MATHEMATICAL PHYSICS

(3 hours)

Closed Book.
Do all problems.

1. a. Define and give an example of the following: entire function, branch point, non-isolated essential singularity, saddle point. (15)

   b. Evaluate the integral (15)

$$\int_0^\infty \frac{\log\,(a^2+x^2)}{a^2+x^2}\,dx$$

2. Polynomials of a certain kind satisfy the differential equation

$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0$$

They have the following contour integral representation:

$$L_n(x) = \frac{1}{2\pi i} \oint \frac{e^{-\frac{xz}{1-z}}}{(1-z)z^{n+1}}\,dz$$

where the contour of integration is a closed curve that encloses the origin and does not enclose $z = 1$.

(10) a. Find the generating function $\sum_{n=0}^{\infty} L_n(x)z^n$

b. Derive the recurrence relations

(10)
$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$$

(10)
$$x\,L_n'(x) = nL_n(x) - nL_{n-1}(x)$$

3. The Green's function $G(\underset{\sim}{x})$ satisfies

$$(\nabla^2 - k^2)G(\underset{\sim}{x}) = -\delta(\underset{\sim}{x})$$

and the boundary condition $\lim\limits_{|\underset{\sim}{x}| \to \infty} G(\underset{\sim}{x}) = 0$

a. Find solutions in one, two, and three dimensions which correspond to a plane source, line source and point
(24) source respectively.

b. By considering the relation between a point source and
(8) a line source, show that

$$\int_{r=\rho}^{\infty} \frac{e^{-kr}}{\sqrt{r^2 - \rho^2}}\, dr = K_o(k\rho)$$

c. By considering the relation between a line source and
(8) a plane source show that

$$\int_{\rho=x}^{\infty} \frac{K_o(k\rho)\,\rho d\rho}{\sqrt{\rho^2 - x^2}} = \frac{\pi}{2}\ \frac{e^{-k|x|}}{k}$$

Useful Relations:

$$J_o(x) = \frac{1}{2\pi}\int_o^{2\pi} d\varphi\ e^{ix\cos\varphi}$$

$$K_o(x) = \int_o^{\infty} y\ dy\ \frac{J_o(y)}{x^2 + y^2}$$

QUANTUM MECHANICS I (Quantitative)

($4\frac{1}{2}$ hours)

Open Book:
One standard reference text allowed.

1. A particle is known to be localized in the left half of a box (infinite square well) with sides at $x = \pm a$. If all values of x in the left half side are equally probable, a) what wave function describes the particle at t=o? b) Will the particle remain localized at later times? c) Calculate the probability that an energy measurement yields the ground state energy; d) the energy of the first excited state.

---

2. Two particles of mass m are placed in an infinite square well potential of side a in the lowest energy state of the system compatible with the conditions below. Assuming that the particles interact with each other according to the potential $V=V_o a\delta(x_1-x_2)$, use first order perturbation theory to calculate the energy of the system under the following conditions:

    a) particles not identical
    b) identical particles of spin 0
    c) identical particles of spin $\frac{1}{2}$ with spins anti-parallel
    d) identical particles of spin 1/2 with spins parallel.

Also, give the unperturbed wave function for the system in each case.

3. The weakness of gravitational compared to electrostatic interaction is dramatically illustrated by considering a system of two neutrons under the sole influence of their mutual gravitational attraction. The gravitational potential is

$$V_G(r) = -\frac{Gm_n^2}{r} \qquad V_{E\&M}(r) = -\frac{e'^2}{r}$$

where $G(=6.67 \times 10^{-11} N\text{-}m^2\text{-}kg^{-2})$ is the gravitational constant, and $m_n(=1.6748 \times 10^{-27} kg)$ is the neutron mass.

a) Give expressions for the bound state energies and for the "Bohr radius" for such a system.

b) Estimate, to the nearest power of ten (order of magnitude) the numerical value of the ground state energy (in electron volts), and of the Bohr radius (in cm).

---

4. Consider two relativistic particles of equal mass μ, moving along the z-direction, and interacting via a one dimensional Coulomb potential. (The "hydrogen" atom in one dimension.) The Hamiltonian of the system is given as (in units of C=1):

$$H = \sqrt{P_1^2 + \mu^2} + \sqrt{P_2^2 + \mu^2} + \gamma|x_1 - x_2| \quad ; \quad \gamma > 0$$

a) Make the transformation to the center of mass and relative coordinates (R,P) and (x,k). Is P a constant? Why?

b) Plot the classical closed orbits in phase space (x,k space) for a given energy E and center of mass momentum P=0. Discuss the motion as given in your diagram.

c) Apply the WKB method of quantization to find the approximate quantized energy levels for fixed P=0. What is the effective mass (M) of the total system ($M^2 = E^2 - P^2$)? Give your answers in the form of an integral. Evaluate the integral for $\mu = 0$.

5. The new ψ-particle spectroscopy can be described by a "charmonium" model. The assumptions of this model are that spin $\frac{1}{2}$ quarks c (charge 2/3) and $\bar{c}$ (charge -2/3) are bound together strongly and:

   a) The $c\bar{c}$ levels can be obtained from the non-relativistic solutions to the Schrödinger equation for an e'^2/r type potential ($e' \neq e$).

   b) The p-levels have "fine" structure due to a spin-orbit term given by $(\vec{L} \cdot \vec{S})$ in the potential.

   c) The spin-spin forces are small.

   d) In the absence of the spin-orbit term, there is a small perturbation term that would put the 2p levels between the 1s and 2s levels.

   e) The ψ (3100 MeV) is to be identified with the 1^3S_1 level, the ψ' (3700 MeV) is to be identified with the 2^3S_1 level.

   Evaluate from this model:

   1) The mass of the charmed quark.

   2) The coupling strength as an "effective fine structure constant" $\alpha' = \frac{e'^2}{\hbar c}$.

   3) The effective radius of the ψ-meson in cm.

   4) The relative splittings of the P levels due to the $(\vec{L} \cdot \vec{S})$ coupling.

   5) The 2^3S_1 (ψ') shows strong radiative transitions to levels at 3552, 3508, 3415 MeV. Evidence suggests that $\chi(3552)$ has $J \geq 2$ and $\chi(3415)$ has $J=0$. Is this consistent with 4)?

6. A scattering experiment is carried out where positrons are scattered through the Coulomb interaction from neutral atoms of the element with Z protons and Z electrons with a point nucleus:
   i) What is the Hamiltonian for this problem?
   ii) Discuss briefly the Hartree-Fock approximation for the treatment of the atomic Hamiltonian.
   iii) Show that the scattering cross section is given in first Born approximation by

$$\frac{d\sigma_{fi}}{d\Omega} = \frac{4Z^2}{\left(\underset{\sim}{q}^2\right)^2} \left(\frac{e^2}{\hbar c}\right)^2 \left(\frac{m_e c}{\hbar}\right)^2 \left(\frac{k_f}{k_i}\right) \left| F_{fi}\,(\underset{\sim}{q}^2) - \delta_{fi}\right|^2 .$$

Here $\underset{\sim}{k}_i, \underset{\sim}{k}_f$ are the initial and final positron wave number; $\underset{\sim}{q} \equiv \underset{\sim}{k}_f - \underset{\sim}{k}_i$ is the three-momentum transfer; the subscripts i, f refer to the initial and final atomic states (with angular momentum quantized along the $\underset{\sim}{q}$ axis); and F_{fi} is the atomic form factor

$$F_{fi}(\underset{\sim}{q}^2) = \frac{1}{Z} \int e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}} \left\langle f \middle| \hat{\rho}(\underset{\sim}{x}) \middle| i \right\rangle d\underset{\sim}{x} .$$

where $\hat{\rho}(\underset{\sim}{x})$ is the atomic charge density operator.

CLASSICAL MECHANICS

(3 hours)

Open Book: One standard reference text allowed.
Answer any 5 problems. All problems have equal weight.

1. A particle under the influence of a central force moves along a spiral trajectory described in polar coordinates by $r = e^{a\theta}$, with the origin at the center of force. Derive the form of the force law.

2. A beam of particles is scattered by a central potential $V(r) = A/r^n$, $(n > 0)$. Consider the limiting situation in which the deflection is small, and derive an expression for the form of the cross-section for small-angle scattering. In particular obtain the form of the dependence of $\frac{d\sigma}{d\theta}$ on momentum and angle θ as $\theta \to 0$.

3. A mass, m, under the influence of gravity, is hung from a point on the edge of a horizontal circular platform of radius a, by means of a massless rigid rod of length ℓ. The rod is constrained so that it can swing only in a radial direction with respect to the platform. The platform is rotated with a small, constant angular velocity ω, about a vertical axis through its center.
   a. Write the Lagrangian for the system in appropriate coordinates;
   b. Find the equation of motion;
   c. Obtain the solution for the motion of the mass, m, assuming only small displacements from equilibrium.

4. A particle of mass m undergoes orbital motion in a circle of radius a under the influence of a central potential of form $V(r) = Ar^n$. Suppose the particle is slightly perturbed and executes small radial oscillations about the circular orbit.

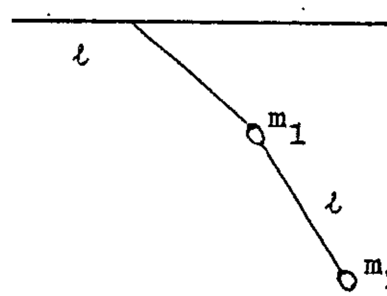
a. Find the frequency of the radial oscillations and compare it to the orbital frequency.

b. What conditions must A and n satisfy for such motion to be possible?

5. Two particles of masses m_1 and m_2 are subject to non-uniform potentials V_1 and V_2, respectively, which differ by a constant factor: $V_2/V_1 = \beta$. Find a relation between the times t_1 and t_2 which the two particles take to traverse the same path. State your reasoning clearly.

6. Two particles of masses m_1, m_2 are suspended from massless rods of equal length ℓ, to form a double pendulum (see Figure). Assume motion is confined to a plane. Find the frequencies and normal modes of the system undergoing small oscillations.

<u>Figure</u>:



Physics Qualifying Examination UCI September 27, 1977

ELECTRICITY AND MAGNETISM

(3 hours)

Open Book: One standard reference text allowed.

Do 4 of the 6 problems.

1. Consider an ionization chamber bounded on opposite sides by two parallel conducting plates of area A and separation d, differing in potential by V_o. Ionization produces electron-ion pairs uniformly through the volume at a rate R m^{-3} sec^{-1}. Now suppose that the electrons are collected in a negligible time by the positive plate, but that the ions drift slowly with velocity $\vec{V} = \mu\vec{E}$, where μ is their mobility. Determine (neglecting edge effects and ion recombination) steady state expressions for the following quantities as a function of position in the chamber.
   a) $V(x)$
   b) $E(x)$
   c) $\rho_+(x)$ (charge density of positive ions)
   d) $J_+(x)$ (current density of positive ions)

2. A thin conducting disk of thickness h, diameter D, and conductivity σ is placed in a uniform alternating magnetic field $B = B_o \sin \omega t$ parallel to the axis of the disk. Neglecting radiation and the self-interaction of currents in the disk, find expressions for:
   a) The average power dissipated in the disk.
   b) The total magnetic field, at distances $r \gg D$ away from the disk.
   c) The conditions that h, D, σ, B_o and ω must satisfy in order to justify the neglect of current self interaction.

3. Fact: The equipotential surfaces associated with two parallel infinite line charges having opposite charge densities $+\lambda$ and $-\lambda$ coulombs/meter and separated by a distance 2a, are circular cylinders of radius R with axes at a distance x from the midpoint between line charges:

$$R = a \operatorname{csch}(2\pi\epsilon_o V/\lambda)$$

$$x = a \coth(2\pi\epsilon_o V/\lambda)$$

   a) Determine the capacitance per unit length of a system consisting of an infinite conducting cylinder of radius R, with axis a distance d away from a parallel infinite conducting plane.
   b) Determine the charge density distribution on the plane in terms of the linear charge density on the cylinder.

4. Consider the following solution to the energy crisis: A charge Q is uniformly distributed over the surface of a spherical shell of radius R. An electric dipole $\vec{p}$ is made to approach the sphere from a large distance, oriented radially so that it is drawn by the field gradient, doing extractable work. When it reaches the surface of the sphere, it passes through a small hole into the interior field-free region, where its direction is reversed at no cost in energy. It is then removed to a large distance, now oriented so that again the field gradient is doing usable work on the dipole. At a remote point its direction is again reversed, and the procedure cycled repeatedly.
   a) Determine the work done on the dipole in approaching the sphere from infinity.
   b) Find the flaw in the scheme, and quantitatively account for the missing energy.

5. A charge of mass m and charge q is subject to a Newtonian friction. Frictional force

$$= - \gamma v ,$$

where γ = constant, v = instantaneous velocity. Upon it is incident a circularly polarized electromagnetic plane wave of intensity I and angular frequency ω in the z direction.

   a) Calculate (in the steady state) the rate at which energy is absorbed from the wave by the particle.
   b) Calculate directly (i.e., by use of the Lorentz force equation) the rate at which angular momentum is absorbed.
   c) How much angular momentum is absorbed per absorbed photon?

6. A radiating quadrupole consists of a square of side a with charges $\pm$ q at alternate corners. The square rotates with angular velocity ω about an axis normal to the plane of the square and through its center. Calculate in a long-wave length approximation:

   a) The quadrupole moments
   b) The radiation fields
   c) The angular distribution of radiation.

Physics Qualifying Examination UCI September 28, 1977

THERMODYNAMICS & STATISTICAL MECHANICS

(3 hours)

Closed Book.

A. Thermodynamics -- do two of the three problems

1. Derive the following relations:

(a) $$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$$

where S is the entropy, V is the volume, p is the pressure and T is the absolute temperature

(b) $$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

where E is the internal energy.

(c) $$C_p - C_V = \frac{TV\alpha^2}{\kappa}$$

where C_p and C_V are the heat capacities at constant pressure and volume, respectively, α is the coefficient of volume expansion, $\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$

and κ is the isothermal compressibility, $-\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$.

2. Consider a gas whose equation of state is

$$p(V-b) = RT\left(1-\frac{a}{RTV}\right) \quad .$$

(a) Prove that C_V is independent of the volume and hence depends only on the temperature.

(b) Derive an expression for the internal energy E which exhibits explicitly the dependence of E on V.

(c) Derive an expression for the entropy S which reveals explicitly the dependence of S on V.

3. Ten grams of a paramagnetic salt obeying Curie's law are in a magnetic field of 100,000 oersteds and at a temperature of 10^{o}K. Assume that the heat capacity is constant at 0.10 cal/gm deg. and that the Curie constant is 0.05 deg./gm.

(a) If the field is reduced reversibly and isothermally to zero, calculate the heat transferred.

(b) If the field is reduced reversibly and adiabatically to zero, calculate the temperature change.

B. Statistical Mechanics -- do two of the three problems

1. The Fermi-Dirac distribution law for the probability f_k that a state with energy ϵ_k is occupied is

$$f_k = \frac{1}{e^{(\epsilon_k - \mu)/k_B T} + 1}$$

where μ is the Fermi energy. Consider a perfect Fermi gas of spin $\frac{1}{2}$ particles characterized by the energy

$$\epsilon_k = \frac{\hbar^2\left(k_x^2 + k_y^2 + k_z^2\right)}{2m}$$

and by the degeneracy parameter

$$y = \frac{nh^3}{2(2\pi m k_B T)^{3/2}}$$

where n is the number density, m is the mass, T is absolute temperature, and $\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$ is the wave vector.

(a) Taking the components of $\vec{k}$ to be specified by periodic boundary conditions applied to a large cubic container, derive a relation between y and μ. This relation should involve an integral over energy. Evaluate this relation explicitly in the limits of weak degeneracy and of strong degeneracy.

(b) Calculate the total kinetic energy and the pressure in the limit of strong degeneracy.

2. A one-dimensional Ising model with cyclic boundary conditions has the Hamiltonian

$$H = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} \quad , \quad \sigma_{N+1} = \sigma_1$$

where J is the exchange constant for nearest neighbor interactions and the possible values of σ_i are ± 1.

(a) Calculate the partition function

(b) In the limit $N \to \infty$, calculate the specific heat at constant volume. Make a rough plot of specific heat versus absolute temperature.

(c) Is there a phase transition?

3. Consider a system of spins each of which has an energy of interaction with an effective magnetic field H_e given by

$$E = g\mu H_e S_z \quad .$$

The possible values of S_z are $0, \pm 1, \pm 2, \ldots \pm S$.

(a) Show that the ensemble average of S_z is given by

$$\langle S_z \rangle = SB_S(\alpha)$$

where $B_S(\alpha)$ is the Brillouin function,

$B_S(\alpha) = \left(1 + \frac{1}{2S}\right) \coth \left(S + \frac{1}{2}\right)\alpha - \frac{1}{2S} \coth \frac{\alpha}{2}$, and

$\alpha = g\mu H_e / k_B T$.

(b) If the effective field H_e is given by

$$H_e = \frac{cJ}{g\mu} \langle S_z \rangle$$

where c is the number of nearest neighbors of a given spin and J is the exchange constant, show that a critical temperature T_c exists defined by $k_B T_c = \frac{1}{3} cJS(S+1)$.

(c) For temperatures just below the critical temperature, show that

$$\langle S_z \rangle \sim (T_c - T)^{\frac{1}{2}} \quad .$$

Note: The following expansion may be useful:

$$\coth x = \frac{1}{x} + \frac{1}{3} x - \frac{1}{45} x^3 \quad .$$

Physics Qualifying Examination September 28, 1977

QUANTUM MECHANICS II (Qualitative)

($1\frac{1}{2}$ hours)

Closed Book.

1. Consider the helium atom. Estimate the ground state energy either by using perturbation theory or by the variational method.

   Potentially useful integrals and formulas are:

$$\int d^3r \quad e^{-2\kappa r} = \frac{\pi}{\kappa^3}$$

$$\int d^3r \quad \frac{e^{-2\kappa r}}{r} = \frac{\pi}{\kappa^2}$$

$$\int d^3r_1 \; d^3r_2 \quad \frac{e^{-2\kappa(r_1+r_2)}}{|\vec{r}_1 - \vec{r}_2|} = \frac{5}{8}\;\frac{\pi^2}{\kappa^5}$$

$$\nabla_r^2 = \frac{1}{r^2}\;\frac{\partial}{\partial r}\;(r^2\,\frac{\partial}{\partial r})$$

2. Given the potential energy

$$V(x) = V_o(x-a)^2(x+a)^2$$

$$V_o = \frac{\kappa a^2}{8}$$

   discuss and sketch the eigen-functions and associated energy eigenvalues for $E<<V_o$ and $E>>V_o$. Discuss the effect on adjacent energy levels as a becomes very large.

3. Molecules sometimes behave like rotators. If rotational spectra are characterized by radiation of wavelength of order 10^7 Å and this is used to estimate interatomic distances in a molecule like H_2, what kind of separations (in Å) are obtained?

4. The Hamiltonian for the Zeeman effect in the presence of spin orbit coupling is given approximately by

$$\hat{H} = \beta(\hat{\vec{L}} + 2\hat{\vec{S}}) \cdot \vec{B} + 2\,\alpha\,\hat{\vec{L}} \cdot \hat{\vec{S}}$$

where β is the Bohr magneton; $\hat{\vec{L}}$ and $\hat{\vec{S}}$ are orbital and spin angular momentum operators respectively and α is the spin-orbit coupling constant. Note that

$$[\hat{H}, \hat{L}^2] = [\hat{H}, \hat{S}^2] = 0$$

so that no mixing occurs between states of different (ℓ, s). Consider <u>one</u> electron in a p-state ($\ell=1, s=\frac{1}{2}$). Describe in words how you would find the energy levels of this electron in the weak field limit and the strong field limit. What basis functions would you use in each limit? Sketch the energy levels and label them with appropriate quantum numbers.

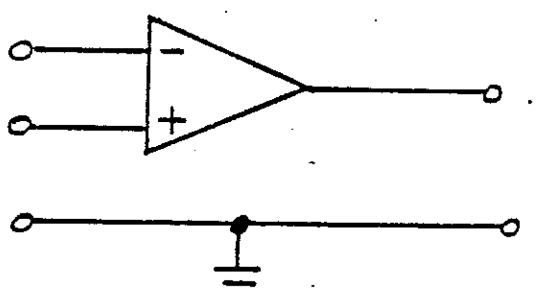
GENERAL PHYSICS

(Strive to do ten of the twelve questions. If you do ten, you are doing extremely well!)

1. Can a photon in vacuum decay into a system of n massive particles? Show your math.

2. Why will an ultra-cold niobium ball levitate in a magnetic field in defiance of gravity? Explain how, by applying a square wave voltage at the natural frequency of oscillation of the levitating ball, and utilizing a SQUID detector, one can look for quarks. Why would a niobium sphere be preferable to a hydrocarbon sphere for this purpose?

3. An ideal operational amplifier has infinite gain, infinite input impedance and zero output impedance. Its symbol is shown at the left. Show how to use an op amp to build:



a) An inverting variable gain amplifier. (Why is it stable?)
b) A non-inverting amplifier
c) A differential amplifier
d) A summation circuit
e) An integrator
f) A differentiator
g) Show how to combine four diodes and a reference oscillator to create a phase-sensitive detector (i.e. heart of a lock-in amplifier).

4. What is a Legendre transformation? Why is it relevant to
   a) the definition of thermodynamic free energy,
   b) the connection between Hamiltonian and Lagrangian mechanics.

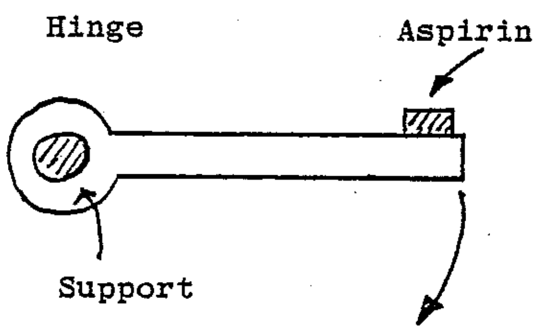
5. Consider a spherical universe of constant mass density ρ. Suppose it expands with a velocity $v(R)$ where

$$v(R) = H_R \quad ,$$

H being the Hubble constant and R the radius. Expansion will continue forever if the kinetic energy of the matter exceeds the potential energy. Find the value of ρ for which this is so, in terms of H.

6. The 1976 Nobel Prize in Physics was awarded to B. Richter of SLAC and S.C.C. Ting of MIT. Describe the nature of their discoveries.

7. A rigid rod pendulum, hinged at one end, is held in place in horizontal position with an aspirin resting on the end. When the rod is let go the end of the rod initially accelerates faster than the aspirin. Explain.



8. A polar semi-conductor such as InSb is doped so that it contains a certain concentration of free carriers. What is the qualitative nature of the optical absorption spectrum from the far infrared to the far ultraviolet?

9. Discuss how quarks make up baryons and mesons. What different kinds of quarks do you know of and what kinds of particles do they make up? What are the angular momenta of the s-wave baryon and meson states in the quark model?

10. A plasma in a fusion device has a number density of 10^{14} cm^{-3} of electrons and ions, each. Their temperature is 10 keV. Make some assumption, then compute the magnetic field necessary to contain the plasma.

11. The earth is slightly flattened at the poles and it bulges at the equator. The plane of the equator is inclined at an angle of about 25^o with respect to the plane of the earth's rotation about the sun. In one model of the ice ages this gives rise to periodic advance and retreat of glaciers from the pole, with a time scale of about 25,000 years. Without doing math, but exploiting the analogy to a well known problem in mechanics, explain the connection. (Hint: It is related to the fact that the Pole star changes from Polaris to Lyra and back on the same 25,000 year time scale.)

12. a) Why is the sky blue? Why is a sunset red?
    b) If a layer of liquid is heated from below what changes occur as the thermal gradient between bottom and top increases?
    c) List the four known fundamental forces of nature in order of increasing strength.