Physics Qualifying Examination September 28, 1987

8:30 - 11:30 a.m.

CLASSICAL MECHANICS

1. This problem is designed so that the successive parts can be done (making the necessary assumptions and approximations) even if the preceding ones have not been completely solved. If you are doing one of the later parts without having solved the earlier ones, explain your assumption and notation. (Total points 40)

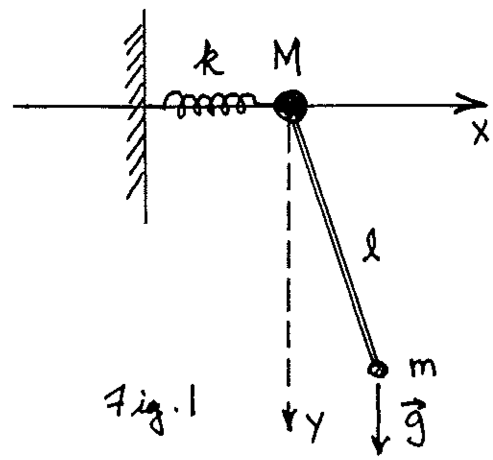


Fig. 1

Consider a dynamical system (in the plane) consisting of a large mass M constrained to move along a horizontal line (x-axis) and attached by a spring of negligible mass and spring constant to a rigid wall. A simple pendulum (consisting of a mass $m \ll M$ and a bar of length ℓ and negligible mass) is attached to the mass M and can oscillate around it (see Fig. 1)

(Points)

a) (5) Determine the number of degrees of freedom of this sytem, choose appropriate generalized coordinates and write down the complete Lagrangian for this system.

b) (5) Without making any approximations, write down the Lagrange equations of motion and say what kind of equations they are (linear, nonlinear, first order, second order, etc.).

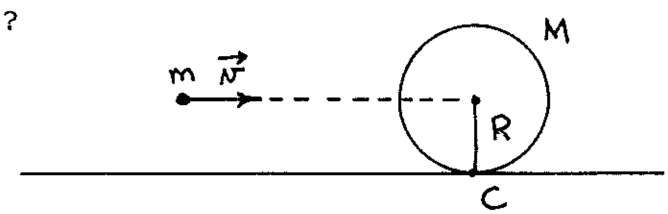
c) (5) Now consider small oscillations around the equilibrium position (defined by the minimum of the potential energy) and write down the appropriate bilinear (quadratic) approximation to the Lagrangian and the corresponding linear equations of motion.

d) (5) Find the generalized momenta and the Hamiltonian in this approximation.

e) (10) Find the normal modes (eigenfrequencies and normal coordinates) -- either by diagonalizing the Lagrangian or by solving the coupled linear equations. You may make the simplifying assumption of "weak coupling", i.e., $m \ll M$.

f) (5) Assuming that $\frac{k}{M} = \frac{g}{\ell}$ (what is the meaning of this assumption?), describe qualitatively the motion of the system if the bob of the pendulum is released with no initial velocity from an angle $\phi_0 \neq 0$ and M is at its equilibrium position.

g) (5) What effects do you expect (qualitatively) for the frequencies of the coupled system when nonlinearities are taken into account to lowest order in the kinetic and potential energies?



2. (Total points 15)

A uniform cylinder of mass M and radius R is initially at rest on a horizontal plane. A ball of mass m and initial velocity $\underline{v}$ parallel to the plane and perpendicular to the axis of the cylinder collides elastically with the cylinder, striking it on a diameter passing through the cylinder's center of mass (distance R from the contact point C with the plane).

(Points)

a) (5) Which dynamical quantities are conserved in this collision and which are not? Explain your answer(s).

b) (5) Assuming pure rolling motion of the cylinder (and using only conservation laws) determine the angular and linear velocity of the cylinder and the velocity of the ball after the collision. (The principal moment of inertia of a cylinder with respect to its axis is $I_3 = 1/2\ MR^2$.)

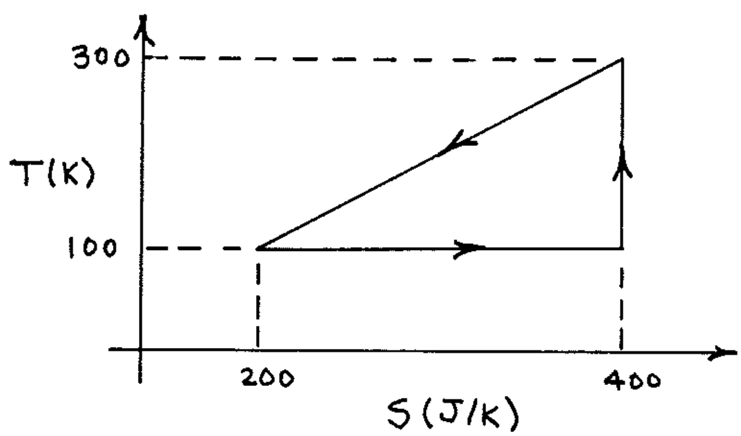
c) (5) For what value of the ratio M/m does the ball come to rest after the collision?

3. (Total points 10) Answer one of the following two questions.

   a) Explain (qualitatively) the precession and nutation of a heavy symmetric top and give a quick derivation of the precession frequency for the case when the kinetic energy of rotation is large compared to the gravitational potential energy.

   b) State the conditions on a transformation of generalized coordinates and momenta to be a canonical transformation.

4. A reversible thermodynamic machine executes the following cycle:



a) (5) Is the machine an engine or a refrigerator? Explain

b) (10) The efficiency, η, of an engine is the ratio of the work done to the heat absorbed per cycle. The efficiency of a refrigerator is the ratio of the heat absorbed to the work done on the refrigerator per cycle. Calculate the efficiency of this machine.

2. A system consisting of N spin 1/2 particles lies in a constant external magnetic field. The particles with S_z = 1/2 have energy ε, the particles with S_z = - 1/2 have energy 0. The system is completely isolated so that the total energy of the spins, U, is constant. Stirlings' approximation is

$$\ln(N!) = N \ln N - N \text{ (for large N)}$$

a) (5) How many states are available to the spins?

b) (5) What is the entropy of the spins?

c) (5) What is the temperature of the spins?

d) (5) Use your answer to a or b and the second law to show that U can never exceed $N\varepsilon/2$ if energy is added to the system only by placing it in contact with arbitrarily hot objects.

3. A certain gas obeys the van der Waals equation of state

$$\left(P + \frac{a}{v^2}\right)(v - b) = RT ,$$

where P is the pressure, v the molar volume, T the Kelvin temperature, and a, b, and R are constants.

a) (10) Calculate the Helmholtz free energy of the gas as completely as possible.

b) (5) How could the arbitrary function of T that appears in your answer to a) be experimentally determined?

4) Consider a system of noninteracting, spin zero bosons confined to move in a plane. There are N bosons per unit area.

a) (10) Calculate the chemical potential as a function of the temperature for fixed N.

b) (5) Does the system undergo a Bose-Einstein condensation? If so, calculate T_{BE}. In either case, explain your answer.

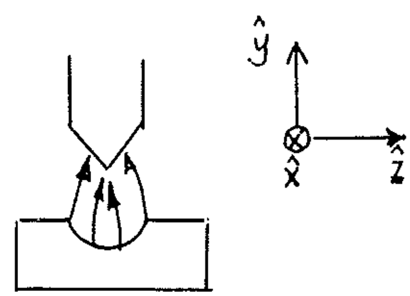
Physics Qualifying Examination — September 28, 1987, 3 - 6 p.m.

QUANTUM MECHANICS

You are allowed one textbook (either Schiff or Baym) and a book of math tables. Work all five problems; show all steps.

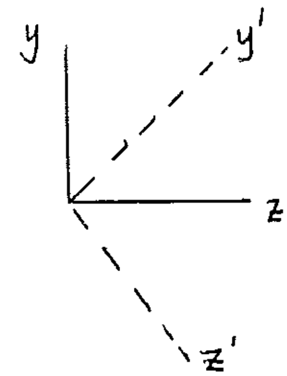
1. Given the potential $V(r) = \beta\delta(\underline{r})$.
   a) In Born approximation, calculate the phase shifts $\delta_\ell(E)$ for a particle of mass m scattering off this potential.
   b) Calculate and sketch total cross section $\sigma(E)$ versus energy E from a).
   c) Using conservation of probability, at what energy must your result b) break down?
   d) For $\beta = 100$ MeV$(f)^3$ and $mc^2 = 10^3$MeV, what is the break down E in c)?
   NOTE: $1\ f = 10^{-13}$ cm. Take $\hbar c = 200$ MeV f.

2. A beam of neutral spin 1/2 particles travelling in the x direction is prepared with spin polarization with respect to the z-axis described by the spinor $\sqrt{\frac{1}{10}}\begin{pmatrix} 2+i \\ i(2-i)\end{pmatrix}$. The beam is passed between the poles of a magnet with field pointing in the y-direction, as shown.



   a) What is the fraction of particles deflected to the positive y-direction?

   b) The co-ordinate system is rotated 45° clockwise about the x-axis. What is the spinor in the new reference frame xy'z'?



3. Given a quadrupole tensor operator $Q_m^{\ell=2}$

a) What is the minimum angular momentum j that a state must have for a non-vanishing diagonal matrix element of Q_m^2 ?

b) Given $|\langle j_2 = 2\ m_2 = 1|Q_o^2|j_1 = 1\ m_1 = 1\rangle|^2 = 17$
Use the Wigner-Eckart theorem and the attached table of Clebsch-Gordan coefficients to evaluate

$$|\langle 22|Q_1^2|11\rangle|^2$$

$$|\langle 2-1|Q_{-1}^2|10\rangle|^2$$

$$|\langle 21|Q_o^2|10\rangle|^2 \ .$$

4. Given the free Dirac equation

$$(E - c\underline{\alpha} \cdot \underline{p} - \beta mc^2)\psi = 0 \ . \qquad \text{(a)}$$

Write $\psi = \binom{\phi_1}{\phi_2}$ where ϕ_i $i = 1,2$ each have 2 components. Derive from (a) 2 equations for ϕ_2 in terms of ϕ_1. Show by direct manipulation that they are equivalent.

5. At $t < 0$ an electron is known to be in the ground state of a 1D infinite square well potential which extends from $x = 0$ to $x = L$. At $t = 0$ a uniform electric field is applied, pointing in the direction of increasing x; it is turned off after a short time τ. What is the probability that the electron will be in the first excited state for $t > \tau$? Plot the probability as a function of τ.

CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A $\sqrt{\ }$ is to be understood over every coefficient; e.g., for -8/15 read $-\sqrt{8/15}$.

Notation:

	J	J ...
	M	M ...
m_1 m_2		
m_1 m_2	Coefficients	
. .		

1/2 × 1/2

m_1	m_2	J=1, M=+1
+1/2	+1/2	1

m_1	m_2	J=1, M=0	J=0, M=0
+1/2	-1/2	1/2	1/2
-1/2	+1/2	1/2	-1/2

m_1	m_2	J=1, M=-1
-1/2	-1/2	1

1 × 1/2

m_1	m_2	J=3/2, M=+3/2
+1	+1/2	1

m_1	m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1	-1/2	1/3	2/3
0	+1/2	2/3	-1/3

m_1	m_2	J=3/2, M=-1/2	J=1/2, M=-1/2
0	-1/2	2/3	1/3
-1	+1/2	1/3	-2/3

m_1	m_2	J=3/2, M=-3/2
-1	-1/2	1

2 × 1/2

m_1	m_2	J=5/2, M=+5/2
+2	+1/2	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	-1/2	1/5	4/5
+1	+1/2	4/5	-1/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1	-1/2	2/5	3/5
0	+1/2	3/5	-2/5

m_1	m_2	J=5/2, M=-1/2	J=3/2, M=-1/2
0	-1/2	3/5	2/5
-1	+1/2	2/5	-3/5

m_1	m_2	J=5/2, M=-3/2	J=3/2, M=-3/2
-1	-1/2	4/5	1/5
-2	+1/2	1/5	-4/5

m_1	m_2	J=5/2, M=-5/2
-2	-1/2	1

3/2 × 1/2

m_1	m_2	J=2, M=+2
+3/2	+1/2	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+3/2	-1/2	1/4	3/4
+1/2	+1/2	3/4	-1/4

m_1	m_2	J=2, M=0	J=1, M=0
+1/2	-1/2	1/2	1/2
-1/2	+1/2	1/2	-1/2

m_1	m_2	J=2, M=-1	J=1, M=-1
-1/2	-1/2	3/4	1/4
-3/2	+1/2	1/4	-3/4

m_1	m_2	J=2, M=-2
-3/2	-1/2	1

2 × 1

m_1	m_2	J=3, M=+3
+2	+1	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+2	0	1/3	2/3
+1	+1	2/3	-1/3

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	-1	1/15	1/3	3/5
+1	0	8/15	1/6	-3/10
0	+1	6/15	-1/2	1/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1	-1	1/5	1/2	3/10
0	0	3/5	0	-2/5
-1	+1	1/5	-1/2	3/10

m_1	m_2	J=3, M=-1	J=2, M=-1	J=1, M=-1
0	-1	6/15	1/2	1/10
-1	0	8/15	-1/6	-3/10
-2	+1	1/15	-1/3	3/5

m_1	m_2	J=3, M=-2	J=2, M=-2
-1	-1	2/3	1/3
-2	0	1/3	-2/3

m_1	m_2	J=3, M=-3
-2	-1	1

1 × 1

m_1	m_2	J=2, M=+2
+1	+1	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+1	0	1/2	1/2
0	+1	1/2	-1/2

m_1	m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1	-1	1/6	1/2	1/3
0	0	2/3	0	-1/3
-1	+1	1/6	-1/2	1/3

m_1	m_2	J=2, M=-1	J=1, M=-1
0	-1	1/2	1/2
-1	0	1/2	-1/2

m_1	m_2	J=2, M=-2
-1	-1	1

3/2 × 1

m_1	m_2	J=5/2, M=+5/2
+3/2	+1	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	-2/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2	-1	1/10	2/5	1/2
+1/2	0	3/5	1/15	-1/3
-1/2	+1	3/10	-8/15	1/6

m_1	m_2	J=5/2, M=-1/2	J=3/2, M=-1/2	J=1/2, M=-1/2
+1/2	-1	3/10	8/15	1/6
-1/2	0	3/5	-1/15	-1/3
-3/2	+1	1/10	-2/5	1/2

m_1	m_2	J=5/2, M=-3/2	J=3/2, M=-3/2
-1/2	-1	3/5	2/5
-3/2	0	2/5	-3/5

m_1	m_2	J=5/2, M=-5/2
-3/2	-1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\ \cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\ \sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\ \sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\ \sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\ Y_\ell^m e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

3/2 × 3/2

m_1	m_2	J=3, M=+3
+3/2	+3/2	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	-1/2

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2	-1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	-2/5
-1/2	+3/2	1/5	-1/2	3/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2	-3/2	1/20	1/4	9/20	1/4
+1/2	-1/2	9/20	1/4	-1/20	-1/4
-1/2	+1/2	9/20	-1/4	-1/20	1/4
-3/2	+3/2	1/20	-1/4	9/20	-1/4

m_1	m_2	J=3, M=-1	J=2, M=-1	J=1, M=-1
+1/2	-3/2	1/5	1/2	3/10
-1/2	-1/2	3/5	0	-2/5
-3/2	+1/2	1/5	-1/2	3/10

m_1	m_2	J=3, M=-2	J=2, M=-2
-1/2	-3/2	1/2	1/2
-3/2	-1/2	1/2	-1/2

m_1	m_2	J=3, M=-3
-3/2	-3/2	1

$$d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2} \qquad d^1_{0,0} = \cos\theta$$

2 × 3/2

m_1	m_2	J=7/2, M=+7/2
+2	+3/2	1

m_1	m_2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	-3/7

m_1	m_2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	-1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	-2/5
0	+3/2	2/7	-18/35	1/5

m_1	m_2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2	-3/2	1/35	6/35	2/5	2/5
+1	-1/2	12/35	5/14	0	-3/10
0	+1/2	18/35	-3/35	-1/5	1/5
-1	+3/2	4/35	-27/70	2/5	-1/10

m_1	m_2	J=7/2, M=-1/2	J=5/2, M=-1/2	J=3/2, M=-1/2	J=1/2, M=-1/2
+1	-3/2	4/35	27/70	2/5	1/10
0	-1/2	18/35	3/35	-1/5	-1/5
-1	+1/2	12/35	-5/14	0	3/10
-2	+3/2	1/35	-6/35	2/5	-2/5

m_1	m_2	J=7/2, M=-3/2	J=5/2, M=-3/2	J=3/2, M=-3/2
0	-3/2	2/7	18/35	1/5
-1	-1/2	4/7	-1/35	-2/5
-2	+1/2	1/7	-16/35	2/5

m_1	m_2	J=7/2, M=-5/2	J=5/2, M=-5/2
-1	-3/2	4/7	3/7
-2	-1/2	3/7	-4/7

m_1	m_2	J=7/2, M=-7/2
-2	-3/2	1

2 × 2

m_1	m_2	J=4, M=+4
+2	+2	1

m_1	m_2	J=4, M=+3	J=3, M=+3
+2	+1	1/2	1/2
+1	+2	1/2	-1/2

m_1	m_2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	-3/7
0	+2	3/14	-1/2	2/7

m_1	m_2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	-1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	-1/14	-3/10
0	+1	3/7	-1/5	-1/14	3/10
-1	+2	1/14	-3/10	3/7	-1/5

m_1	m_2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2	-2	1/70	1/10	2/7	2/5	1/5
+1	-1	8/35	2/5	1/14	-1/10	-1/5
0	0	18/35	0	-2/7	0	1/5
-1	+1	8/35	-2/5	1/14	1/10	-1/5
-2	+2	1/70	-1/10	2/7	-2/5	1/5

m_1	m_2	J=4, M=-1	J=3, M=-1	J=2, M=-1	J=1, M=-1
+1	-2	1/14	3/10	3/7	1/5
0	-1	3/7	1/5	-1/14	-3/10
-1	0	3/7	-1/5	-1/14	3/10
-2	+1	1/14	-3/10	3/7	-1/5

m_1	m_2	J=4, M=-2	J=3, M=-2	J=2, M=-2
0	-2	3/14	1/2	2/7
-1	-1	4/7	0	-3/7
-2	0	3/14	-1/2	2/7

m_1	m_2	J=4, M=-3	J=3, M=-3
-1	-2	1/2	1/2
-2	-1	1/2	-1/2

m_1	m_2	J=4, M=-4
-2	-2	1

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2} \qquad d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2} \qquad d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2 \qquad d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1) \qquad d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Sign convention is that of Wigner (Group Theory, Academic Press, New York, 1959), also used by Condon and Shortley (The Theory of Atomic Spectra, Cambridge Univ. Press, New York, 1953), Rose (Elementary Theory of Angular Momentum, Wiley, New York, 1957), and Cohen (Tables of the Clebsch-Gordan Coefficients, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The signs and numbers in the current tables have been calculated by computer programs written independently by Cohen and at LBL. (Table extended April 1974.)

Physics Qualifying Examination September 29, 1987

8:30 - 11:30 a.m.

ELECTRICITY AND MAGNETISM

Do 4 of the 6 problems. If you attempt more than 4, please indicate clearly which 4 problems you wish to have graded.

You may refer to one textbook of your choosing. Answers are to be expressed in CGS=Gaussian units (the ones used in Jackson).

1. Two large parallel conducting plates, each of area A, are separated by distance d. All edge effects may be taken to be negligible. A homogeneous anisotropic dielectric fills the space between the plates. The dielectric permittivity tensor ε_{ij} relates the electric displacement $\vec{D}$ and the electric field $\vec{E}$ according to $D_i = \sum_{j=1}^{3} \varepsilon_{ij} E_j$.

The principal axes of this permittivity tensor are:

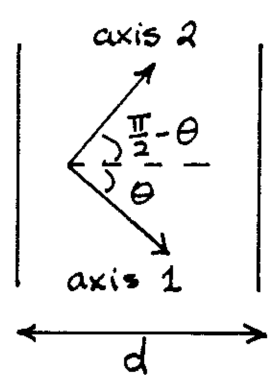
Axis 1 (with eigenvalue ε_1) is in the plane of the paper at an angle θ below the horizontal.

Axis 2 (with eigenvalue ε_2) is in the plane of the paper at an angle $\frac{\pi}{2} - \theta$ above the horizontal.

Axis 3 (with eigenvalue ε_3) is perpendicular to the plane of the paper.

a) Free charges $+ Q_F$ and $- Q_F$ are uniformly distributed on the left and right conducting plates, respectively. Find $\vec{D}$ and $\vec{E}$ within the dielectric. Specify the horizontal and vertical components of these vectors and express your answers in terms of Q_F, A, ε_i, and θ. HINT: The $\vec{D}$ field will not be perpendicular to the plates.

b) Calculate the capacitance of this system in terms of A, d, ε_i, and θ.



2. A large number N of electrons move ultrarelativistically in a region of space permeated by a uniform magnetic field B. The electrons all have the same Lorentz factor γ ($\gamma \gg 1$), but the distribution of velocity directions is isotropic. The effect of the electrons on each other may be neglected. Under these circumstances the power radiated by the electrons is directly proportional to the magnetic energy density associated with the external field B.

Prove this result and find the constant of proportionality.

3. A thin horizontal disk of radius R and height h << R has a uniform magnetization $\vec{M}$ which points up, parallel to the axis of the disk.

a) Find the fields $\vec{B}$ and $\vec{H}$ (magnitude and direction) at the center of the disk.

b) Find the fields $\vec{B}$ and $\vec{H}$ (magnitude and direction) at a point P on the axis of the disk a distance d above the center. You may assume d >> h, but d is the same order of magnitude as R.

4. A perfectly flat metallic sheet that is also perfectly conducting coincides with the xy plane. A second identical sheet is parallel to the xy plane at z = D.

In the region between z = 0 and z = D, electromagnetic waves can propagate. Obtain a description of these waves. It is easiest to suppose they propagate parallel to the x direction, so all field components exhibit a dependence on x and t of the form $\exp(ikx-i\omega t)$. Deduce a relationship between ω and k, and show that if the frequency $\omega < c\pi/D$, such a set of parallel plates acts as a polarizer.

5. A loop of copper wire has radius a, and carries a DC current I. It is placed a distance d above the flat surface of a material with magnetic permeability μ and dielectric constant ε. Find an expression for the force on the loop, and comment on its direction. Confine your attention to the case where $d \gg a$.

6. A spatially uniform, infinitely extended plasma is placed in a static uniform magnetic field $\vec{B}$ parallel to the z-axis. It then can be described by the frequency dependent dielectric tensor

$$\underset{\approx}{\varepsilon}(\omega) = \begin{pmatrix} \varepsilon_1(\omega) & +i\varepsilon_2(\omega) & 0 \\ -i\varepsilon_2(\omega) & \varepsilon_1(\omega) & 0 \\ 0 & 0 & \varepsilon_3(\omega) \end{pmatrix}$$

where $\varepsilon_1(\omega) = 1 + \frac{\omega_p^2}{\omega_c^2 - \omega^2}$, $\varepsilon_2(\omega) = \frac{\omega_p^2 \omega_c}{\omega} \frac{1}{\omega_c^2 - \omega^2}$, and

$\varepsilon_3(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$. Here $\omega_p^2 = \frac{4\pi n e^2}{m}$ is the plasma frequency, and

$\omega_c = eB/mc$ is the cyclotron frequency.

a. Discuss the nature of the electromagnetic waves which propagate parallel to the magnetic field.

b. An electromagnetic wave of frequency ω propagates along the magnetic field. In the xy-plane, it has the electric field E_0 parallel to the x-axis. Find an expression for the electric field everywhere else along the z-axis. Describe the nature of the wave, after examining your result.

Physics Qualifying Examination September 29, 1987
3 - 6 p.m.

MATHEMATICAL PHYSICS

1. Evaluate $\int_0^{2\pi} \frac{\cos\theta \, d\theta}{(1+a\cos\theta)}$ where $1 > a > 0$.

Show all work.

2. Find the solution $u(x,y)$ of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} = 1$$

subject to the boundary conditions

$$u(0,y) = \phi(y)$$

$$\frac{\partial u}{\partial x}(0,y) = \psi(y) \quad .$$

3. Find the resolvent kernel $R(x,y,\lambda)$ for the integral equation

$$f(x) = g(x) + \lambda\int_0^1 (1+xy)f(y)dy \quad .$$

4. Obtain a variational estimate of the two smallest eigenvalues of the following boundary-value problem:

$$xy'' + y' + \lambda xy = 0$$

$$y(0) = 1, \; y'(1) = 0 \quad .$$

5. Obtain the Green's function $G(x|y)$ that satisfies

$$\left(x\frac{d^2}{dx^2} + \frac{d}{dx} - \frac{\nu^2}{x}\right)G(x|y) = \delta(x-y)$$

$$0 < x, \; y < 1$$

subject to the boundary conditions

$$G(0|y) = G(1|y) = 0 \quad .$$

Assume that ν is greater than zero.

6. The generating function for the Laguerre polynomials $\{L_n(x)\}$ is

$$\frac{e^{-\frac{xt}{1-t}}}{1-t} = \sum_{n=0}^{\infty} L_n(x)t^n \qquad |t| < 1 \quad .$$

Evaluate the integral

$$I_n = \int_0^{\infty} dx\ e^{-x}x^n L_n(x) \quad .$$