Scattering of Electromagnetic Waves by a Strongly Correlated Plasma

- Cross-sections of scattering; structure factors; correlation functions; radar back-scattering from the ionosphere; plasma critical opalescence; Laue patterns with super-cooled Coulomb glasses; ionic quasi-crystals in a laser-cooled non-neutral plasma

Multi-particle Correlation, Equations of State, and Phase Diagrams

- RPA correlation with superposition of dressed test charges; correlation energy beyond RPA; phase diagrams of hydrogen: metallization, solidification, and magnetization; magnetic white dwarfs; phase diagram of nuclear matter
Scattering of Electromagnetic Waves by a Strongly Correlated Plasma


Multi-particle Correlation, Equations of State, and Phase Diagrams


Scattering Cross-section

Cross-section of Thomson scattering

\[ \sigma_T = \frac{8\pi}{3} \left( \frac{e^2}{mc^2} \right)^2 = 6.553 \times 10^{-25} \text{ (cm}^2\text{)} \]

With polarization: \( \eta_1, \eta_2 \)

\[ \frac{d^2Q}{d\Omega d\omega} = \frac{3V}{8\pi} \sigma_T (\hat{\eta}_1 \cdot \hat{\eta}_2)^2 S(k, \omega) \]

where \( k = k_2 - k_1 \) and \( \omega = \omega_2 - \omega_1 \).

Without polarization: \( \eta_1, \eta_2 \)

\[ \frac{d^2Q}{d\Omega d\omega} = \frac{3V}{8\pi} \sigma_T \left( 1 - \frac{1}{2} \sin^2 \theta \right) S(k, \omega) \]

\( \theta \) is the scattering angle between \( k_1 \) and \( k_2 \).

DYNAMIC STRUCTURE FACTOR

Spectral distribution in \( k \) and \( \omega \) of electron density fluctuations

\[ S(k, \omega) = \frac{1}{2\pi V} \int_{-\infty}^{\infty} dt \langle \rho_k(t + t')\rho_{-k}(t') \rangle \exp (i\omega t) \]
DYNAMIC STRUCTURE FACTOR

\[ S(k, \omega) = \frac{1}{2\pi V} \int_0^\infty dt \langle \rho_k(t) \rho_{-k}(t) \rangle \exp(i\omega t) . \]

\[ \rho_k(t) = \int dr \left\{ \sum_{j=1}^N \delta[\mathbf{r} - \mathbf{r}_j(t)] - n \right\} \exp(-i\mathbf{k} \cdot \mathbf{r}) \]

\[ = \sum_{j=1}^N \exp[-i\mathbf{k} \cdot \mathbf{r}_j(t)] - N\delta(k,0) , \]

Spectral distribution of density fluctuations for \( N \) electrons with trajectories \( \mathbf{r}_j(t) \)

Fourier transform of density-density correlation between electrons
IONOSPHERIC BACKSCATTERING

IONOSPHERIC F-LAYER
Altitude: 200〜500 km

$n = 10^5 – 10^6 \text{ cm}^{-3}$

$T = 1,500 \text{ K}$

Bowles’ Experiment (1958)

RADAR PULSES: $f = 40.92 \text{ MHz}$
width 120 $\mu$s
repetition 25〜40 s$^{-1}$
peak power: 1 MW

$
\Delta f_e \sim 82 \text{ kHz}$

Pineo et al’s Experiment (1960)

$f = 440 \text{ MHz}$
$k << k_D \sim 5.3 \text{ cm}^{-1}$
collective motion!!

Scattering by electrons co-moving with ions, or by “dressed” ions!!
Scattering of electromagnetic waves by electron density fluctuations associated with an onset of acoustic-wave instability due to drift motion $V_d$ of electrons relative to ions

$$S = \sim V_i \left( \frac{T_e}{T_i} \right)^{1/2} : \text{ion-acoustic velocity}$$

ION-ACOUSTIC WEAVE : $\omega_k = s k$

a well-defined excitation, when $T_e \gg T_i$ ; for $k < k_D$

Fig. 4. Boundary between growing waves and damped waves for a helium plasma with $(n_i/m_i) = 4 \times 10^{-6}, \left( T_e/T_i \right) = 20$. Curve A represents the boundary when ion short-range collision is neglected, curve B the case for $\omega_n \tau_e = 100$, curve C the case for $\omega_n \tau_e = 10$; $\tau_e$ is the relaxation time of ions due to short-range collision.

The form factor $S(k)$ at the critical point $T_e/T_i = 10$, for $\cos \chi = 1$ and the
STATIC STRUCTURE FACTOR

Spectral distribution of spatial density fluctuations or spatial configurations of density distribution such as lattice structures

\[
\frac{dQ}{d\omega} = \left(\frac{3N}{8\pi}\right) \sigma_T \left(1 - 0.5 \sin^2 \theta\right) S(k)
\]

Observation of Laue patterns (1914: M. von Laue) ⇔ X-ray crystallography (1915: W. H. Bragg & W. L. Bragg)

Short-ranged crystalline order at the nearest-neighbor separations
We report the first observation of layered structures in the computer-simulated Coulomb glasses produced by rapid quenching of one-component plasmas in the absence of an external force field. Degrees of polycrystalline nucleation and the nature of local order developed in the glasses are elucidated through analyses of the intralayer and interlayer particle correlations and by means of the Laue patterns formed by scattering of plane waves. Stages of evolution for the glass transition in Coulombic systems are conjectured.

\[
\Gamma = \frac{(Ze)^2}{k_B T} = 2.69 \times 10^{-5} Z^2 \left[ \frac{n}{10^{12} \text{ cm}^{-3}} \right]^{1/3} \left[ \frac{T}{10^6 \text{ K}} \right]^{-1}
\]

Dynamic evolution and formation of Coulomb glasses by the Monte Carlo simulation method, starting with equilibrated fluid state at \( \Gamma = 160 \)

Probability of acceptance for the Monte Carlo configurations generated by random displacements of particles

\[
P = \begin{cases} 
\exp \left( -\frac{\Delta E}{k_B T} \right), & \text{if } \Delta E > 0, \\
1, & \text{if } \Delta E \leq 0,
\end{cases}
\]

FIG. 2. Normal projections of most closely packed layers: open circles, upper layer; closed circles, middle layer; crosses, lower layer.
**FIG. 4.** Two-dimensional radial distribution functions between intralayer particles in the glasses (A)–(C). The bottom of the figure shows the peak positions for the fcc-hcp (solid lines) and bcc (dashed lines) lattices.

fcc = face-centered cubic st.  
hcp = hexagonal close-packed st.  
bcc = body-centered cubic cubic st.

(A) sudden quench to $\Gamma = 400$  
(B) gradual quench stepwise with $\Delta\Gamma = 10$  
from $\Gamma = 160$ at $c = 0$ to $\Gamma = 400$ at $c = 2.3 \times 10^7$ configurations  
(C) sudden quench to $\Gamma = 300$

$\chi = \text{the scattering angle between } k_1 \text{ and } k_2$  
$\phi = \text{the azimuthal angle around } k_1$
Penning-trapped, laser-cooled, strongly coupled, non-neutral ion plasma

5-fold symmetry exhibited in Laue pattern with ultra-cold $1.2 \times 10^5 \, ^9\text{Be}^+ \text{ions (1999)}$

(a) Bragg signal in k-space
(b) CT scan picture for 4 layers of ion configuration

“ionic quasi-crystals?”


- a non-neutral OCP in the co-rotating frame of reference – can be maintained stably for many hours
  - $n = 2\sim 3 \times 10^7 \, \text{cm}^{-3}$
  - $T_c = 10\sim 60 \, \text{mK}$, $T_z = 70\sim 150 \, \text{mK}$
  - $\Gamma = 5\sim 10$

Penning trap
Collective vs. Individual Particles
Aspects of Fluctuations

Bohm-Pines Theory (1951〜1953) of “A Collective Description of Electron Interactions”

The density fluctuations may be split into two approximately independent components. The collective component, that is, the plasma oscillation, is present only for wavelengths greater than the Debye length. The individual particles component represents a collection of individual electrons surrounded by co-moving clouds of screening charges; collisions between them may be negligible under the random-phase approximation (RPA).

1. Collective Mode

\[ \varepsilon (k, \omega) = 0 \Rightarrow \omega = \omega (k) - i \gamma (k) \]

Individual Electrons

\[ r_i (t) = r_j + v_i \cdot t \]

\[ \rho_j^{(0)}(k, \omega) = 2\pi \exp (-ik \cdot r_j) \delta (\omega - k \cdot v_j) . \]

2. Dressed Electrons

\[ \rho_j^{(S)}(k, \omega) = \frac{\rho_j^{(0)}(k, \omega)}{\varepsilon (k, \omega)} . \]
RPA Structure Factors

Superposition of Dressed Test Charges
(Rostoker and Rosenbluth, 1960; Ichimaru, 1962)

\[ S(k, \omega) = \frac{S^{(0)}(k, \omega)}{|\epsilon(k, \omega)|^2}, \]

\[ S^{(0)}(k, \omega) = \int dp F(p) \delta(\omega - k \cdot v) \]

\[ S^{(0)}(k, \omega) = \frac{n}{k} \sqrt{\frac{m}{2 \pi T}} \exp \left( -\frac{m \omega^2}{2Tk^2} \right) \]
RADIAL DISTRIBUTION FUNCTION and CORRELATION ENERGY

\[ g(r) = 1 + \frac{1}{n} \int \frac{dk}{(2\pi)^3} [S(k) - 1] \exp(i k \cdot r) \]

radial distribution function = a joint probability density of finding two particles at a separation \( r \)

Coulomb correlation energy

\[ U_{\text{int}} = 2\pi(Ze)^2 n \int \frac{dk}{(2\pi)^3 k^2} [S(k) - 1] \]

\[ = \frac{(Ze)^2}{2} \int dr \frac{1}{r} [g(r) - 1] \]

RPA internal energy density

\[ \frac{U}{nk_B T} = \frac{3}{2} - \frac{\sqrt{3}}{2} \Gamma^{3/2} \]  

(\( \Gamma << 1 \))

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kinetic + Debye-Hückel
MULTI-PARTICLE CORRELATION
and OCP INTERNAL ENERGY

\[ u_{\text{ex}}^{\text{ABE}} = -\frac{\sqrt{3}}{2} \Gamma^{3/2} - 3 \Gamma^3 \left[ \frac{3}{8} \ln(3\Gamma) + \frac{\gamma}{2} - \frac{1}{3} \right], \]

\( \Gamma < 0.1 \)

\[ \gamma = 0.57721\ldots \]

Euler’s constant

- giant cluster expansion calculation (Abe, 1959)
- expansion in \( \Gamma \) of the BBGKY hierarchy (O’Neil and Rostoker, 1965)
- multiparticle correlation in the convolution approximation
  (Totsuji and Ichimaru, 1973)
- density-functional formulation of multi-particle correlations
  (Ichimaru, Iyetomi and Tanaka, 1987 - review)
- Monte Carlo simulations (Slattery, Doolen and DeWitt, 1982; Ogata and Ichimaru, 1987)

\[ u_{\text{ex}}^{\text{OCP}}(\Gamma) = -0.898004\Gamma + 0.96786\Gamma^{1/4} + 0.220703\Gamma^{-1/4} - 0.86097. \]

\( 1 < \Gamma < 180 \)

\[ u_{\text{ex}}(\Gamma) = \frac{u_{\text{ex}}^{\text{ABE}}(\Gamma) + (3 \times 10^3)\Gamma^{5.7}u_{\text{ex}}^{\text{OCP}}(\Gamma)}{1 + (3 \times 10^3)\Gamma^{5.7}}. \]

\( \Gamma < 180 \)
Equations of State for the Metallic Hydrogen

Metallic fluid = itinerant electrons (fermions with spin $\frac{1}{2}$) and itinerant protons (classical, or fermions with spin $\frac{1}{2}$) with strong e-i coupling

Metallic solid = itinerant electrons (fermions with spin $\frac{1}{2}$) and a body-centered cubic (bcc) array of protons (classical) with harmonic and anharmonic lattice vibrations and with strong e-i coupling

Fluid-solid transitions $\Leftrightarrow$ comparison of the Gibbs free energy: $G = F + PV$; at a constant pressure: $P = -\left(\frac{\partial F}{\partial V}\right)_T$
Equations of State for an Insulator Phase of Hydrogen

Molecular fluid = an ideal Bose gas, short-range repulsive interaction between molecular cores, attractive (dipolar) van der Waals forces, molecular rotation (roton), intramolecular vibration (vibron), and the ground-state energy of a H$_2$ molecule ($E_{H_2}$)

Molecular solid = cohesive energy with a hexagonal-close-packed structure, lattice vibration (phonon), roton, vibron, and the ground-state energy of a H$_2$ molecule ($E_{H_2}$)

Fluid-solid transitions:

\[ P_{\text{mol.fl.}}(\rho_{\text{fl}}, T) = P_{\text{mol.sol.}}(\rho_{\text{sol}}, T) \]
\[ G_{\text{mol.fl.}}(\rho_{\text{fl}}, T) = G_{\text{mol.sol.}}(\rho_{\text{sol}}, T) \]
Mixture of atomic and molecular fluid

\[ f_{\text{at.mol.fl}} = \frac{F_{\text{at.mol.fl}}}{N_H k_B T} \]

\[ = \frac{n_m}{n_H} \left( f_{m}^{\text{id}} + f_{\text{rot}} + f_{vib} + \frac{E_{H_2}}{k_B T} \right) \]

\[ + \frac{n_a}{n_H} \left( f_{a}^{\text{id}} + \frac{E_{H}}{k_B T} \right) \]

\[ + \frac{n_m + n_a}{n_H} (f_{\text{HS}} + f_{\text{attr}}), \]

\[ E_{H_2} = \text{the ground-state energy of a H}_2 \text{ molecule} \]
Equations of State for Hydrogen


\[
f_{\text{tot}}(n_{\text{H}}, T; \langle Z \rangle, \alpha_d) \equiv \frac{F}{N_{\text{H}}k_BT} = \frac{(1 - \alpha_d)(1 - \langle Z \rangle)}{2} \left( f_{\text{id}}^{\text{m}} + f_{\text{vib}} + f_{\text{rot}} + \frac{E_{\text{H}_2}}{k_BT} \right) + \alpha_d(1 - \langle Z \rangle) \left( f_{\text{id}}^{\text{a}} + \frac{E_{\text{H}}}{k_BT} \right) + \langle Z \rangle f_{\text{met.fl}}(\bar{n}, T) + \frac{(1 + \alpha_d)(1 - \langle Z \rangle)}{2} (f_{\text{HS}} + f_{\text{attr}}).
\]

At a given \( n_{\text{H}} \) and \( T \), the chemical equilibrium may be achieved through the conditions that \( f_{\text{tot}} \) be minimized with respect to \( \langle Z \rangle \) and \( \alpha_d \), the degrees of ionization and dissociation.

Metal-Insulator Transitions

\[
P(\rho_{\text{M}}, T) = P(\rho_{\text{I}}, T), \quad G(\rho_{\text{M}}, T) = G(\rho_{\text{I}}, T).
\]
Metallic fluid and solid = itinerant electrons (fermions with spin $\frac{1}{2}$) and itinerant or body-centered cubic (bcc) array of protons (classical)

Molecular fluid and solid hydrogen

Mixture of atomic and molecular hydrogen

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PHASE DIAGRAM OF HYDROGEN


PROTONS

fermions with spin \( \frac{1}{2} \) in the ground state
Magnetization and Solidification of Metallic Hydrogen

TABLE I. Physical parameters at the fluid–solid critical point ($C_{FS}$) and the magnetic critical point ($M_C$) in the phase diagrams of Fig. 1 for the liquid-metallic hydrogen.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$C_{FS}$</th>
<th>$M_C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_m$ (g/cm$^3$)</td>
<td>$4.0 \times 10^3$</td>
<td>$1.4 \times 10^6$</td>
</tr>
<tr>
<td>$R_i$</td>
<td>161</td>
<td>22.4</td>
</tr>
<tr>
<td>$T$ (K)</td>
<td>$6.6 \times 10^3$</td>
<td>$6.6 \times 10^5$</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>0.159</td>
<td>0.317</td>
</tr>
<tr>
<td>$B_M$ (G)</td>
<td>$2.6 \times 10^5$</td>
<td>$8.0 \times 10^6$</td>
</tr>
<tr>
<td>$\Gamma_s$</td>
<td>517</td>
<td>34.4</td>
</tr>
</tbody>
</table>

$R_m \approx 3.8 \times 10^{16} \left( \frac{R_{WD}}{5000 \text{ km}} \right)^2 \left( \frac{\omega_{WD}}{2\pi \text{/day}} \right)$

$\sim 1.2 \times 10^{12}$ (Jovian activities)

$\sim 1.0 \times 10^8$ (solar activities)
Phase Diagram of Nuclear Matter

deconfinement of quarks ⇔ metallization

hadrons
quark gluon plasmas
insulators
metals

From K. Yagi, T. Hatsuda & Y. Miake "QUARK-GLUON PLASMA"

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Concluding Remark

It is clear from what we have surveyed on the two subjects:

“Scattering of Electromagnetic Waves by a Strongly Correlated Plasma”

and

“Multi-particle Correlation, Equations of State, and Phase Diagrams”

that Norman Rostoker made outstanding contributions to the advancement of the fundamental plasma physics.