k can be both *positive* or *negative*.

a) Set $\lambda = \frac{1}{6}$ and sketch the potential energy versus x for three cases: (1) $k > 0$, (2) $k = 0$ and (3) $k < 0$.

b) For $k > 0$ find an expression for $E_0(k)$ which is accurate to lowest order in λ.

c) For $k = 0$, construct a simple variational estimate of E_0. Evaluate this by taking $\hbar = m = 1$ and $\lambda = \frac{1}{6}$.

d) Take $\hbar = m = 1$ and $\lambda = \frac{1}{6}$. Find the leading behavior for $E_0(k)$ when $k \ll 1$ and the next correction.

e) For $\hbar = m = 1$ and $\lambda = \frac{1}{6}$, use the results you have found to plot $E_0(k)$ versus k for $-3 \leq k \leq 3$. Label the axes.

Useful integrals:

$$\int_0^\infty x^{2a} e^{-r^2 x^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2a-1)}{2^{a+1} r^{2a+1}} \sqrt{\pi}$$

II. A quantum-mechanical system is described by the Hamiltonian

$$H = \omega \, (a^\dagger a + b^\dagger b)$$

where the operators a and b obey $aa^\dagger - a^\dagger a = 1$ and $bb^\dagger + b^\dagger b = 1$. The operators a and b are supposed to be independent of each other, so the two a's commute with the two b's. The eigenstates of the Hamiltonian are

$$|n_a, n_b\rangle = \frac{1}{\sqrt{n_a!}} (a^\dagger)^{n_a} (b^\dagger)^{n_b} |0, 0\rangle$$

with n_a any nonnegative integer, and $n_b = 0, 1$.

(a) Define the operator $Q = b^\dagger a$. Show that $QQ^\dagger + Q^\dagger Q = \beta H$ and determine β. Set $Q = Q_1 + iQ_2$ and $Q^\dagger = Q_1 - iQ_2$ and express $Q_i Q_j + Q_j Q_i$ in terms of H.

(b) Show that $Q|1, 0\rangle$ is proportional to $|0, 1\rangle$.

(c) Can the operator b be represented by a suitable combination of Pauli matrices? Show explicitly why, or why not.

III. For times $t \leq 0$, a particle which moves in one dimensions resides in the ground state of the infinitely deep square well potential

$$V(x) = \begin{cases} 0, & \text{for } -\frac{d}{2} \leq x \leq +\frac{d}{2}; \\ \infty, & \text{for } |x| > \frac{d}{2}. \end{cases}$$

At time $t = 0$, an electric field parallel to the x axis is switched on, with time dependence

$$E(t) = E_0 \exp(-t/\tau) \quad .$$

Regard the electric field as very weak. This introduces into the Hamiltonian the perturbation

$$V(x,t) = -eE(t)x \quad .$$

(a) Derive an expression for finding the particle in the n-th excited state of the potential well, at time $t = \infty$. Express your answer in terms of the appropriate matrix element between the ground state wave function $\psi_0(x)$ and that $\psi_n(x)$ of the n-th excited state, i.e. do not waste time evaluating this matrix element.
(b) From what you know about the nature of the ground state and the excited states of this potential, discuss selection rules that apply to this problem.

IV. In the $1S$ configuration of positronium (e^+ - e^-), interactions between the electron and positron magnetic moments serves to remove the spin degeneracy that would otherwise occur. The effect can be represented by a weak perturbation

$$H' \;=\; \lambda\, \sigma_1 \cdot \sigma_2 \;\; ,$$

where $\lambda > 0$ and σ_1 and σ_2 are the electron and positron Pauli spin operators.

a) Compute the level separation between the singlet and the triplet states in the $1S$ configuration.

b) An external magnetic field $\mathbf{B_0}$ is applied. Graph the energies of the eigenstates in the $1S$ configuration versus the magnetic field. Give the formula in terms of λ and $\mathbf{B_0}$.

UC Irvine Department of Physics & Astronomy
Fall 2015 Comprehensive Examination: Statistical Physics

S.A. Parameswaran

(Dated: Wednesday, September 9$^{\text{th}}$, 2015, 2:30PM - 4:00PM)

- This qualifying exam is open book. You may not refer to your notes, the Internet, or other resources, but may use a hard copy of Kardar's *Statistical Physics of Particles* or Pathria's *Statistical Mechanics*.
- There are 3 problems on this exam, all worth equal points; attempt any 2 for full credit.
- You may **not** submit 3 problems and ask that the best 2 be counted; if you do so, I will automatically grade any two at my discretion (and they may be those that get you the lowest total!)
- Show all your work; absolutely no credit will be given for answers devoid of reasoning. I have been lenient about this on homework assignments but have repeatedly commented on inadequate logic; no such consideration will be given on this exam.
- **Please return this cover page with your exam, with your name and signature**

Name: **Signature:**

1. **Fermi Gas Model of a Nucleus.** A simple picture of an atomic nucleus is to imagine that it is a sphere filled with a degenerate Fermi gas of nucleons moving freely within the nuclear volume. Consider a nucleus containing Z protons and N neutrons, and let $A = N + Z$ denote the total number of nucleons. Each nucleon may be in one of two different spin states. The energy levels of the nucleus can be approximated by those of a 3D square well whose side is the same as the nuclear radius, given by $R = R_0 A^{1/3}$ where $R_0 = 1.2$ fm. Assume that protons and neutrons see an identical potential well and work at $T = 0$.

   (a) Determine the Fermi momenta $p_F^{(N)}$, $p_F^{(Z)}$ of the neutrons and protons, respectively, in terms of N, Z and A.

   (b) Obtain a formula for the total kinetic energy of the nucleons.

   (c) *Heavy* nuclei have an excess of neutrons. Assuming that the excess is small relative to the total number of nucleons, show that the total energy can be approximately expressed in the form

   $$E_{\text{tot}}(N, Z) \approx aE_F\left(A + b\frac{(N-Z)^2}{A}\right) \tag{1}$$

   where E_F is the Fermi energy of the protons (or the neutrons) alone in the symmetric case for which $N = Z = A/2$, and a, b are constants; determine a, b.

   (d) Using the fact that a nucleon has a mass of 938 MeV, compute E_F and hence the 'asymmetry energy' abE_F above.

2. **The Recombination Epoch.** Consider an extremely dilute gas of partially ionized atomic hydrogen, such as occurred in the early universe. The binding energy of an electron and proton in the atomic ground state is $\epsilon \approx 13.6$ eV. Assume that this dilute plasma is neutral, with equal numbers of electrons and protons, and that everything is close to thermal equilibrium: in other words, we have

   $$e + p \rightleftharpoons H \tag{2}$$

   Let n_e, n_p, n_H respectively denote the densities of electrons, protons, and hydrogen atoms. You will use statistical mechanics to infer the density of protons and electrons in the early universe.

   (a) The equilibrium described by (2) is assumed to occur at constant temperature and pressure. Under these conditions, use the extremization of an appropriate free energy to obtain a condition that relates the equilibrium value of an intensive state variable for e, p and H.

   (b) Use the results above to determine a formula for the ratio $n_e n_p / n_H$. (Leave your answer in terms of T, ϵ, fundamental constants and the electron mass.) This is known as the *Saha equation*.

   (c) If the fraction of ionized atoms is 1/2, then n_e, n_p, n_H are not independent. How are they related? Use this and the result of part (b) to obtain a formula for the density of protons n_p in the early Universe.

   (d) For $\epsilon/(k_B T) = 100$, give a numerical density, in units of m^3, correct to the nearest order of magnitude.

   Make and justify any appropriate approximations that will simplify your calculation.

3. **Defects in a Crystal.** In a crystal lattice, a defect is created when an atom hops from a lattice site to an interstitial site. The ground state is a configuration with no defects. However, when the lattice is in equilibrium at a finite temperature T, defects appear spontaneously.

   Consider the case where the number of atoms N is equal to the number of lattice sites and the number of possible interstitial sites is N_i (of the same order as N). Consider the thermodynamic limit where $N, N_i \to \infty$ at constant $N_i/N = \rho$. The energy required to create a defect is ϵ. Denoting by K the number of defects, let $n = K/N$ be their density.

   (a) Compute the entropy S as a function of the defect density n and the interstitial fraction ρ.

   (b) Obtain a formula that yields the density of defects as a function of temperature, $n(T)$. You may leave the result as an implicit expression relating some function $f(n, \rho)$ to a function of ϵ and T.

   (c) Give an explicit expression for $n(T)$ in the limit $T \to 0$, and also in the limit of very high temperature, $k_B T \gg \epsilon$, where it takes simple forms.

   (d) Compute the defect contribution to the heat capacity in the limit $T \to 0$, and sketch the result.

Possibly Useful Information

- The fundamental thermodynamic identity, valid for reversible, quasi-static processes, is

$$dE = TdS + \mathbf{J} \cdot d\mathbf{x} + \mu \cdot d\mathbf{N}.$$

Here, E is the internal energy, T and S denote the temperature and the entropy, $\mathbf{J}$, $\mathbf{x}$ are vectors of generalized forces and displacements, and μ, $\mathbf{N}$ are vectors that describe the chemical potentials and particle number of the different species. For the case of an ideal gas with a single species, we may take $J = -P$, $x = V$ and find

$$dE = TdS - PdV + \mu dN.$$

- The different thermodynamic potentials for the ideal gas are given by

$$\begin{aligned} &E && \text{(internal energy)} \\ H &= E + PV && \text{(enthalpy)} \\ F &= E - TS && \text{(Helmholtz free energy)} \\ G &= H - TS && \text{(Gibbs free energy)} \\ \mathcal{G} &= E - TS - \mu N && \text{(grand potential)} \end{aligned}$$

- The differential changes in the thermodynamic potentials are obtained by using the Liebniz rule ($d(AB) = AdB + BdA$) in conjunction with the thermodynamic identity.

- The different ensembles that we consider are summarized in the following table:

Ensemble	Macrostate (M)	$p(\mu_s)$	Normalization
Microcanonical	$(E, \mathbf{x}, N)$	$\delta_\Delta(\mathcal{H}(\mu_s) - E)/\Omega$	$S(E, \mathbf{x}, N) = k_B \ln \Omega$
Canonical	$(T, \mathbf{x}, N)$	$\exp(-\beta\mathcal{H}(\mu_s))/Z$	$F(T, \mathbf{x}, N) = -k_B T \ln Z$
Gibbs Canonical	$(T, \mathbf{J}, N)$	$\exp(-\beta\mathcal{H}(\mu_s) + \beta\mathbf{J} \cdot \mathbf{x})/\mathcal{Z}$	$G(T, \mathbf{J}, N) = -k_B T \ln \mathcal{Z}$
Grand Canonical	$(T, \mathbf{x}, \mu)$	$\exp(-\beta\mathcal{H}(\mu_s) + \beta\mu N(\mu_s))/\mathcal{Q}$	$\mathcal{G}(T, \mathbf{x}, \mu) = -k_B T \ln \mathcal{Q}$

Here, $p(\mu)$ is the probability of microstate μ_s (note the subscript that distinguishes it from μ, the chemical potential), $\mathcal{H}(\mu_s)$ is the Hamiltonian that assigns energy to the microstates, $\delta_\Delta(x - E)$ is a delta function that ensures that $x \in (E - \Delta, E + \Delta)$, and $\beta = 1/k_B T$. The different thermodynamic potentials are labeled consistent with their definition above. Ω is the *number of microstates*, Z is the *partition function*, $\mathcal{Z}$ is the *Gibbs partition function* and $\mathcal{Q}$ is the *grand partition function*; the grand canonical ensemble is written for a single species. As usual, to obtain results for the ideal gas, we set $\mathbf{x} = V$, $\mathbf{J} = -P$.

- The quantum partition function is obtained by computing the trace of the density matrix , $Z = \text{tr} e^{-\beta\mathcal{H}}$, taking care to compute the trace in a basis that appropriately incorporates the particle statistics. This makes it frequently more convenient to compute the properties using the grand canonical ensemble.

- The Bose-Einstein and Fermi-Dirac distributions describe the occupancy of single-particle energy levels $\varepsilon_\mathbf{k}$ by bosons and fermions, respectively. Using $\eta = \pm 1$ for bosons and fermions, we have

$$\langle n(\varepsilon_\mathbf{k}) \rangle = \frac{1}{e^{\beta(\varepsilon_\mathbf{k} - \mu)} - \eta}$$