Physics Qualifying Exam

Classical Mechanics/Thermal Physics

Monday, September 18, 1989

8:30 - 11:30 A.M.

Part A: Classical Mechanics

(Closed book, closed notes)

Classical Mechanics

Closed book

Problem 1: The Kinetic energy of a system of N particles in cartesian coordinates is

$$T = \sum_{i=1}^{N}[\frac{1}{2}m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)] \quad (1)$$

(a). What is the most general expression for the kinetic energy in terms of generalized coordinates and generalized velocities, $(q_1, q_2...q_{3N})$ and $(\dot{q}_1, ...\dot{q}_{3N})$, where the x's are related to the q's by

$$x_i = x_i(q_1, q_2...q_{3N}, t) \quad (2)$$

(b) Under what conditions is T only a quadratic function of genralized velocities ?

(c) If the Lagrangian is not an explicit function of time then show that the quantity A

$$A = \sum_k \dot{q}_k \partial L/\partial \dot{q}_k - L \quad (3)$$

is a constant of motion.

(d) Give an example where the above quantity is not the total energy of the system and explain why.

Problem 2:

(a) Define the Hamiltonian of a system with generalized coordinates q and $\dot{q}$ in terms of its Lagrangian.

(b) Derive Hamilton's canonical equations of motion for the system in terms of Poissan brackets.

(c) Is the following transformation canonical ?

$$P = q \times cot(p) \tag{4}$$

$$Q = ln(q^{-1} \times sin(p)) \tag{5}$$

(d) For a simple harmonic oscillator whose Hamiltonian is given by

$$H = \frac{p^2}{2m} + \frac{k}{2} \times q^2 \tag{6}$$

carry out a canonincal transformation from q,p to Q,P such that the new Hamiltoninan is cyclic in in Q and solve for P(t), Q(t) and q(t).

Problem 3: Consider small oscillations of a pendulum of mass m and lenght l on the surface of the earth at a latitude of 30°. Take $\hat{z}$ to be the local zenith, $\hat{x}$ to be east and $\hat{y}$ to be north. Neglect the vertical displacement of the pendulum as being a quantity of second order in smallness.

(a) Write down the equation of motion for the pendulum in a coordinate system fixed on the earth's surface.

(b) If Ω is the angular velocity of earth's rotation, omitting terms of the order of Ω^2, derive the coupled equations for x and y motions.

(c) Assuming $\Omega_z \ll \omega_0$ the natural frequency of the pendulum obtain an equation for $\eta = x + iy$, where $i = \sqrt{-1}$ and solve for the motion. Explain why the solution implies rotation of the plane of oscillation of the pendulum.

(d) Estimate the value of Ω_z and compare it with the natural period of a pendulum of 30 m length.

Physics Qualifying Exam

Classical Mechanics/Thermal Physics

Monday, September 18, 1989

8:30 - 11:30 A.M.

Part B: Thermal Physics

(Closed book/closed notes)

I. General Knowledge

1. State the temperaure dependence of the following:

   a. the internal energy of an ideal gas

   b. the heat capacity of a Fermi gas at low temperature.

   c. the heat capacity of an insulating crystal at low temperature

II.

A thermally insulated container is divided into two parts by a thermally insulated partition. Both parts contain ideal gases. One of these parts contains N_1 particles of gas at a temperature T_1 and pressure P_1; the other contains N_2 particles of gas at a temperature T_2 and pressure P_2. The partition is now removed and the system is allowed to come to equilibrium.

1. Find the final pressure.

2. Find the change of total entropy ΔS if the gases are different.

3. Find ΔS if the gases are identical.

III.

Consider a crystal composed of N particles. Each particle has two energy states with energies 0 and ϵ .

1. Calculate the Helmholtz free energy F.

2. Calculate the entropy S.

3. Calculate the internal energy E.

4. Calculate the heat capacity at constant volume C_V. Analyze the behavior of this quantity for high and low temperatures; state clearly what high and low mean in this case.

Physics Department Qualifying Exam

Electromagnetic Theory Section

Fall 1989

* There are six problems. Do two out of the first three, and two out of the second three problems. Four problems total.

Materials Allowed

(Jackson, Math. Handbook; closed notes)

Problem 1:

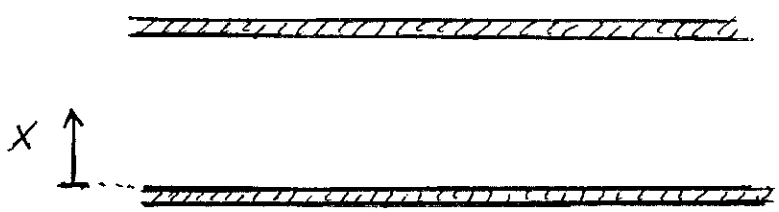
(a) A dielectric is in the form of a hollow sphere. The inner radius is R_1, the outer radius is R_2, and ϵ is

(50%) the dielectric constant of the material. There is also a uniform, positive charge density throughout the dielectric; its total charge is Q.

Find expressions for the electric field everywhere in space.

(b) One plate of a parallel plate capacitor is kept at the potential $\phi = 0$ and the other at $\phi = V$. The

(50%) capacitor contains a space charge density between the plates $\rho = \frac{1}{2} k x^2$, where k is a constant, and x is measured from the plate where $\phi = 0$ (See sketch).



Determine the potential ϕ as a function of x.

Problem 2

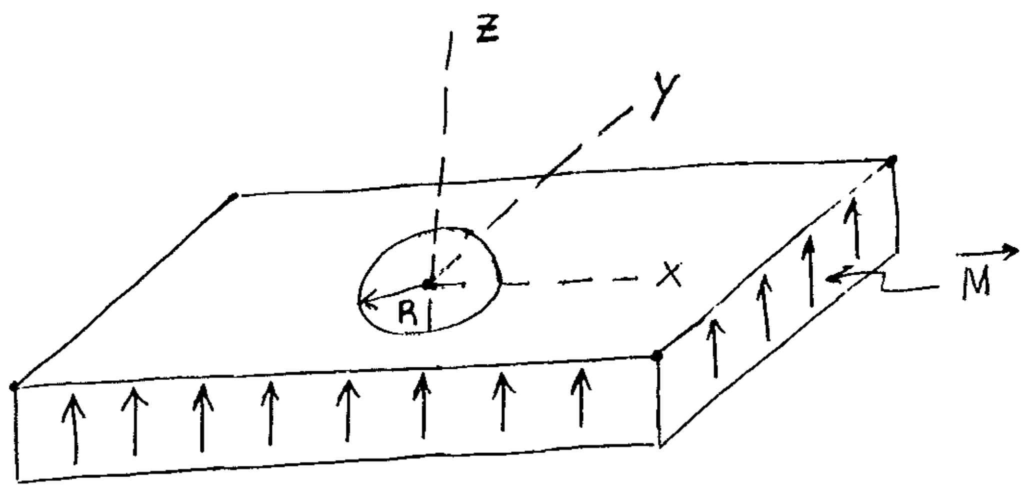
A radio tuning capacitor has maximum capacitance of 100pF ($1pF = 10^{-12}$ F). By rotation of the moving plates, the capacity can be reduced to 10pF. Assume the capacitor is charged to a potential difference of 300V, at maximum capacitance. The capacitor is isolated, and then the tuning knob is rotated to minimum capacitance.

(50%) (a) What is the final value of the potential difference.

(50%) (b) How much mechanical work is done in rotating the knob?

Problem 3

A slab of thickness L is fabricated from ferromagnetic material with magnetization $\vec{M}$ that is constant in magnitude and direction everywhere. The magnetization is perpendicular to the two surfaces, as shown. A hole of radius R is drilled in the slab. Find an expression for $\vec{H}$ and $\vec{B}$ everywhere on the z axis.



The upper and lower surfaces of the slab are planes of infinite extent, parallel to the xy plane.

Problem 4

A straight piece of wire has length ℓ, and carries current $I(t) = I_0 \exp(-i\omega t)$. The wire is parallel to the z direction. Clearly, it emits electromagnetic radiation. The length ℓ is small compared to the wavelength of the radiation emitted.

(20%) (a) Draw a sketch which shows the wire, and which defines the "near zone" and the "far zone" of electromagnetic radiation theory.

(60%) (b) Obtain an expression for the electric and magnetic field in the far zone, and use these to calculate the power per unit solid angle emitted by the wire. Identify the nature of the emitted radiation (electric dipole, magnetic dipole, electric quadrupole, ...)

(20%) (c) Find an expression for $\vec{B}$ in the near zone.

Problem 5

Consider a plasma with plasma frequency ω_p, occupying the half space $z > 0$, while vacuum occupies the half space $z < 0$. Investigate solutions of Maxwell's equations in which the field components exhibit the space and time variation

$$e^{i(kx-\omega t)} \cdot e^{-\alpha z} \qquad , \qquad z > 0$$

$$e^{i(kx-\omega t)} \cdot e^{+\alpha_0 z} \qquad , \qquad z < 0$$

corresponding to a wave localized at the surface of the plasma, and propagating parallel to it, with wave vector k and frequency ω. Confine your attention to TM waves (magnetic field perpendicular to the xz plane), and show that a dispersion relation (relation between frequency ω and wave vector k) follows by requiring all electromagnetic boundary conditions be obeyed at the surface.

Problem 6

An extremely thin, electrically neutral plasma sheet coincides with the xy plane. Within the sheet, a plasma wave of wave vector q and frequency ω is present. As a consequence, there is a current density

$$\vec{J}\,(x,z) = \hat{x}\, A\, \delta(z)\, \cos(qx-\omega t)$$

everywhere in the xy plane.

(20%) (a) Show that there must also be a non-zero charge density which accompanies the wave, and obtain an explicit expression for it.

(80%) (b) Find expressions for the electric and magnetic fields everywhere. From the nature of the solution, state the condition that must be satisfied for the plasma to lose energy by radiation.

(Hint: Don't just write down all possible Maxwell equations and slug away. Use physical reasoning to decide which components of $\vec{E}$ and $\vec{B}$ are non-zero.)

Qualifying Examination - Mathematical Physics

Tuesday, September 19, 1989; 2:00–5:00 pm

Do 6 out of the 7 problems; show all your work and reasoning. Partial credit will be given for correct reasoning without final results! Some problems are constructed in such a manner that the individual parts can be solved without having done all the preceding parts.

Put the solutions to problems 1–4 in one bluebook, and the solutions to problems 5–7 in another.

1. Locate and classify the singularities of the following functions of a complex variable:

(i) $f(z) = \sqrt{z-1}$,

(ii) $f(z) = \tan\left(\frac{1}{z}\right)$,

(iii) $f(z) = \frac{z-2}{z^2}\sin\left(\frac{1}{1-z}\right)$,

(iv) $f(z) = \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x-z}dx$.

2. The Hermite polynomials have the following contour integral representation

$$H_n(z) = \frac{n!}{2\pi i}\oint e^{-t^2+2tz}\frac{dt}{t^{n+1}}.$$

a. Find the generating function

$$f(x,s) = \sum_{n=0}^{\infty} \frac{H_n(x)\, s^n}{n!}. \tag{1}$$

b. Derive the recurrence relations

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{2}$$

$$H_n'(x) = 2nH_{n-1}(x). \tag{3}$$

c. Show that the Fourier transform of the Weber-Hermite function $\psi_n(x) = e^{-x^2/2}H_n(x)$ is

$$\tilde{\psi}_n(x) = \sqrt{2\pi}i^{-n}\psi_n(k) = \sqrt{2\pi}i^{-n}H_n(k). \tag{4}$$

3. Find the Green's function for an elastic medium $G_{ij}(\mathrm{x})$ so that

$$u_i(\mathrm{x}) = \int G_{ij}(\mathrm{x}\text{-}\mathrm{x}')F_j(\mathrm{x})d\mathrm{x}' \tag{5}$$

is the solution of the differential equation

$$\frac{\partial^2}{\partial x_i \partial x_j} u_i(\mathrm{x}) + \frac{1}{1-2\sigma}\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) = -\frac{2(1+\sigma)}{E}F_i(\mathrm{x}), \tag{6}$$

where σ and E are Poisson's ratio and the elastic modulus, respectively. Both are assumed to be constant. The boundary conditions are

$$\lim_{|\mathrm{x}|\to\infty} u_i(\mathrm{x}) = \lim_{|\mathrm{x}|\to\infty} F_i(\mathrm{x}) = 0. \tag{7}$$

$F_i(\mathrm{x})$ is a given force field. Summation over repeated indices goes from 1 to 3; $\mathrm{x} = (x_1, x_2, x_3)$.

4. Consider the symmetry of an equilateral triangle that is described by the group of symmetry operations — a 3-fold axis of rotation and three 2-fold axes. This six-element group is called D_3 or $(3,2)$. Consider an ion in a d-state which is 5-fold degenerate; the wave functions are of the form $Y_l^m(\theta,\varphi)f_l(r)$, with $l = 2$ and $m = -2,-1,0,1,2$. A perturbing potential with the symmetry of D_3 will split this level or remove some of the degeneracy. Determine by group theory methods how this level will be split by the perturbing potential.

For reference the following characters are given for irreducible representations:

D_3 Group:

	I	A	C
Γ_2	2	-1	0
Γ_1	1	1	-1
I	1	1	1

Rotation Group:

$$\chi_l(\varphi) = \sum_{m=-l}^{l} e^{im\varphi} = \frac{\sin(l+\frac{1}{2})\varphi}{\sin\frac{\varphi}{2}}$$

I is the identity; A means rotation through 120° about the 3-fold axis; C means rotation through 180° about a 2-fold axis.

5. A perfectly flexible homogeneous string of linear mass density ρ and length l rotates uniformly in a horizontal plane around an axis, with angular velocity ω. Neglecting gravity, its equilibrium position in rotating coordinate system is assumed to be along the x-axis.

a. Show that the equation governing small vibrations around this "equilibrium" (denoting by $u(x,t)$ the transverse displacement of the string at position x and time t) is

$$\frac{\partial}{\partial x}\left[\left(l^2 - x^2\right)\frac{\partial u}{\partial x}\right] = \frac{2}{\omega^2}\frac{\partial^2 u}{\partial t^2}, \tag{8}$$

with the boundary conditions

$$u(0,t) = 0, \quad u(l,t) = \text{finite}. \tag{9}$$

b. Find a complete set of normal modes for this string.

Hint: Reduce to a Legendre equation by scaling $\xi = \frac{x}{l}$ and don't overlook the boundary conditions.

c. Write down the formal solution (as expansion in normal modes) for the initial value problem:

$$u(x,0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x). \tag{10}$$

d. Express the results of question **c** in terms of a Green's function.

e. Find the solution of the problem with the initial condition:

$$u(x,0) = 0.1\left(\frac{x}{l}\right)^3, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0. \tag{11}$$

Useful formulas:

Normalization integral for Legendre polynomials:

$$\int_{-1}^{+1} \left[P_n(y)\right]^2 dy = \frac{2}{n+1}$$

Needed Legendre polynomials:

$$P_1(y) = y, \quad P_3(y) = \frac{5}{2}y^3 - \frac{3}{2}y$$

6. Consider the integral equation:

$$y(s) - \lambda \int_{-1}^{+1} (st + s^2t^2)\, y(t)\, dt = f(s).$$

a. What kind of equation is it and what do you expect to find by just looking at the kernel?

b. Determine the eigenvalues and eigenfunctions of the homogeneous equation.

c. Find the solution of the inhomogeneous equation for $\lambda = \frac{5}{2}$ and $f(s) = s^3$. Why was it possible to find the solution and what is its degree of arbitrariness?

7. Consider the 2-dimensional Riemannian manifold with the metric

$$ds^2 = dr^2 + r^2 d\phi^2, \qquad 0 < r < \infty, \qquad 0 \le \phi < 2\pi.$$

a. Write down the differential equations of the geodesics for this metric.

b. Show that the following two expressions are first integrals (i.e., are constant along the geodesics) of these equations:

$$r^2 \frac{d\phi}{ds} = R_0 = \text{const.}, \qquad \text{(A)}$$

$$\left[\frac{dr}{ds}\right]^2 + r^2 \left[\frac{d\phi}{ds}\right]^2 = 1. \qquad \text{(B)}$$

c. Use the two equations (A) and (B) to find a first-order differential equation for the geodesics in the form $r = r(\phi)$, and show that the solutions of this equation are straight lines expressed in polar coordinates.

Quantum Physics Qualifying

9/19/89 8:30 - 11:30

Do all five problems.

You may bring one textbook, no notes.

Problem 1

(a) Consider a Hamiltonian H, with eigenstate ψ, and eigenvalue E:

$$H\psi = E\psi$$

Suppose H depends on a real parameter λ. Derive an expression for $\frac{\partial E}{\partial \lambda}$ in terms of the expectation value of $\frac{\partial H}{\partial \lambda}$.

(b) For the harmonic oscillator

$$H = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 x^2$$

use the theorem in part (a) and the known energy eigenvalues to evaluate the expectation value of x^2 in an arbitrary eigenstate of H.

Problem 2

Consider a harmonic oscillator interacting with a spin-1/2 system. Neglecting zero point energy, the Hamiltonian is

$$H = \hbar\omega a^{+}a + \hbar\omega \left(\frac{1+\sigma_z}{2}\right) + \hbar\lambda \left(a^{+}\sigma_- + a\sigma_+\right)$$

where, as usual, a, a^{+} are annihilation and creation operators for the oscillator, and

$$\sigma_{\pm} \equiv \frac{\sigma_x \pm i\sigma_y}{2}$$

with σ_x, σ_y, σ_z ≡ Pauli spin matrices.

(a) Find an operator (other than H) which corresponds to a conserved quantum number of the system.

(b) Suppose we start at t = 0 in a state with one quantum of energy in the oscillator and the spin-1/2 system with spin pointing down $(\sigma_z = -1)$. Find the probability that, at some arbitrary later time t, the spin is pointing up.

Problem 3

The electrostatic potential of a neutral atom of atomic number Z is, to a good approximation, a screened Coulomb potential:

$$\Phi(r) = \frac{Ze}{r} e^{-r/a}$$

Consider scattering of a non-relativistic particle of charge Q and mass m from this potential.

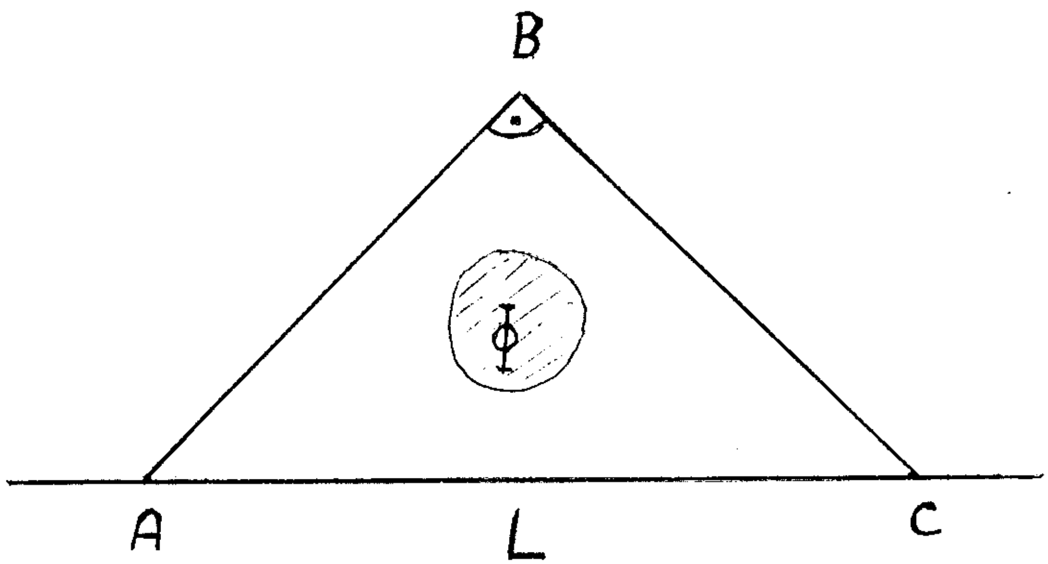
(a) Using the Born Approximation, find the differential scattering cross-section expressed explicitly as a function of θ.

(b) Find the total cross-section.

(c) Identify and interpret the results found in (a) above in the limit $a \to \infty$.

Problem 4

At point A an electron beam is split into two beams. One travels undeflected, while the other one is bent by 45^o. The latter is deflected through 90^o at point B and intersects the original beam at point C.



L is the AC distancs. A magnetic flux Φ (perpendicular to the paths) is present and confined so as not to intersect the beams.

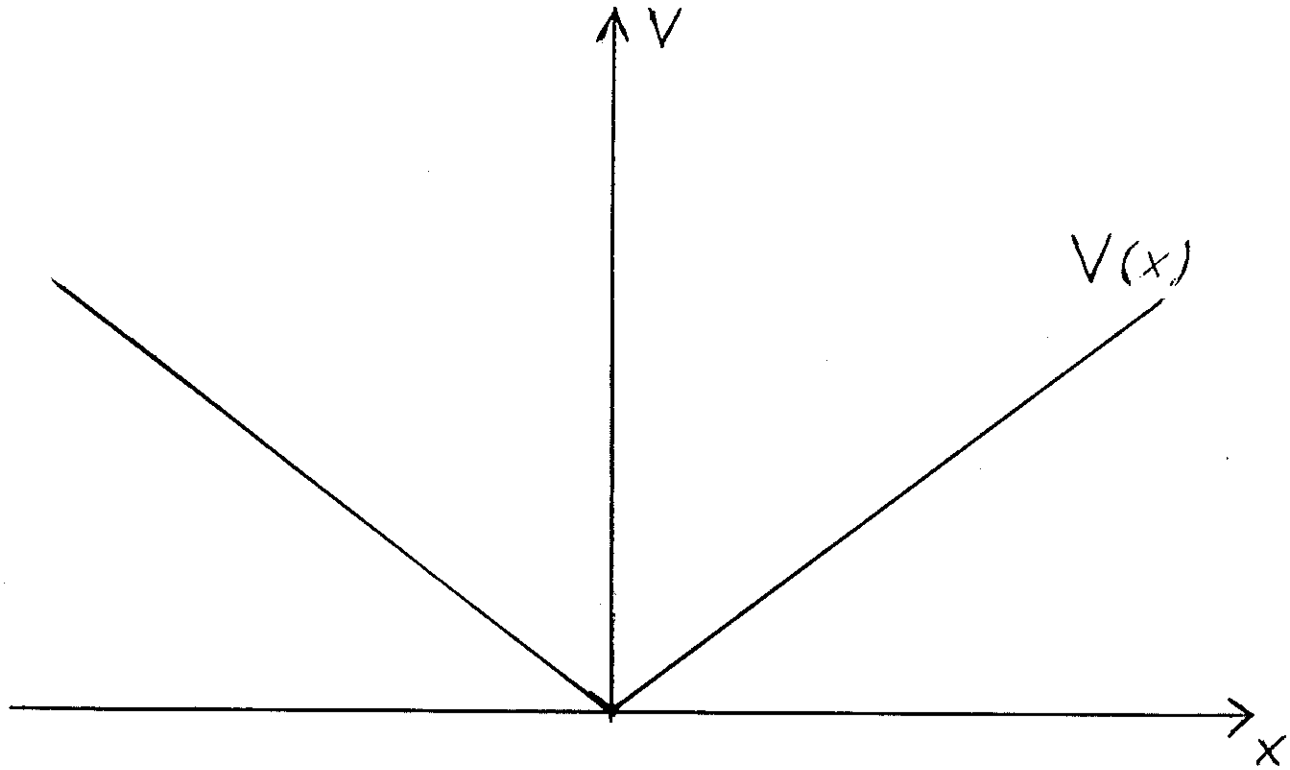
(i) Find the intesity at point C as a function of L, Φ, charge e, and the beam momentum p.

(ii) For what values of Φ will this intensity be a maximum?

(iii) a minimum?

Problem 5

A particle of mass m moves in a potential $V(x) = a|x|$



Use semiclassical methods to:

(1/3 credit) a. Estimate the ground state energy, E_0.

(2/3 credit) b. Estimate the magnitude of the wave function at a point x, where $x > E_0/|a|$.