This is a short summary to review the two main kinematics equations and compare the use of components versus vector diagrams.

Equation 1: \( x = v_0 t + \frac{1}{2} a t^2 \).

Method 1: vector diagram: Here we are applying the graphical rules for adding vectors to construct a triangle. The addition we are doing is \( \mathbf{c} = \mathbf{a} + \mathbf{b} \), where \( \mathbf{c} = x \), \( \mathbf{a} = v_0 t \), and \( \mathbf{b} = \frac{1}{2} a t^2 \). So, from addition of vectors, we know that the tails of \( \mathbf{c} \) and \( \mathbf{a} \) must touch, the head of \( \mathbf{a} \) touches the tail of \( \mathbf{b} \), and the heads of \( \mathbf{c} \) and \( \mathbf{b} \) must touch. This tells us how to connect \( x \), \( v_0 t \), and \( \frac{1}{2} a t^2 \). So, the main question is which to draw first? The best procedure is to apply the following three guidelines:

1) Attempt to draw the position vector first. Generally, you know something about the initial and final position of the object, this tells you how to draw \( x \), as it just connects the initial and final position. The situation where you would not know this to begin with is when you are being asked to find the initial or final position.

2) For projectile motion, draw \( \frac{1}{2} a t^2 \) next as it is always vertical, and by construction must end at the final position. It starts where \( v_0 t \) ends.

3) By construction, \( v_0 t \) starts from the initial position and goes at an angle that is either given, or you need to find, and then meets up with \( \frac{1}{2} a t^2 \).

The types of picture you will end up with have the following characteristics:
For the cases where the vector addition does not give a right triangle, a dashed line is drawn to indicate the horizontal. In these cases, the difference in initial and final height combined with the $\frac{1}{2} a t^2$ vector will give the length of one side of the right triangle. How does this compare to components?

Component equations: $x = v_{ox} t + \frac{1}{2} a_x t^2$ and $y = v_{oy} t + \frac{1}{2} a t^2$.

These two equations are the ones that you would get from using sine and cosine of the angle between $v_0$ and the horizontal. Notice, by drawing the triangle, you also have the option to immediately use tangent, Pythagorean theorem, or complete the diagram as a rectangle and equate opposite sides of the rectangle. If you go back and look at the example problems, extra problems, and exam problems, you will find cases where this was useful.

**EQUATION 2: $v = v_o + at$**

For this equation, if it is projectile motion, the best way to draw it is as follows:

1) Draw $v_o$ first. Is it up, down, or horizontal? Generally, even when you are solving for the angle, you at least know if it was thrown up or down.

2) Draw $at$ coming down from $v_o$.

3) Draw in $v_f$. Start $v_f$ from $v_o$ and draw it up, down, or horizontal based on the information you have about its final direction. If this is unknown, pick a likely direction, and use the information to solve for the unknown angle. Connect $at$ to $v_f$.

**WARNING:** In this case, whether or not $at$ goes above or below the horizontal line has nothing to do with the initial and final position of the object, only the initial and final direction of motion.

Two example diagrams for an object launched up, and the relationship to the actual motion:

The path of the object is shown as a dashed line. The correct diagrams for the velocity for the two cases are:

**FINAL NOTE:** There are only two angles that are guaranteed to be the same in the position and velocity diagrams. These are: (1) the angle in the velocity diagram between $v_o$ and $at$ is the same as the angle in the position diagram between $v_0 t$ and $\frac{1}{2} at^2$; (2) the angle between $v_o$ and the horizontal in the velocity
diagram is the same as the angle between \( v_0 \) and the horizontal in the position diagram. This is because multiplication by a scalar does not change the direction of a vector.

Here are four example questions that you should try to work out. All four questions are based on the diagram and situation below:

\[
\begin{array}{c}
10 \text{ m} \\
20 \text{ m} \\
\end{array}
\]

You are standing on a 10 m high cliff. A tube is 20 m away in the horizontal direction. The tube is 2 m high. Answer the following questions, ignoring air resistance.

1) Suppose that you throw a ball up at an angle of 30° from the horizontal, how long before it hits the top of the tube?

2) Given that throw, what was the initial speed and its final velocity?

3) What initial speed is required if the ball is launched horizontally?

4) How high is the cliff if the ball is launched at an angle of 5° below the horizontal and it lands just in front of the tube traveling at an angle of 15° from the vertical? (The distance from the cliff to the tube is still 20 m.)

A DISCUSSION OF THE POSSIBLE WAYS TO ANSWER TO ALL QUESTIONS ARE ON THE NEXT PAGE. NOTE: THESE DO NOT REPRESENT SOLUTIONS IN THE SENSE THAT THEY ARE NOT PRESENTED IN THE SOLUTION FORM. THIS IS DONE TO PROVIDE A LITTLE EXTRA DISCUSSION OF HOW TO DO EACH PROBLEM.
1) The two diagrams for this situation are:

Some things to note are the following. In the position diagram, the vector $x$ starts at the initial height of 10 m and goes to the top of the tube at 2 m. Therefore, the dashed line representing its vertical component is only 8 m long. As discussed, the vector $\frac{1}{2} at^2$ must start at the head of the vector $v_o t$. Since this vector starts initially in the upward direction, $\frac{1}{2} at^2$ must be LONGER than the vertical component of the vector $x$. However, by drawing a horizontal line, we are able to make a right triangle with a vertical side opposite the $30^\circ$ angle whose length is given by $h$, a quantity that we can calculate. The velocity diagram is relatively straightforward for this problem.

For the following calculations, all variables are defined in the vector diagrams. For the first problem, we want to find time. No single equation will give us time as $h$, $v_o$, and $v_f$ are all unknown. However, we can get two equations with two unknowns using the upper right triangle in the position diagram:

\[
(v_o t) \cos(30) = 20 \text{ m}
\]

\[
(v_o t) \sin(30) = h = \frac{1}{2} at^2 - 8 \text{ m}
\]

Dividing the second equation by the first gives:

\[
\tan(30) = \left(\frac{\frac{1}{2} at^2 - 8}{20}\right)
\]

Solving for time gives:

\[
t = \frac{(20 \tan(30) + 8)}{5} \text{ or } t = 2 \text{ s.}
\]

This method required a minimum of 2 equations with two unknowns. We had the possibility of many ways of getting these equations and unknowns. If you used components for a problem like this, you still would have two equations. However, now you would have to make sign choices. Choosing up and to the right as positive, the two equations become:

\[
x = 20 \text{ m} = [v_o \cos(30)]t
\]

\[
y = -8 \text{ m} = [v_o \sin(30)]t - \frac{1}{2} at^2
\]

Notice, this assumes that you use $a = 10 \text{ m/s}^2$ and put the minus sign explicitly in the equation. If you use $a = -10 \text{ m/s}^2$, then the second equation would have $+1/2 at^2$. Therefore, in the component method you have to think carefully about minus signs. In the graphical method, the direction is accounted for when you draw the vectors. Minus signs only occur due to geometry, as in the $h = 1/2 at^2 - 8 \text{ m}$ equation. All variables become magnitudes because they represent lengths of the vectors.
2) For the first part, the equation, \( v_o = \frac{20}{[2 \cos(30)]} = 11.5 \text{ m/s} \), is easily recognized by either method. You just use the equations you already had in part 1. To find \( v_f \) by the graphical method, we consider the triangle for the velocity vectors:

\[
\begin{align*}
\vec{v}_o & \quad \vec{v}_f \\
\theta & \quad 30^\circ \quad 60^\circ \quad \text{at}
\end{align*}
\]

VELOCITY DIAGRAM

Now we just apply the law of cosines to find the magnitude of \( v_f \) and the law of sines to find \( \theta \).

\[
v_f^2 = (v_o)^2 + (at)^2 - 2(v_o)(at)\cos(60)
\]

\[
= 11.5^2 + 20^2 - 2(11.5)(20)(0.5) = 302.25
\]

\[v_f = 17.4 \text{ m/s}\]

\[
\frac{\sin(\theta)}{20 \text{ m/s}} = \frac{\sin(60)}{17.4 \text{ m/s}} \quad \text{or} \quad \sin(\theta) = \frac{20\sin(60)}{17.4}
\]

\[\theta = 84.5^\circ\]

So, the final answer is a velocity of 17.4 m/s to the right and down 54.5° from the horizontal (notice the direction of \( v_f \) is not \( \theta \) but \( \theta - 30^\circ \)). Other options included solving for the angle in the lower right of the triangle and giving the direction of \( v_f \) relative to the vertical. This would involve the law of sines applied to the \( v_o \) and \( v_f \) sides of the triangle.

Again, by components one would have slightly additional work:

\[
v_{fx} = v_o \cos(30)
\]

\[
v_{fy} = v_o \sin(30) - at
\]

\[v_f^2 = v_{fx}^2 + v_{fy}^2
\]

\[
\tan(\theta) = \frac{v_{fy}}{v_{fx}}
\]

Here one has to make a sign choice again. The only advantage is that in this case \( \theta \) is actually the direction of \( v_f \). One does not have to subtract from 30°.
3) This one is relatively straightforward by either components or a diagram. One only needs the position diagram:

\[ 8 \text{ m} = \frac{1}{2} at^2 \text{ (notice, here the sign choice is straightforward)} \]
\[ t = 1.3 \text{ s} \]
\[ v_0 t = 20 \text{ m gives } v_0 = 15.4 \text{ m/s} \]

4) This is the problem that shows the full power of the graphical method. This method will always be better when information about angles is the main information in the problem. In this case, we are given the initial and final direction of travel – so we know the angles in the velocity diagram (as indicated). Please note that a common error here is to use the final angle of travel in the position diagram. THIS IS INCORRECT. The angle in the position diagram for \( x \) has nothing to do with the direction of travel, only the relation between its horizontal and vertical components. See the above diagrams of a trajectory to convince yourself of this.

Some comments on the velocity diagram, because this is where all the work for this problem really is. First, it is not to scale. Second, the angles of 5° and 15° are what are given in the problem. Since \( at \) is always vertical, we know where the 15° angle goes. Second, using that \( at \) is perpendicular to the horizontal, we can put in the 75° angle. Finally, using the fact that alternate interior angles for a line
crossing two parallel lines are equal, we can conclude that the angle in the triangle is 70°. As you will see, this method eliminates are large amount of algebra in exchange for remembering some geometry.

So, how to do this problem? The goal is to find h. If we know time and the initial speed, we have h from the position diagram. So we have 3 unknowns. Two more equations involving only h, v₀, and t are required. One can come from the position diagram. The other comes from the velocity diagram using the Law of Sines. Therefore, we have 3 equations and 3 unknowns. Slightly harder than our other problems, but not too bad as the last equation we simply have to plug into:

From the position diagram:

\[ v_0 t \sin(5) = h - \frac{1}{2} at^2 \]
\[ v_0 t \cos(5) = 20 \text{ m} \text{ or } v_0 = 20/[t \cos(5)] \]

From the velocity diagram:

\[ \frac{v_0}{\sin(15)} = \frac{at}{\sin(70)} \]
\[ t = \frac{\sin(70)v_0}{a \sin(15)} \text{ using equation 2 from the position diagram} \]
\[ t^2 = \frac{\sin(70)20}{a \sin(15) \cos(5)} = 7.3 \text{ s}^2 \text{ or } t = 2.7 \text{ s} \]

Plugging back into equation 2 gives:

\[ v_0 = 7.4 \text{ m/s} \]

Plugging into equation 1 gives:

\[ h = v_0 t \sin(5) + \frac{1}{2} at^2 = 38.2 \text{ m} \]

If you used components, your only real choice is to use the following four equations and four unknowns:

- horizontal position: \[ v_0 \cos(5)t = 20 \text{ m} \]
- vertical position: \[ v_0 \sin(5)t + \frac{1}{2} at^2 = h \]
- horizontal velocity: \[ v_f \cos(75) = v_0 \cos(5) \]
- vertical velocity: \[ v_f \sin(75) = v_0 \sin(5) + at \]

Again, you have to make the correct choices of sign.

The bottom line is the following:

I would always draw the vector addition diagrams. This will help you make sure you know what is happening. Then, decide which method you are more comfortable with. If you are comfortable with trig and geometry, read the most useful equations off of the diagrams, and you will not have to worry about signs and much algebra. If you are much more comfortable with algebra, then proceed to write down the equations for the components. Either way, what the vector diagram help with is correctly locating the relevant angles in the problem, so that for either method, you do not use an incorrect angle.

**FINAL NOTE:** Only use the third equation: \[ v_f^2 = v_0^2 + 2ax \] when the vectors a and x are in the SAME DIRECTION. Again, the vector diagram will make clear when this occurs!