Physics Qualifying Examination

October 2, 1984
2:30 - 5:30 p.m.

QUANTUM MECHANICS

Four problems, equally weighted.

Comment: Please present your solutions in a clear and coherent fashion. You may lose credit if the approach is hard to follow or you present several solutions, only one of which proceeds along the correct lines. So please read each problem, and think it over a bit before you proceed.

1. A spin ½ particle is described by a spin independent Hamiltonian H_o at times $t < 0$, and in fact resides in a wave function that is the ground state of H_o, with energy E_o. For $t < 0$, its spin is aligned along the z direction.

   Between time $t = 0$ and $t = T$, a spatially uniform and time independent magnetic field parallel to the x axis is turned on, i.e., the particle is subject to a term in the Hamiltonian (in appropriate units)

$$V(t) = \begin{cases} 0 & t < 0 \\ h\sigma_x & 0 < t < T \\ 0 & t > T\,. \end{cases}$$

   Find the wave function at all times, and calculate with it the expectation value of the x, y, and z components of magnetic moment of the particle, again at all times.

2. Consider, in three dimensions, a spherically symmetric attractive potential with the form of a thin shell, $V(r) = -V_o\delta(r-r_o)$. Here $\delta(x)$ is the Dirac delta function of argument x.

(a) Find the s wave phase shift, and s wave scattering amplitude for this potential.

(b) From the properties of either the phase shift or scattering amplitude, deduce a criterion for the occurrence of bound states, of s wave character. What are the constrains on V_o for bound states to occur?

(c) Consider two cases: (i) V_o is so weak that there are no bound states. (ii) V_o is just barely strong enough to create a bound state.

For these two cases, sketch the variation with energy of the s wave phase shift and its contribution to the cross section.

3. Consider a hydrogen atom placed in a uniform electric field E_o parallel to the z axis. The field is static, and assumed to be very weak. Discuss the splittings that occur in the n = 3 level. The hydrogen atom is here modeled as an electron of mass m, moving in the potential of an infinitely massive nucleus, $V(r) = -e^2/r$, i.e., complexities of effects such as spin orbit coupling are to be ignored. Do not take the time to evaluate any radial integrals you encounter. Give them names, and express your answer in terms of them. Note the following table:

$Y_{oo} = \frac{1}{\sqrt{4\pi}}$

$Y_{1,\pm1} = \mp(\frac{3}{8\pi})^{1/2}\sin\theta e^{\pm i\phi}$

$Y_{10} = (\frac{3}{4\pi})^{1/2}\cos\theta$

$Y_{2\pm2} = \frac{1}{4}(\frac{15}{2\pi})^{1/2}\sin^2\theta e^{i2\phi}$

$Y_{2\pm1} = \mp(\frac{15}{8\pi})^{1/2}\sin\theta\cos\theta e^{\pm i\phi}$

$Y_{20} = (\frac{5}{4\pi})^{1/2}(\frac{3}{2}\cos^2\theta - \frac{1}{2})$

$Y_{3,\pm3} = \mp \frac{1}{4}(\frac{35}{4\pi})^{1/2}\sin^3\theta e^{\pm i3\phi}$

$Y_{3\pm2} = \frac{1}{4}(\frac{105}{4\pi})^{1/2}\sin^2\theta\cos\theta e^{\pm i2\phi}$

$Y_{3\pm1} = -\frac{1}{4}(\frac{21}{4\pi})^{1/2}\sin\theta(5\cos^2\theta - 1)e^{\pm i\phi}$

$Y_{30} = (\frac{7}{4\pi})^{1/2}(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta)$.

When you finish the analysis, draw an energy level diagram, labeling the degeneracy which remains for each level.

4. (Note that you should be able to answer part (b), even if (a) is incomplete).

We have an attractive potential well in one dimension, $V(x)$. At time $t = -\infty$, a particle resides in the ground state of the well. It is then subjected to a (weak) time dependent electric field of strength E_o and frequency ω, turned on and off by an instrument that has time constant γ. That is, the particle is subjected to the perturbation

$$V(x,t) = -eE_o x \cos(\omega t)e^{-\gamma|t|} ,$$

where $\gamma = 1/\tau$.

(a) There is a certain probability that such a field pulse will eject the particle from the well, into a continuum state. Let $(dP/dE)dE$ be the probability the particle is found with energy between E, and $E + dE$ at time $t = \infty$. Find an expression for (dP/dE). Express it in terms of the matrix element $\langle\psi_k|x|\psi_o\rangle$, where $|\psi_k\rangle$ is the (continuum) wave function of the final state and $|\psi_o\rangle$ that of the ground state, i.e., you need not try to evaluate this matrix element.

(b) A spatially uniform electric field, possibly time dependent, parallel to the x axis, may be described by the electrostatic potential $\phi(x) = -eE_o(t)(x-x_o)$, where x_0 is arbitrary. The form used in part (a) above makes the very special choice $x_o = 0$. Show that the answer of part (a) is independent of the choice of x_o.

Physics Qualifying Examination

October 2, 1984
9:00 - 11:00 a.m.

THERMODYNAMICS AND STATISTICAL PHYSICS

Closed Book/Closed Notes
Problems have equal numerical weight
i.e., each one is 25% of the total

1. Subject: Thermodynamics

The following are experimentally determined properties of water (H_2O):

at 1 atm. water freezes at 0^oC with a latent heat of freezing of 3.34×10^5 J/Kgm. and a volume change of -9.05×10^{-5} m^3/Kgm. n.b. 1 atm = 1.01×10^5 Newtons/m^2.

a) At approximately what pressure at -2^oC does water melt?

b) Will ice skates work at -2^oC? -40^oC? Make reasonable assumptions about body weight and blade size.

2. Subject: Elementary Probability

A frequently used generator of gaussian random numbers is the following:

let $x_1 \ldots\ldots x_{12}$ be random numbers distributed uniformly in [0,1]

let $$y = \sum_{i=1}^{12} (x_i - 1/2)$$

a) Characterize this distribution by its first few moments

$\langle y\rangle$, $\langle y^2\rangle$, $\langle y^3\rangle$, $\langle y^4\rangle$.

b) What gaussian distribution does it resemble? What pattern of moments does a gaussian distribution have? Is the approximation close to the gaussian distribution? What are cumulants?

3. Subject: Classical Ideal Gas and Boltzmann Distribution

A certain pair of atoms A and B form a molecule AB with a bindng energy of 0.01 eV. The atomic weights of the atoms are 10 and 20 respectively. A container of size 1 m^3 contains 0.05 moles of A and B initially. The container is in equilibrium at room temperature (300^oK). A, B, and AB are gasses.

a) What is the partial pressure of the molecule AB?

b) At what temperature, if any, are the three species at an equal partial pressure?

4. Subject: Ideal Fermi Systems.

A crude model of a metal treats the conduction electrons as an ideal Fermi gas contained inside the metal ignoring Coulomb effects. Model copper (Cu) with a density of 9 gm/cm^3 and atomic weight 63.5 and one conduction electron per atom in this way.

a) What is the pressure of this gas?

b) What range of kinetic energies are seen for the electrons?

CONSTANTS AND CONVERSION FACTORS

The constants given below, together with their probable errors, are the corrected 1947 values, computed by J. W. M. DuMond and E. Richard Cohen.*

Normal specific volume of ideal gas	$(22.4146 \pm 0.0006)\,m^3\,mol^{-1}$
Gas constant R	$(8.31436 + 0.00038) \times 10^3$ joule $mol^{-1}\,deg^{-1}$
Avogadro's number N_0	$(6.0251 \pm 0.0004) \times 10^{26}$
Electronic charge e	$(1.60199 \pm 0.00016) \times 10^{-19}$ coul
Electron mass m	$(9.1055 \pm 0.0012) \times 10^{-31}$ kgm
Specific electronic charge e/m	(1.75936 ± 0.00018) coul kgm^{-1}
Proton mass	$(1.67229 \pm 0.0004) \times 10^{-27}$ kgm
Planck's constant h	$(6.6237 \pm 0.0011) \times 10^{-34}$ joule sec
Boltzmann's constant k	$(1.38032 \pm 0.00011) \times 10^{-23}$ joule deg^{-1}
Bohr magneton μ_B	$(0.92736 \pm 0.00017) \times 10^{-23}$ amp m^2
Normal atmospheric pressure	1.013246×10^5 n m^{-2}

1 inch	$= 2.540 \times 10^{-2}$ meter
1 ft	= 0.3048 meter
1 lb (mass)	= 0.4536 kgm
1 kgm-cal	= 4182 joules
1 ft-lb	= 1.356 joules
1 Btu	= 1055 joules
1 lb (force)	= 4.4482 newtons
1 lb/in^2	$= 6.895 \times 10^3$ newtons/m^2
1 atm	$= 1.013 \times 10^5$ newtons/m^2
1 "cm Hg"	= 1333 newtons/m^2
1 lb/ft^3	= 16.02 kgm/m^3

*J. W. M. DuMond and E. Richard Cohen, *Rev. Mod. Phys.* 21, 651(1949).

Physics Qualifying Examination

October 1, 1984
10:00 a.m. - 1 p.m.

MATHEMATICAL PHYSICS

One Textbook - Do all Problems

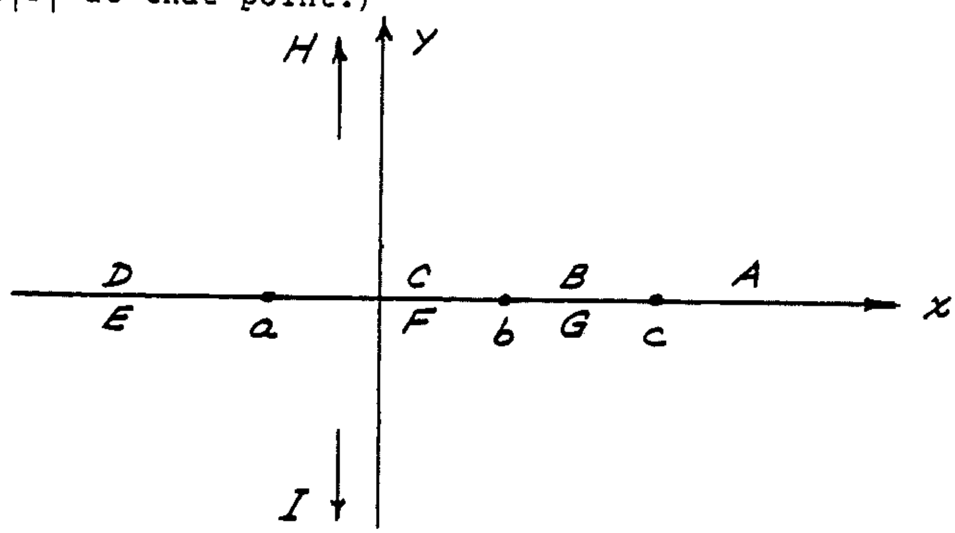
Comment: Please present your solutions in a clear and coherent fashion. You may lose credit if the approach is hard to follow or you present several solutions, only one of which proceeds along the correct lines. So please read each problem and think it over a bit before you proceed.

Points

15 1. Consider the following function of the complex variable z:

$$F(z) = \sqrt{(z - a)(z - b)(z - c)}$$

where a, b, c are real, and a < b < c. Assume that each of the branch cuts required to make this function single-valued goes along the negative real axis to $-\infty$ from the corresponding branch point. Indicate the behavior of F(z) at the several points A, B, C, ... marked on the diagram below in terms of $|F|$. (Thus if F(z) is positive imaginary at one of these points denote this by writing $i|F|$ at that point.)



10 2. What is the interval of convergence of the series

$$f(x) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^n \quad ?$$

Points

20 3. Obtain the first two terms in the small x expansion of the integral

$$I(x) = \int_x^\infty \frac{e^{-u}}{u}\,du \quad .$$

15 4. The Maradudin polynomials $\{M_n(x)\}$ are defined by the generating function

$$\frac{1}{(1 - 3xt + t^2)^{1/3}} = \sum_{n=0}^{\infty} t^n M_n(x) \quad .$$

a) Using only the generating function obtain the relation between $M_n(-x)$ and $M_n(x)$.

b) Obtain the three-term recurrence relation satisfied by these polynomials.

20 5. As an exercise in Fourier transforms and the solution of integral equations solve

$$f(x) = g(x) + \lambda \int_0^\infty \cos xy f(y)\,dy \quad ,$$

where $g(x)$ is an arbitrary, Fourier transformable, function of x. For which values of λ does no solution exist?

20 6. A semi-infinite wire occupies the positive x-axis. Its displacement as a function of time $y(x,t)$ satisfies the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \quad ,$$

where v is called the wave velocity. Find $y(x,t)$ if the string is initially at rest,

$$y\Big|_{t=0} = 0 \quad , \quad \frac{\partial y}{\partial t}\Big|_{t=0} = 0 \quad ,$$

but for times $t > 0$ the end $x = 0$ is oscillated up and down according to

$$y(0,t) = 3\sin 2t \qquad t > 0 \quad .$$

Physics Qualifying Examination

October 1, 1984
2:30 - 5:30 p.m.

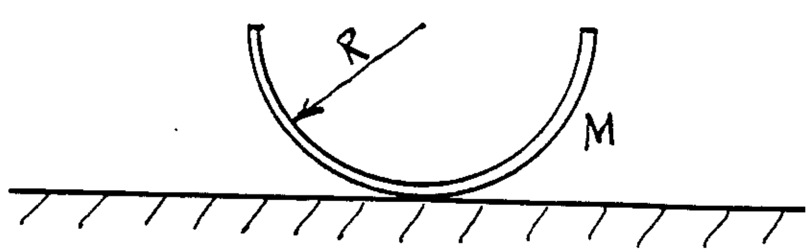
CLASSICAL MECHANICS

This is a three-hour, closed book, closed notes exam. Each problem is worth 20 points. There are 5 problems. To maximize your score, make sure that your solutions are presented clearly. Insert brief comments into your work where appropriate and cross off any false starts or side tracks.

1. An hourglass filled with sand is placed on a very sensitive scale. With all of the sand in the bottom half of the hourglass, the weight indicated by the scale is W. The hour glass is then inverted and placed on the scale. What weight does the scale indicate while the sand is falling? Prove it. Assume each grain of sand has mass M and that the rate which grains leave the top half of the hourglass is $\dot{n}$. Explicitly define any other variables you use. Ignore the "jitter" in the indicated weight that can be made arbitrarily small by designing the scale properly.

2. A hollow, semicylindrical shell of radius R and mass M rests round side down on a horizontal, frictionless table. If given a light tap, it will execute small oscillations.
   a) What is the frequency of the oscillations?
   b) If the table were rough so that the shell rolled without sliding, would the frequency be higher, lower, or unchanged? Briefly explain your answer. (No calculation required.)

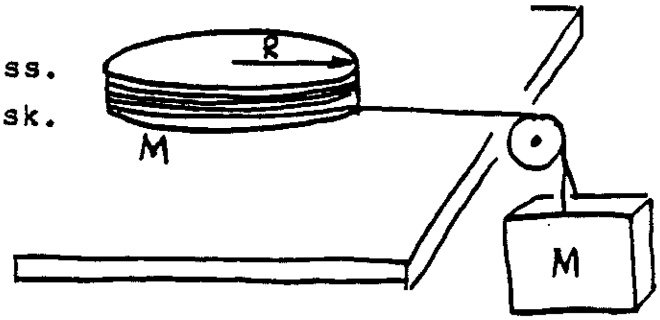
(Hint: The parallel axis theorem says $I = I_{cm} + Md^2$, where I_{cm} is the moment of inertia about an axis through the center of mass and I is the moment about a parallel axis a distance d away.)



3. A flat disk of radius R and mass M, lies flat side down on a horizontal frictionless table. It is free to move on the table. A massless string is wrapped around the disk many times. The free end of the string passes over a massless, frictionless pulley at the table edge and is tied to a weight of mass M. Calculate:

   a) The acceleration of the mass.

   b) The acceleration of the disk.

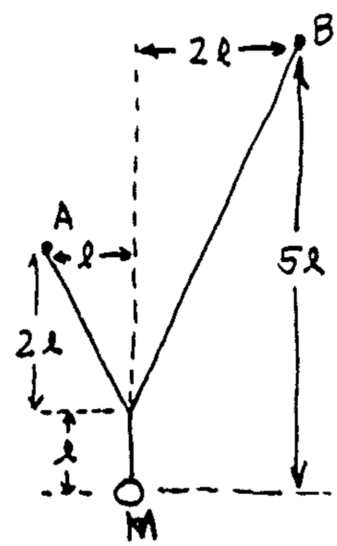


4. Three massless strings are tied together to support a mass, M. The length of the string between the mass and the (massless) knot at C is ℓ. The leftmost string is supported at A. A is a vertical distance 2ℓ above C and a horizontal distance ℓ to the left of C. The rightmost string is supported at B. B is a vertical distance 5ℓ above the knot at C and a horizontal distance 2ℓ to the right of it. Only consider motions small enough so that all three strings remain taut.

   a) Describe the normal modes. (No calculation required)

   b) Write down the exact Lagrangian.

   c) Calculate the normal mode frequencies for small oscillations.



5. A particle of charge q and mass m is constrained to move on the surface of a frictionless sphere of radius R. The sphere is in a region of constant magnetic field $\vec{B} = B\hat{z}$. (Recall that the presence of a B field adds a term $\frac{q}{c}\vec{A} \cdot \vec{v}$ to the Lagrangian. $\vec{A}$ is the vector potential such that $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{v}$ is the particle's velocity.) Ignore the self fields of the particle.

A. Write down the Lagrangian in terms of ϕ, the azimuthal angle, and θ , the polar angle measured from the z axis.

B. Find the momenta conjugate to ϕ and θ, P_ϕ and P_θ.

C. Find $H(\theta, \phi, P_\theta, P_\phi)$.

D. Let $\vec{J}$ be the total angular momentum of the particle. Is J^2 conserved? Explain. Use the appropriate Poisson bracket to prove your claim.

E. At $t = 0$, $\theta = \frac{\pi}{2}$, $\phi = 0$, $\dot{\theta} = \dot{\theta}_0$ and $\dot{\phi} = 0$. Show that $\theta_{min} < \theta(t) < \theta_{max}$. Find expressions for θ_{min} and θ_{max}.

Physics Qualifying Examination

October 3, 1984
10 a.m. - 12 noon

ELECTRICITY AND MAGNETISM

The exam period is two hours and is to be taken without the use of books or notes. All three problems carry the same point possibility. Problem #2 will probably require more calculations than the other problems. In the event you cannot complete a problem, outline in as much detail as possible how you would go about solving the problem if more time were available.

Good luck and may you have a field day.

Table 3

Conversion table for symbols and formulas

The symbols for mass, length, time, force, and other not specifically electromagnetic quantities are unchanged. To convert any equation in Gaussian variables to the corresponding equation in mks quantities, on both sides of the equation replace the relevant symbols listed below under "Gaussian" by the corresponding "mks" symbols listed on the right. The reverse transformation is also allowed. Since the length and time symbols are unchanged, quantities which differ dimensionally from one another only by powers of length and/or time are grouped together where possible.

Quantity	Gaussian	mks
Velocity of light	c	$(\mu_0\epsilon_0)^{-1/2}$
Electric field (potential, voltage)	$\mathbf{E}(\Phi, V)$	$\sqrt{4\pi\epsilon_0}\,\mathbf{E}(\Phi, V)$
Displacement	$\mathbf{D}$	$\sqrt{\frac{4\pi}{\epsilon_0}}\,\mathbf{D}$
Charge density (charge, current density, current, polarization)	$\rho(q, \mathbf{J}, I, \mathbf{P})$	$\frac{1}{\sqrt{4\pi\epsilon_0}}\rho(q, \mathbf{J}, I, \mathbf{P})$
Magnetic induction	$\mathbf{B}$	$\sqrt{\frac{4\pi}{\mu_0}}\,\mathbf{B}$
Magnetic field	$\mathbf{H}$	$\sqrt{4\pi\mu_0}\,\mathbf{H}$
Magnetization	$\mathbf{M}$	$\sqrt{\frac{\mu_0}{4\pi}}\,\mathbf{M}$
Conductivity	σ	$\frac{\sigma}{4\pi\epsilon_0}$
Dielectric constant	ϵ	$\frac{\epsilon}{\epsilon_0}$
Permeability	μ	$\frac{\mu}{\mu_0}$
Resistance (impedance)	$R(Z)$	$4\pi\epsilon_0 R(Z)$
Inductance	L	$4\pi\epsilon_0 L$
Capacitance	C	$\frac{1}{4\pi\epsilon_0}C$

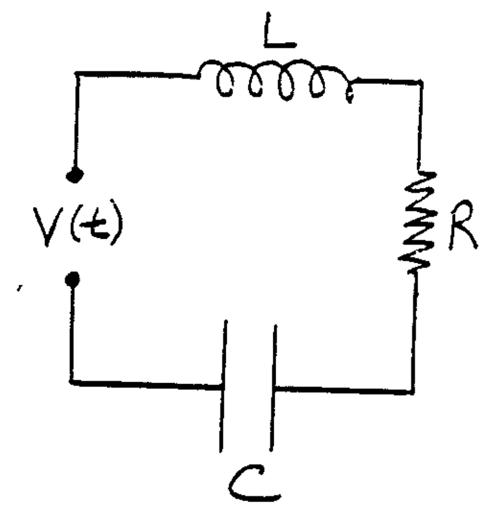
Table 4

Conversion table for given amounts of a physical quantity

The table is arranged so that a given amount of some physical quantity, expressed as so many mks or Gaussian units of that quantity, can be expressed as an equivalent number of units in the other system . Thus the entries in each row stand for the same amount, expressed in different units. All factors of 3 (apart from exponents) should, for accurate work, be replaced by (2.997930 ± 0.000003), arising from the numerical value of the velocity of light. For example, in the row for displacement (D), the entry ($12\pi \times 10^{5}$) is actually ($2.99793 \times 4\pi \times 10^{5}$). Where a name for a unit has been agreed on or is in common usage, that name is given. Otherwise, one merely reads so many Gaussian units, or mks units.

Physical Quantity	Symbol	Rationalized mks	Gaussian	
Length	l	1 meter (m)	10^{2}	centimeters (cm)
Mass	m	1 kilogram (kg)	10^{3}	grams (gm)
Time	t	1 second (sec)	1	second (sec)
Force	F	1 newton	10^{5}	dynes
Work Energy	W U	1 joule	10^{7}	ergs
Power	P	1 watt	10^{7}	ergs sec^{-1}
Charge	q	1 coulomb (coul)	3×10^{9}	statcoulombs
Charge density	ρ	1 coul m^{-3}	3×10^{3}	statcoul cm^{-3}
Current	I	1 ampere (coul sec^{-1})	3×10^{9}	statamperes
Current density	J	1 amp m^{-2}	3×10^{5}	statamp cm^{-2}
Electric field	E	1 volt m^{-1}	$\frac{1}{3} \times 10^{-4}$	statvolt cm^{-1}
Potential	Φ, V	1 volt	$\frac{1}{300}$	statvolt
Polarization	P	1 coul m^{-2}	3×10^{5}	dipole moment cm^{-3}
Displacement	D	1 coul m^{-2}	$12\pi \times 10^{5}$	statvolt cm^{-1} (statcoul cm^{-2})
Conductivity	σ	1 mho m^{-1}	9×10^{9}	sec^{-1}
Resistance	R	1 ohm	$\frac{1}{9} \times 10^{-11}$	sec cm^{-1}
Capacitance	C	1 farad	9×10^{11}	cm
Magnetic flux	ϕ, F	1 weber	10^{8}	gauss cm^{2} or maxwells
Magnetic induction	B	1 weber m^{-2}	10^{4}	gauss
Magnetic field	H	1 ampere-turn m^{-1}	$4\pi \times 10^{-3}$	oersted
Magnetization	M	1 ampere m^{-1}	10^{-3}	magnetic moment cm^{-3}
*Inductance	L	1 henry	$\frac{1}{9} \times 10^{-11}$	

1.) Consider the circuit shown at right, subjected to $V(t) = V_o \cos \omega t$. Find the following:



i) the charge, q, on the capacitor as a function of ω and plot the result. Note: q is a real quantity;

ii) the maximum power dissipated by the resistor;

iii) the Q of the circuit if $R = 10^{-1}\Omega$, $L = 10^{-5}$H, $C = 10^{-6}$F.

iv) Give an example of where and why such a circuit might be used for a physics experiment.

2.) Consider the intensity of sunlight as it passes through the atmosphere of the earth to an observer on the ground. What fraction of blue light ($\lambda = 4500Å$) is scattered out of the observer's direct line of sight to the sun when the sun is directly overhead and when the sun is a few degrees above the horizon? Comment qualitatively on why clouds are white (hint: compare atomic dimensions to light wavelengths). Why does smog lead to pretty sunsets?

Notes to help you:

1.) Several justifiable approximations and estimates must be made to get the final numbers.

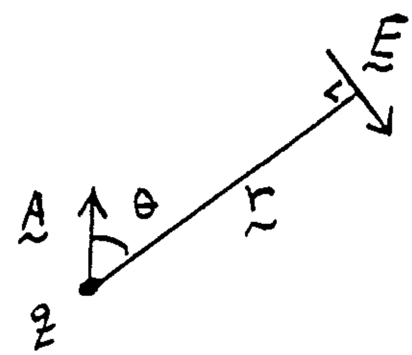
2.) The natural frequencies, ω_0, of electrons in air atoms are such that $\omega_0 >> \omega$ of sunlight. Thus, the scattering cross section is modified from the Thompson cross section.

3.) Far field electric field (in MKS) of a charge accelerated as shown in the figure is approximately

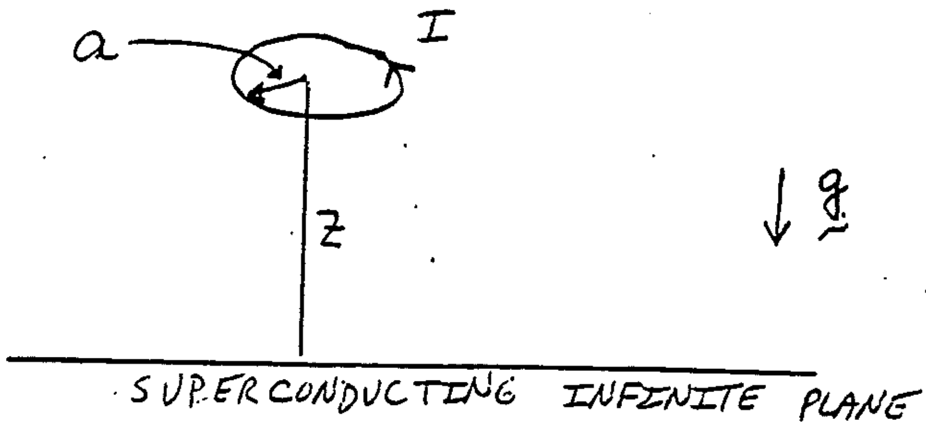
$$E(t) = \frac{-qA_{RET}\sin\theta}{4\pi\epsilon_0 c^2 r} \quad \text{MKS}$$

$$= \frac{-qA_{RET}\sin\theta}{c^2 r} \quad \text{CGS}$$

where A_{RET} is the acceleration at the retarded time $t-r/c$.



3.) Consider a circular current loop brought from infinity to a distance z above an infinite superconducting plane, as shown in the figure. There is a uniform, constant gravitational field as shown. For a loop with large current, I, and small mass, m, is there an equilibrium position (and if so, what is it?) for a << z? If there is an equilibrium position, is it stable?



NOTE: A current loop in free space has spherical coordinates, see figure below, of the magnetic field given by:

$$B_r \simeq \frac{I\pi a^2}{c} \cos\theta \frac{(2a^2 + 2r^2 + ar\sin\theta)}{(a^2 + r^2 + 2ar\sin\theta)^{5/2}}$$

$$B_\theta \simeq \frac{-I\pi a^2}{c} \sin\theta \frac{(2a^2 - r^2 + ar\sin\theta)}{(a^2 + r^2 + 2ar\sin\theta)^{5/2}}$$

$B_\varphi = 0$ in CGS units

in the limit $a \gg r$, $a \ll r$, or $\theta \ll 1$.