MASTER

Qualifying Examination

Quantum Mechanics

September 18, 1990 - 8:30-11:30 A.M.

<u>One open book</u>

<u>Do all problems</u>

Physics Qualifying Exam

Thermal Physics

Answer all problems as completely as possible. If additional assumptions or mathematical approximations are required to make progress on a problem, be sure to state them clearly. Liberal partial credit will be given for correct reasoning.

1. A torsional oscillator has moment of inertia I and the restoring torque τ is linearly related to the angular deflection θ, with $\tau = k\theta$. If the oscillator (which can be considered a classical object) is in thermal equilibrium with a temperature bath at temperature T, what is the probability distribution of θ ? At what temperature does the assumption of classical behavior break down?

2. The residual pressure in a common laboratory vacuum device is 10^{-6}torr. At this pressure and room temperature, estimate the mean free path due to molecular collisions. (10^3 Torr$\approx$1atmo$\approx 10^6$ dyne/cm^2)

3. Sketch a plot of the entropy of a crystalline solid as a function of temperature. Specify the low and high temperature behaviors. What does low and high temperature mean?

4. Sketch a plot of the thermal conductivity of an ideal gas as a function of the pressure. Specify the low and high pressure behavior. What does low and high pressure mean?

5. Two phases of a one component system coexist at temperature T_x and pressure P_x . Sketch a plot of the chemical potential of the two phases (at $P=P_x$) as a function of temperature. What is the physical significance of the slopes of the curves?

6. $\left(\frac{\partial P}{\partial T}\right)_V$ is related to another partial derivative via a Maxwell's relation. What is the related derivative?

7. Consider a system of non-interacting particles of mass m which are confined to move on the surface of a sphere of radius R. The Hamiltonian for a single particle involves the angular part of the Laplacian operator. The eigenfunctions are simply Y_{lm}/R ,where the Y_{lm} are the standard spherical harmonics. The energy eigenvalues are $E=h^2 l(l+1)/2m$ with degeneracy $2l+1$. A statistical analysis of a gas of such particles utilizes the density of states $\rho(E)$. The number of single particle states with energy below E is given by

$$\int_0^E \rho(E)\,dE$$

a) What is $\rho(E)$ for this system?

b) If the particles are bosons and the chemical potential is fixed by a reservoiur at $\mu=\bar{\mu}$, write an expression for the internal energy of the gas as a function of T in terms of $\rho(E)$.

c) If the particles are fermions, write an expression for the total number of particles N on the sphere as a function of $\bar{\mu}$ and T in terms of $\rho(E)$.

Table 4
Conversion Table for Given Amounts of a Physical Quantity

The table is arranged so that a given amount of some physical quantity, expressed as so many MKSA or Gaussian units of that quantity, can be expressed as an equivalent number of units in the other system. Thus the entries in each row stand for the same amount, expressed in different units. All factors of 3 (apart from exponents) should, for accurate work, be replaced by (2.99792456), arising from the numerical value of the velocity of light. For example, in the row for displacement (D), the entry ($12\pi \times 10^5$) is actually ($2.99792 \times 4\pi \times 10^5$). Where a name for a unit has been agreed on or is in common usage, that name is given. Otherwise, one merely reads so many Gaussian units, or MKSA or SI units.

Physical Quantity	Symbol	Rationalized MKSA	Gaussian	
Length	l	1 meter (m)	10^2	centimeters (cm)
Mass	m	1 kilogram (kg)	10^3	grams (gm)
Time	t	1 second (sec)	1	second (sec)
Frequency	ν	1 hertz (Hz)	1	hertz (Hz)
Force	F	1 newton	10^5	dynes
Work Energy	W U	1 joule	10^7	ergs
Power	P	1 watt	10^7	ergs sec^{-1}
Charge	q	1 coulomb	3×10^9	statcoulombs
Charge density	ρ	1 coul m^{-3}	3×10^3	statcoul cm^{-3}
Current	I	1 ampere (amp)	3×10^9	statamperes
Current density	J	1 amp m^{-2}	3×10^5	statamp cm^{-2}
Electric field	E	1 volt m^{-1}	$\frac{1}{3} \times 10^{-4}$	statvolt cm^{-1}
Potential	Φ, V	1 volt	$\frac{1}{300}$	statvolt
Polarization	P	1 coul m^{-2}	3×10^5	dipole moment cm^{-3}
Displacement	D	1 coul m^{-2}	$12\pi \times 10^5$	statvolt cm^{-1} (statcoul cm^{-2})
Conductivity	σ	1 mho m^{-1}	9×10^9	sec^{-1}
Resistance	R	1 ohm	$\frac{1}{9} \times 10^{-11}$	sec cm^{-1}
Capacitance	C	1 farad	9×10^{11}	cm
Magnetic flux	ϕ, F	1 weber	10^8	gauss cm^2 or maxwells
Magnetic induction	B	1 tesla	10^4	gauss
Magnetic field	H	1 ampere-turn m^{-1}	$4\pi \times 10^{-3}$	oersted
Magnetization	M	1 ampere m^{-1}	10^{-3}	magnetic moment cm^{-3}
*Inductance	L	1 henry	$\frac{1}{9} \times 10^{-11}$	

*There is some confusion prevalent about the unit of inductance in Gaussian units. This stems from the use by some authors of a modified system of Gaussian units in which current is measured in electromagnetic units, so that the connection between charge and current is $I_m = (1/c)(dq/dt)$. Since inductance is defined through the induced voltage $V = L(dI/dt)$ or the energy $U = \frac{1}{2}LI^2$, the choice of current defined in Section 2

Physics Qualifying Exam
UCI

Fall 1990

Electricity & Magnetism

Closed book, closed notes exam. Math tables provided upon request. Do all problems. Problems equally weighted in score.

1.) Suppose you need to measure, at a distance, the electromagnetic flux from a sudden, short-time duration event. Your receiver near the event must transmit wideband (in frequency) data down a transmission line promptly.

a.) Why would coaxial line be a top candidate for the transmission line? Suggest some other transmission lines and briefly discuss their strengths and weaknesses for this situation.

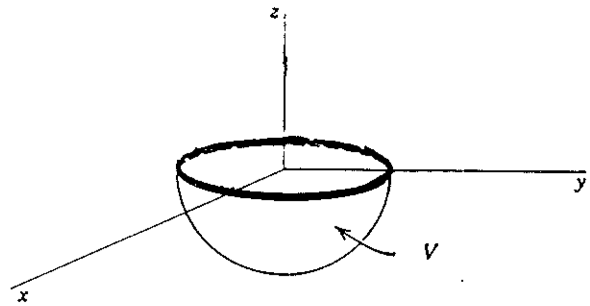
b.) What are the capacitance (F) and inductance (H) per meter of coaxial line of inner radius, a, and outer radius, b?

c.) What are the group and phase velocities of transmitted signals?

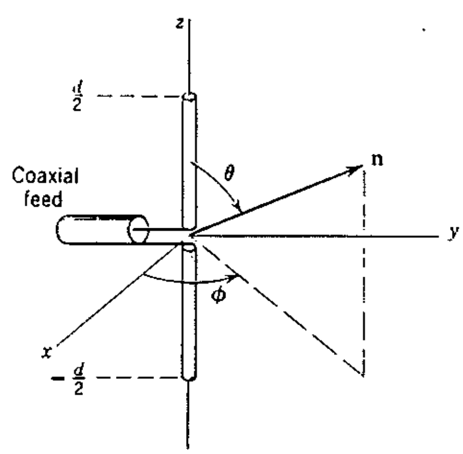
d.) Why is $b/a = 3.6$ in most coax lines?

e.) What size cable would you choose to use and why?

2.) Consider a conducting hemisphere of radius a = 10 cm, held at V = 10^2 V, as shown in the figure. Estimate the potential at x = 0, y = 0, z = 0 and x = 0, y = 0, z = 30 cm. How good is your estimate?



3.) An elementary mobile telephone might use a short, center-fed, linear antenna (as shown in figure) with d/2 = 20 cm at an operating frequency of f = 100 MHz. The antenna radiates 1 W.



At 100 cm from the origin, where do the electric and magnetic fields each reach peak amplitudes and approximately what are these values?

At a geostationary satellite what is the received power in W/m^2 for arbitrary θ,ϕ? What θ,ϕ should be used for the satellite (in other words, how should the antenna be oriented)?

CLASSICAL MECHANICS

Do problem 1 (20 points) and <u>two</u> of the next four problems (15 points each). Put these three mechanics problems in a separate blue book.

1. Consider the system illustrated below. Masses M and m are at the tips of a rigid massless L-shaped frame with equal arm length L, pivoted at point P on the platform shown. A spring (spring constant k) supports the frame at the point at which M is attached. Gravity acts downward.

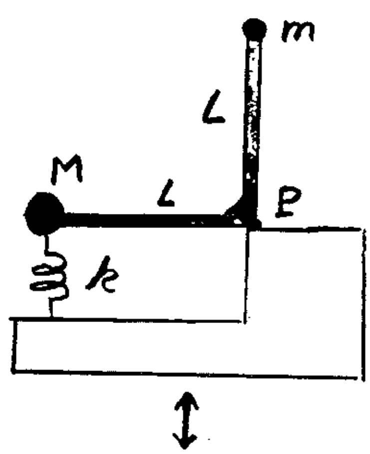
[This system has been proposed as a way to isolate an apparatus -- mass M -- from very low frequency vertical ground vibration. In principle, a spring alone would do the job, but in practice springs capable of supporting a large mass M are too stiff to give a sufficiently low resonant frequency. The effect of gravity on the added mass m is such as to lower the resonant frequency of the system.]

a. Find the resonant frequency of the system if the platform is fixed, and show that it may be made zero for a particular value for m.

b. Suppose that the platform moves vertically with a given motion $x = A\sin\omega t$. Find the Lagrangian for this system, find the equation of motion for the mass M, and solve this equation to find an expression for the vertical position of the mass M as a function of time.

c. Show that for any value of m there is one frequency at which the mass M will not move at all. Try to explain this fact in simple terms (without reference to the results of part b), by considering the low frequency and high frequency limits of the response of the system to the platform's motion.

NOTE: Small angle approximations may be made in all parts of this problem.



DO TWO OF THE PROBLEMS ON THIS PAGE.

2. A proton beam is incident on stationary proton targets.

   a. Find the minimum kinetic energy of the incident proton if an antiproton is to be produced by the process: $pp \rightarrow 3p + \bar{p}$.
   Express your answer in terms of the rest mass energy of the proton.

   b. What is the momentum of this incident proton?

3. A ball is dropped from a height of 100 meters at a latitude of 30°. Find the horizontal displacement of the ball's impact point due to each of the two non-inertial forces acting on the ball due to the earth's rotation. Ignore air drag. The earth's radius is 6,400 km.

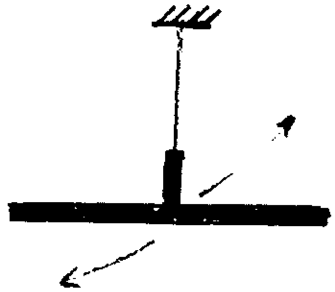
4. Suppose a fixed point mass produces a force on other particles which is radial outward and constant in magnitude at distances up to r_0, and zero at greater distances:

$$\vec{F}=k\hat{r} \text{ for } r\leq r_0, \quad \vec{F}=0 \text{ otherwise.}$$

   Determine the differential scattering cross section for an incident particle of mass m and velocity v, as a function of scattering angle θ. Assume that the momentum of the incident particle is sufficiently high that the scattering angle will be very small, and that the path of the particle while under the influence of the force deviates negligibly from a straight line.

5. Consider a pendulum shaped as shown, which is swinging so that the horizontal bar of the pendulum makes an angle of about 45° with the plane of swing. Curiously, it is found that the orientation of the horizontal bar will change in time, either swinging outward to make an angle of 90° with the plane of swing, or inward so that it lies in the plane of swing, depending on the relative lengths of the vertical and horizontal portions of the pendulum.

   Explain this behavior qualitatively, using Euler's equations.



1. An unperturbed 2-state system, with energy eigenstates $|1\rangle$ and $|2\rangle$, has energy levels which depend on a parameter ρ $(0 \leq \rho \leq 1)$ as follows:

$$E_1 = \rho \ , \ E_2 = 1/2$$

Suppose a small perturbation is introduced which couples the two states, such that

$$\langle 1|H|2\rangle = \langle 2|H|1\rangle = h \ll 1$$

(a) Find and <u>sketch</u> the energy levels of the perturbed system as a function of ρ for ρ between 0 and 1.

(b) Indicate on the sketch the approximate eigenstates (to zeroth order in h) which go with each energy level at ρ = 0 and 1.

2. A particle of mass m moves in a 3-dimensional central, linear potential, $V(r) = ar$. Use a suitable method to find an upper bound on the ground state energy.

3. Consider a one-electron atom with nuclear charge Ze. Suppose the nuclear charge is not located at a point, but is spread out over a small volume; for simplicity, you can consider a uniformly charged sphere of radius R, which is very small compared to the Bohr radius, a_o.

(a) Find, to lowest non-vanishing order in R/a_o, the relative change in the 2P-1S energy difference compared to the energy difference for a point nuclear charge.

(b) Assuming the nucleus is a proton, use for R a reasonable order-of-magnitude approximation for the size of a proton's charge distribution to obtain a numerical estimate of the ratio in part (a).

4. A. Below are five matrix elements, involving states of various angular momenta or combinations of angular momenta; Z_{op}, X_{op} are the position operators in the Z and X directions respectively. Some of these matrix elements are <u>guaranteed</u> to be zero. Indicate which you think are zero and why.

(i) $\langle \ell = 1, m = 1 | \ell_1 = 0, m_1 = 0; \ell_2 = 2, m_2 = 1 \rangle$

(ii) $\langle j = 1/2, m = 1/2 | Z_{op} | j = 3/2, m = 1/2 \rangle$

(iii) $\langle j = 1/2, m = 1/2 | X_{op} | j = 3/2, m = 1/2 \rangle$

(iv) $\langle \ell = 0, m = 0 | Z_{op} | \ell_1 = 1, m_1 = 1; \ell_2 = 2, m_2 = -1 \rangle$

(v) $\langle \ell = 1, m = 0 | Z_{op} | \ell = 1, m = 0 \rangle$

B. A certain matrix element $\langle \ell = 2, m = 0 | Z_{op} | \ell = 1, m = 0 \rangle$ is known,

$$\langle \ell = 2, m = 0 | Z_{op} | \ell = 1, m = 0 \rangle = A_{oo}$$

In terms of A_{oo} evaluate

$$A_{11} = \langle \ell = 2, m = 1 | X_{op} | \ell = 1, m = 0 \rangle$$

5. A particle of mass m and energy E is scattered elastically by some potential. Your professor tells you that the scattering amplitude is some complex function $f_o(\theta)$ of the scattering angle θ. You measure the total cross section and obtain a result σ that is slightly different from σ_o, where σ_o is the total cross section corresponding to $f_o(\theta)$. Your professor (never at a loss for a way out) says that there must be a modification of the potential at very short distances that affects only the $\ell = 0$ phase shift. To first order in $\sigma - \sigma_o$, and in terms of $f_o(\theta)$, calculate the change in the $\ell = 0$ phase shift needed to give the observed total cross section. [Do <u>not</u> use the Born approximation.]

Clebsch-Gordan Coefficients

All the matrix elements outside the dotted lines are equal to zero. It is easily verified that Eqs. (28.5) through (28.11) are in agreement with these matrices.

$j_1 = \frac{1}{2} \qquad j_2 = \frac{1}{2}$:

$$\begin{array}{cc|cccc}
 & & 1 & 1 & 0 & 1 \\
 & & 1 & 0 & 0 & -1 \\
\hline
\frac{1}{2} & \frac{1}{2} & 1 & & & \\
\frac{1}{2} & -\frac{1}{2} & & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \\
-\frac{1}{2} & \frac{1}{2} & & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & \\
-\frac{1}{2} & -\frac{1}{2} & & & & 1
\end{array} \tag{28.12}$$

$j_1 = 1 \qquad j_2 = \frac{1}{2}$:

$$\begin{array}{cc|cccccc}
 & & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \\
 & & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\
\hline
1 & \frac{1}{2} & 1 & & & & & \\
1 & -\frac{1}{2} & & \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & & & \\
0 & \frac{1}{2} & & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} & & & \\
0 & -\frac{1}{2} & & & & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & \\
-1 & \frac{1}{2} & & & & \sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & \\
-1 & -\frac{1}{2} & & & & & & 1
\end{array} \tag{28.13}$$

$j_1 = 1 \qquad j_2 = 1$:

$$\begin{array}{cc|ccccccccc}
 & & 2 & 2 & 1 & 2 & 1 & 0 & 2 & 1 & 2 \\
 & & 2 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -2 \\
\hline
1 & 1 & 1 & & & & & & & & \\
1 & 0 & & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & & & & & & \\
0 & 1 & & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & & & & & & \\
1 & -1 & & & & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} & & & \\
0 & 0 & & & & \sqrt{\frac{2}{3}} & 0 & -\sqrt{\frac{1}{3}} & & & \\
-1 & 1 & & & & \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} & & & \\
0 & -1 & & & & & & & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \\
-1 & 0 & & & & & & & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & \\
-1 & -1 & & & & & & & & & 1
\end{array} \tag{28.14}$$

QUALIFYING EXAMINATION - MATHEMATICAL PHYSICS

Tuesday, September 18, 1990 2-5 PM

1. Define and give an example of the following: branch point, non-isolated essential singularity, saddle point, entire function, analytic function.

2. Evaluate the integral

$$\int_0^\infty \frac{\sqrt{x}\,\ln x}{(x+1)^2}\,dx$$

3. Solve the integral equation

$$f(x) = A + \lambda \int_0^1 (x+y) f(y)\,dy$$

Is there a solution for all values of the constants A and λ?

4. In cartesian coordinates the contravariant components of a vector A^i, the covariant components A_i and the unit vector or physical components are all the same. Derive the relationships of these components in spherical coordinates.

5. A sphere of radius R_o has thermal conductivity k, density ρ and specific heat C. The outer surface is cooled by heat exchange with a fluid of temperature T_o so that at $r = R_o$, $-k\partial T/\partial r = h(T-T_o)$ where h is the heat transfer coefficient. There is a uniform heat source Q_o for $r \le R_o$ and $t \ge 0$ due

for example to nuclear fission. Find the steady state solution $\lim_{t \to \infty} T(r,t)$ and the time dependent solution if the initial temperature is $T(r,0) = T_o$. The equation for T is

$$\frac{\partial T}{\partial t} = D\nabla^2 T + \frac{Q_o}{c\rho}$$

where $D = k/c\rho$.

6. A spinless particle in a spherically symmetric potential is subject to a perturbation $V(x,y,z)$ which has the symmetry properties $V(x,y,z) = V(x,-y,z)$ and $V(x,y,z) = V(-x,y,z)$. Find the multiplication table for the smallest point group with these two symmetries. Find the classes, the character table and the irreducible representations. How is an $\ell = 1$ eigenstate split by this perturbation?