

## Millikan Oil Drop Data Analysis:

The experiment consists of raising a tiny, electrically charged oil drop in an electric field and then lowering it again. To raise it you apply a constant electric field on the drop that forces it upward. To lower the drop you can either turn off the electric field and just let it fall or you can reverse the field and force it downward. Because of air drag forces on the oil drop, the drop very quickly reaches its terminal velocity and moves with a constant velocity. Hence the total force on the oil drop is exactly zero while you observe it. Generally you raise and lower the drop over a fixed distance,  $d$ , and measure the rise and fall times,  $t_r$  and  $t_f$ , respectively. You need to know the electric field as well. This is just the voltage on the capacitor plates,  $V$ , divided by the distance between them,  $D$ . The raw data in the measurement consists of the 5 quantities,  $d$ ,  $t_r$ ,  $t_f$ ,  $V$  and  $D$ . Generally you establish and measure  $d$  and  $D$  once and for all at the beginning of the experiment. ( $D$  is set by the apparatus. The last time I measured it was  $(3.01 \pm 0.03)$  mm. )  $V$  is established and recorded for each drop and the same voltage (except for the sign) is used to raise and lower the drop if you choose to force it down. The two  $t$ 's should be measured many times for each drop; 12 isn't overdoing it. Minimizing random errors and keeping track of them is one of the two most important ingredients of a successful oil drop experiment. If you make  $N$  measurements of one of the  $t$ 's, the best estimate of  $t$  will just be the mean (or average) of those measurements. The uncertainty in  $t$  will be the standard deviation of the  $N$  measurements divided by  $\sqrt{N}$ . You want to make  $N$  large to get a good estimated of the standard deviation of the measurements and to drive the uncertainty of the mean down. The other important ingredient of a successful experiment is the selection of really small drops because they tend to have the fewest number of charges on them. If drops have 1, 2 or just a few charges on them, a 10% precision in the charge measurement will show you charge quantization. To see the difference between 100 and 101 charges, you have to measure the charge to a few parts in 1000, a much harder task.

Below the classical mechanics needed to extract the charge on each drop from the raw data is presented. Finally, a few comments to get you started thinking about how to go from the charges to  $e$  and charge quantization are offered.

The following table contains the symbols we'll use.

**Table 1: Symbols Used in Data Analysis**

| Symbol | Definition            | Comment   |
|--------|-----------------------|---|
| $d$    | Rise/Fall Distance    |   |
| $D$    | Plate spacing         | $(3.01 \pm 0.03)$ mm.                           |
| $t_r$  | Rise time             |   |
| $t_f$  | Fall time             | Free-fall or forced down by field; your choice. |
| $V$    | Voltage across plates |   |

**Table 1: Symbols Used in Data Analysis**

| Symbol   | Definition  | Comment   |
|----------|---|---|
| $v_r$    | Rise velocity   | $v_r = d/t_r$   |
| $v_f$    | Fall velocity   | $v_f = d/t_f$   |
| E        | Electric field in capacitor                           | $E = V/D$   |
| $\rho$   | Mass density of the oil                               | Measure it. It's a little less than 1 g/cm <sup>3</sup> .   |
| $\sigma$ | Mass density of air                                   | Calculate it from what you know about air. (It's small enough that high precision is not important here.) |
| $\rho'$  | $\rho - \sigma$                                       | Including $\sigma$ takes care of "buoyancy forces."   |
| $\eta_0$ | Viscosity of air                                      | It is a function of the temperature of the room. See upcoming discussion.                                 |
| $\eta$   | Viscosity of air corrected for mean free path effects | This matters for the size drops you will use. See the discussion below.                                   |
| a        | Radius of oil drop                                    |   |
| P        | Atmospheric pressure                                  |   |
| T        | Temperature of the room                               | We'll use C (degrees Celsius).  |
| b        | Constant used in calculating $\eta$ from $\eta_0$ .   | See upcoming discussion.<br>$b = 5.908 \times 10^{-3}$ torr-cm  |
| q        | The charge on the drop                                |   |

***Equations of Motion of the Drops***

The data analysis is based on applying  $\vec{F} = m\vec{a}$  to the rising and falling drops. We will take "up" as the positive direction. When the drop is rising in the field

$$ma = 0 = qE - \frac{4}{3}\pi\rho'a^3g - 6\pi\eta av_r \quad (1)$$

In (1) the left side is zero because the drop is moving at its constant terminal velocity;  $qE$  is the upward force on it; the next term is the drop's weight pulling it downward corrected for the buoyancy of the air, and the last term is the viscous air drag on the drop. The last term is called Stokes' law. It is the drag force on a sphere of radius  $a$  moving through a fluid of viscosity  $\eta$ . and is appropriate for small drops at low velocities so that the air flow around the drop is laminar and

therefore not turbulent. If the drop is moving up, the drag on it is downward and therefore comes in with a - sign in (1).

If you just let the drop fall, its equation of motion will be

$$0 = -\frac{4}{3}\pi\rho'a^3g + 6\pi\eta av_f \quad (2)$$

Finally, if you reverse the polarity instead of turning it off and let  $qE$  help gravity pull the drop down, the equation of motion is

$$0 = -qE - \frac{4}{3}\pi\rho'a^3g + 6\pi\eta av_f \quad (3)$$

If you look at (1) long enough, you will discover that after you take the measurements and figure out  $v_r$  and  $E$ , there are two unknowns in it,  $a$  and  $q$ . At this point there are two options depending on whether you let the drop free fall in zero field or forced the drop down by reversing polarity on the plates.

### ***Solving the Equations of Motion for $q$***

#### ***Option 1: Free Fall***

If you measured the fall time in zero field you can solve (2) for  $a$ . The result is

$$a = \left(\frac{9\eta v_f}{2\rho'g}\right)^{\frac{1}{2}} \quad (4)$$

Now we can solve (1). Use (2) in (1) to eliminate the  $mg$  term (second on the right) and then substitute the right side of (4) for  $a$ . The result is

$$q = \frac{6\pi D}{V} \left(\frac{9v_f}{2\rho'g}\right)^{\frac{1}{2}} \eta^{\frac{3}{2}} (v_f + v_r) . \quad (5)$$

The  $D$  and  $V$  in (5) come from writing  $E$  in terms of the plate voltage and separation.

Option 2: Reversing Polarity

In this case you can add (1) and (3) to get

$$\frac{8}{3}\pi\rho'a^3g = 6\pi\eta a(v_f - v_r). \quad (6)$$

Notice that  $v_f$  will always be bigger than  $v_r$  so there can not be sign problems. (6) is solved for  $a$ ;

$$a = \left( \frac{9\eta(v_f - v_r)}{4\rho'g} \right)^{\frac{1}{2}}. \quad (7)$$

Now you can subtract (3) from (1) to get

$$2qE = 6\pi\eta a(v_f + v_r). \quad (8)$$

Then substitute (7) into (8)

$$q = \frac{3\pi D}{V} \left( \frac{9(v_f - v_r)}{4\rho'g} \right)^{\frac{1}{2}} \eta^{\frac{3}{2}} (v_f + v_r) \quad (9)$$

Two comments are worth making. First, you must remember that the  $v_f$  in the free-fall case (Eqs. (2), (4) and (5)) is not the same thing as  $v_f$  in the case where the field is reversed to help force the drop downward (Eqs. (3), (6), (7), (8) and (9)). The fall velocities when the field helps pull the drop down must be bigger than the free-fall velocity. Furthermore, a little thought about the forces involved will (should) convince you that in the free fall case  $v_f$  is proportional to the weight of the drop and that in the field forced drop case  $(v_f - v_r)$  is proportional to twice the weight and hence is twice as big. This is why the 2 in the radicals in (4) and (5) becomes a 4 in (7) and (9). Similarly, in the free fall case,  $(v_f + v_r)$  is proportional to  $qE$ ; in the field aided drop, it is proportional to  $2qE$ . That is why the 6 in (5) becomes a 3 in (9).

There are two things that complicate the results in (5) and (9) and both have to do with the viscosity of air,  $\eta$ . First,  $\eta$  is a function of temperature. To good accuracy

$$\eta_0 = 1.81804 \times 10^{-4} \left( \frac{273.16 + T}{293.16} \right)^{\frac{3}{2}} \left( \frac{273.16 + T + 110.4}{293.16 + 110.4} \right) \quad (10)$$

The units of the viscosity as given in (5) are  $\text{gm cm}^{-1} \text{s}^{-1}$  also called a Poise. In (10)  $T$  is the Celsius temperature. The constant out in front is obviously the viscosity at  $T = 20 \text{ C}$ . You must use (10) as the first step in evaluating the viscosity.

Worse yet, there is another hiding in  $\eta$ . Stokes' law assumes that the fluid that the sphere is moving through is a continuous substance. In a continuous fluid, fluid particles are in immediate contact with their neighbors, and thus in some sense couple to their surroundings in a maximally efficient way. A real gas is not a continuous medium but a swarm of particles that must fly for a distance called the mean free path before they become aware of conditions in their neighborhood. Thus a real gas is less able to organize resistance to an object moving through it than a continuous fluid would be. The difference starts to become important only when the moving object is small, meaning comparable with a mean free path or smaller. Thus the resistance force on a small object moving through a gas is less than what you would calculate based on a measurement of the viscosity made with a big object.  $\eta_0$  is a viscosity measured with big objects, where the mean free path doesn't matter. Your object is going to be a  $1 \mu\text{m}$  (or even less) radius sphere and the mean free path in the surrounding air has been measured as  $0.0673 \mu\text{m}$ . So if you use  $\eta_0$  as the viscosity you will overestimate the drag force and thus get  $q$  wrong. A fairly serious analysis of the situation (M.D. Allen and O. G. Raabe, *Re-evaluation of Millikan's Oil Drop Data for Small Particles in Air*, J. Aerosol Sci. **13**, 537 (1982)) suggests that the viscosity that should be used in our experiment is

$$\eta = \frac{\eta_0}{1 + \frac{b}{aP}} \quad (11)$$

In this equation  $\eta_0$  comes from (10);  $P$  is the pressure of the air in the room in torr (mmHg); and  $b = 5.908 \cdot 10^{-3} \text{ torr-cm}$ . So for  $a = 1 \mu\text{m}$  and  $P = 760 \text{ torr}$ , denominator in (11) corrects  $\eta$  by almost 8%. (Millikan was aware of all this; he just used a slightly larger value of  $b$ .)  $1 \mu\text{m}$  radius drops were the biggest ones I used when I did this experiment a few years ago.)

The easiest thing to do at this point is to substitute (11) into (2) in the free fall case or (6) in the field forced drop case and solve again for a. In both cases you get a quadratic equation that has only one positive root. The results are

$$a = \sqrt{\frac{b^2}{4P^2} + \frac{9\eta_0 v_f}{2\rho'g}} - \frac{b}{2P} \quad (12)$$

in the free-fall case, and

$$a = \sqrt{\frac{b^2}{4P^2} + \frac{9\eta_0(v_f - v_r)}{4\rho'g}} - \frac{b}{2P} \quad (13)$$

when the field helps push the droplet down.

To wrap all of this up, in the free fall case (11) is substituted into (5) to give

$$q = \frac{6\pi D}{V} \left( \frac{9\eta_0^3}{2\rho'g} \right)^{\frac{1}{2}} \frac{1}{\left(1 + \frac{b}{aP}\right)^{\frac{3}{2}}} v_f^{\frac{1}{2}} (v_f + v_r) . \quad (14)$$

and (12) is used to calculate a. In the field forced fall case (11) is substituted into (9) to give

$$q = \frac{3\pi D}{V} \left( \frac{9\eta_0^3}{4\rho'g} \right)^{\frac{1}{2}} \frac{1}{\left(1 + \frac{b}{aP}\right)^{\frac{3}{2}}} (v_f - v_r)^{\frac{1}{2}} (v_f + v_r) . \quad (15)$$

and (13) is used to calculate a.

### ***What else?***

The whole game here is to determine the q on a bunch of different drops. To measure the charge on the drops you want to have drops with only a few charges on them, and the uncertainty in the values of q you derive must be well established by your data analysis and smaller than the charge quantum. If the errors are bigger than that, the charge will look continuous. So you must use the propagation of errors formulas you presumably got taught in P50 something to calculate the uncertainty in each and every q. Needless to say, *Mathematica* (or whatever similar thing you like) makes a world of difference to the labor involved in this. But the error analysis really isn't optional.

*So you've at last got the charges on a bunch of different drops. Now what?*

Once you have the charges on a bunch of drops what do you do? You don't do what most students do at this point. They (1) forget about calculating errors in  $q$ ; (2) divide each charge they measure by  $1.602 \times 10^{-19}$  C; (3) throw out the fractional part of the result of each division; (4) multiply each quotient by  $1.602 \times 10^{-19}$  C; (5) notice that each of the resulting charge values is a perfect integral multiple of  $e$ , and (6) declare victory and perfect agreement with the "accepted" value of  $e$ . There are usually enough extra steps and other nonsense around to camouflage this process to some extent, but this is the heart of what they do. This is an obviously misleading way to process the data. You should resist the temptation to use what you already think is the "right answer" in your data analysis.

Here's an alternative suggestion Find a charge that when divided into each one of your measured charges gives a result within an error bar or two of an integer for each and every charge. The smallest charge you measure or the smallest difference between two charges should be a good candidate for the number to divide by. If you've done a good experiment this will work. Next you want to dither that divisor around just a little to minimize the sum of the square of the differences between the each division result and the integer nearest it. Measure the distance to each nearest integer in units of the error in the division result. (This is minimizing  $\chi^2$  with respect to divisor. Look back at P50 something.) The largest such number you can find is your value of  $e$ .

Confused? You'll have plenty of time to think about it while timing drops. Your instructor may also have even better suggestions, or maybe be able to help you make sense out of this one.