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Invariant Spin Coherent States and the Theory of the Quantum Antiferromagnet in a Paramagnetic Phase[1]

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A consistent theory of the Heisenberg quantum antiferromagnet in the disordered phase with short-range antiferromagnetic order was developed on the basis of the path integral for the spin coherent states. We presented the Lagrangian of the theory in the form that is explicitly invariant under rotations and found natural variables in terms of which one can construct a perturbation theory. The short-wavelength spin fluctuations are similar to the ones in spin-wave theory, and the long-wavelength spin fluctuations are governed by the nonlinear sigma model. We also demonstrated that the short-wavelength spin fluctuations should be considered accurately in the framework of the discrete version in time of the path integral. In the framework of our approach, we obtained the response function for the spin fluctuations for the whole region of the frequency ω and the wave vector $\mathbf{k}$ and calculated the free energy of the system. © *2000 MAIK "Nauka/Interperiodica".*

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The theory of the two-dimensional Heisenberg antiferromagnet (AF) has attracted great interest during the last several years in connection with the problem of AF fluctuations in copper oxides [1–3]. The approach of these papers was based on the sigma model, which describes the long-wavelength fluctuations of the Heisenberg AF in the paramagnetic phase with a short-range antiferromagnetic order. The sigma model is the continuum model for the unit vector $\mathbf{n}(t, \mathbf{r})$, $\mathbf{n}^2 = 1$ in the 1 + 2 time and space dimensions [4, 5]. As a long-wavelength theory, the sigma model can make a lot of physical predictions, such as the structure of the long-wavelength fluctuations and the magnitude of the correlation length [2, 3, 6]. But, up to now, a consistent theory of the spin fluctuations for the quantum AF (QAF) with short-range AF order was absent. This is just the topic of this paper.

Our approach to the description of the QAF is based on the functional integral for the generalized partition function in terms of spin coherent states. We introduce the concept of invariant spin coherent states and, on this basis, we formulate the theory.

We define the invariant spin coherent states (SCS) with the help of relation

$$|\mathbf{n}; \mathbf{m}\rangle = \exp(-i\varphi\hat{S}_z)\exp(-i\theta\hat{S}_y)\exp(-i\psi\hat{S}_z)|ss\rangle. \quad (1)$$

Here, the state $|ss\rangle$ is the state of spin s with the maximal spin projection s. The unit vectors $\mathbf{n}$ and $\mathbf{m}$ are orthogonal: $\mathbf{n}^2 = 1$, $\mathbf{m}^2 = 1$, $\mathbf{n} \cdot \mathbf{m} = 0$; θ, φ are the Euler angles of the unit vector $\mathbf{n} = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$. The dependence on the vector $\mathbf{m}$ is included in the angle ψ only, which, in fact determines only the phase factor in the SCS (1). We can choose the angle ψ in some special manner, which distinguishes this definition from the standard one [7]: $\psi = -k_z/m_z$, where the vector $\mathbf{k} = [\mathbf{n} \times \mathbf{m}]$. This choice has a clear geometrical interpretation. The transformation (1) rotates the reference coherent state that is characterized by the vectors $\mathbf{n}_0 = (0, 0, 1)$ and $\mathbf{m}_0 = (1, 0, 0)$, into the SCS (1). From this geometric interpretation, it is obvious that, upon changing SCS by some rotation $\hat{a}$, we have $|\hat{a}\mathbf{n}; \hat{a}\mathbf{m}\rangle = \hat{U}(\hat{a})|\mathbf{n}; \mathbf{m}\rangle$ without the phase factor, which was introduced and discussed by Perelomov [8]. In this way, the scalar product $\langle\mathbf{n}'; \mathbf{m}'|\mathbf{n}; \mathbf{m}\rangle$ is an invariant, and the matrix element $\langle\mathbf{n}'; \mathbf{m}'|\hat{\mathbf{S}}|\mathbf{n}; \mathbf{m}\rangle$ is a vector under rotations. It seems that the vector $\mathbf{m}$ is an artificial one. However, for the problem of the QAF, it has some real meaning.

We consider the spin system which is described by the Heisenberg Hamiltonian with an interaction of nearest neighbors, $\hat{H}_{Hei}(l, l') = J\hat{\mathbf{S}}_l \cdot \hat{\mathbf{S}}_{l'}$, $\hat{\mathbf{S}}_l \cdot \hat{\mathbf{S}}_l = s(s + 1)$, where $\hat{\mathbf{S}}_l$ are the spin operators; the index l runs over a two-dimensional square lattice; the index l' runs over

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the nearest neighbors of the site l; $J > 0$ is the exchange constant which, since it is positive, corresponds to the AF spin interaction; and s is the spin magnitude. The most efficient method of dealing with a spin system is based on the representation of the generalized partition function Z or the generating functional of the spin Green functions $Z = Tr[\exp(-\beta\hat{H})]$ in the form of a functional integral over spin coherent states:

$$Z = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} D\mu(\mathbf{n}_a, \mathbf{n}_b) \exp(A(\mathbf{n}_a, \mathbf{n}_b)), \tag{2}$$

$$D\mu(\mathbf{n}_a, \mathbf{n}_b) = \prod_{p=a,b;\,\tau,l} \frac{2s+1}{2\pi} \delta(\mathbf{n}_p^2(\tau, l) - 1) d\mathbf{n}_p(\tau, l), \tag{3}$$

where $T = 1/\beta$ is the temperature, τ is the imaginary time, and $A(\mathbf{n})$ is the action of the system. In the continuum approximation, which is valid in the leading order in $1/2s$, the expression of the action $A(\mathbf{n})$ is simplified:

$$A(\mathbf{n}_a, \mathbf{n}_b) = -\int_0^{\beta} \sum_l \mathcal{L}_{tot}(\tau, l) d\tau, \qquad \mathcal{L}_{tot}(\tau, l) = \mathcal{L}_{kin}(\tau, l) + \mathcal{H}(\tau, l), \tag{4}$$

$$\mathcal{H}(\tau, l) = Js^2 \sum_{l' = \langle l \rangle} \mathbf{n}_a(\tau, l) \cdot \mathbf{n}_b(\tau, l'), \qquad B_p(\tau, l) = \langle \mathbf{n}_p; \mathbf{m}_p | \frac{\partial}{\partial \tau} | \mathbf{n}_p; \mathbf{m}_p \rangle, \tag{5}$$

where $\mathcal{L}_{kin}(\tau, l) = B_a + B_b$ and $B_{a,b}$ are the Berry phases for the sublattice $p = a, b$. The idea of the short-range AF order was used in Eqs. (1)–(5), and we split our square lattice into two AF sublattices a and b. For the kinetic part of the action $\mathcal{L}_{kin}$ (which is highly nonlinear), we use the concept of invariant coherent states parametrized by arbitrary vectors $\mathbf{m}_{a,b}$.

In our case, we can define these vectors $\mathbf{m}_{a,b}$ in the following manner: $\mathbf{m}_{a,b} = (\mathbf{n}_{b,a} - x\mathbf{n}_{a,b})/(1 - x^2)^{1/2}$, $x = (\mathbf{n}_a \cdot \mathbf{n}_b)$. As a result, the invariant coherent states have a clear meaning. Substituting these expressions for $\mathbf{m}_{a,b}$ into Eq. (5), we have invariant forms for the Berry phases B_p, which depend on both vectors $\mathbf{n}_{a,b}$ for each sublattice a, b. For $\mathcal{L}_{kin}$, we have an invariant form under rotations:

$$\mathcal{L}_{kin} = \frac{is}{1 - \mathbf{n}_{a\tau l} \cdot \mathbf{n}_{b\tau l}} (\dot{\mathbf{n}}_{a\tau l} - \dot{\mathbf{n}}_{b\tau l}) \cdot [\mathbf{n}_{a\tau l} \times \mathbf{n}_{b\tau l}]. \tag{6}$$

Now we can introduce new, more convenient variables $\mathbf{\Omega}(\tau, l)$ and $\mathbf{M}(\tau, l)$, which realize the stereographic mapping of a sphere:

$$\mathbf{n}_{a,b} = \frac{\pm\mathbf{\Omega}(1 - \mathbf{M}^2/4) - [\mathbf{\Omega} \times \mathbf{M}]}{1 + \mathbf{M}^2/4}, \qquad \mathbf{\Omega}^2 = 1, \quad \mathbf{\Omega} \cdot \mathbf{M} = 0. \tag{7}$$

In terms of these variables, the total Lagrangian $\mathcal{L}_{\Omega M} = \mathcal{L}_{kin} + \mathcal{H}$ has the final form

$$\mathcal{L}_{kin} = \frac{2is\dot{\mathbf{\Omega}} \cdot \mathbf{M}}{1 + \mathbf{M}^2/4},$$

$$\mathcal{H} = Js^2 \times \sum_{l' = \langle l \rangle} \{\mathbf{\Omega} \cdot \mathbf{\Omega}'[(1 - \mathbf{M}^2/4)(1 - \mathbf{M}'^2/4) - \mathbf{M} \cdot \mathbf{M}'] + \mathbf{\Omega} \cdot \mathbf{M}'\mathbf{\Omega}' \cdot \mathbf{M}\}(1 + \mathbf{M}^2/4)^{-1}(1 + \mathbf{M}'^2/4)^{-1}, \tag{8}$$

where $\mathbf{\Omega} \equiv \mathbf{\Omega}_{\tau l}$, $\mathbf{\Omega}' \equiv \mathbf{\Omega}_{\tau l'}$, $\mathbf{M} \equiv \mathbf{M}_{\tau l}$, $\mathbf{M}' \equiv \mathbf{M}_{\tau l'}$. After this change of variables, the measure of integration $D\mu(\mathbf{n})$ (7) becomes

$$D\mu(\mathbf{n}) = \prod_{\tau l} \frac{(2s+1)^2(1 - \mathbf{M}^2/4)}{2\pi^2(1 + \mathbf{M}^2/4)^3} \times \delta(\mathbf{\Omega}^2 - 1)\delta(\mathbf{\Omega} \cdot \mathbf{M}) d\mathbf{\Omega} d\mathbf{M}, \tag{9}$$

where the product in Eq. (9) is performed over the AF (doubled) lattice cells.

The variable $\mathbf{\Omega}$ is responsible for the AF fluctuations, and the variable $\mathbf{M}$ for the ferromagnetic ones. The ferromagnetic fluctuations are small according to the parameter $1/2s$, and, therefore, one can expand the Lagrangian $\mathcal{L}_{\Omega M}$ (8) in $\mathbf{M}$. The vector of the ferromagnetic fluctuations $\mathbf{M}$ plays the role (to the factor $2s$) of the canonical momentum conjugate to the canonical coordinate $\mathbf{\Omega}$. The term of first order in $\mathbf{M}$ coincides (after a change of variables) with previous results [1, 3].

From Eq. (1) one can easily extract the quadratic part of the total Lagrangian in the variables $\mathbf{\Omega}$ and $\mathbf{M}$, $\mathcal{L}_{quad}$:

$$\mathcal{L}_{quad} = 2is(\mathbf{M} \cdot \dot{\mathbf{\Omega}}) + Js^2 \sum_{l' \in \langle l \rangle} [\mathbf{\Omega}^2 - \mathbf{\Omega} \cdot \mathbf{\Omega}' + \mathbf{M}^2 + \mathbf{M} \cdot \mathbf{M}']. \tag{10}$$

The Lagrangian $\mathcal{L}_{quad}$ (10) is very simple, but the measure $D\mu$ (9) is not simple due to the presence of two delta functions. Therefore, we cannot simply perform the Gaussian integration over the fields $\mathbf{\Omega}$ and $\mathbf{M}$. To solve this problem, we shall use the method of the Lagrange multiplier λ together with the saddle point approximation [4, 5] to eliminate $\delta(\mathbf{\Omega}^2 - 1)$. As a result, we shall have an additional integration over λ with the

additional Lagrangian $\mathcal{L}_\lambda(\tau, l) = [i\lambda(\tau, l) + \mu_0^2/2\mathcal{J}][\mathbf{\Omega}^2(\tau, l) - 1]$, where μ_0 is the primary mass of the Ω field and $\mathcal{J} = Jsz$.

To eliminate $\delta(\mathbf{\Omega} \cdot \mathbf{M})$, we shall use some kind of Faddev–Popov trick [5]. As a result of this trick, (*1*) the factor $\delta(\mathbf{\Omega} \cdot \mathbf{M})$ disappears from the measure (9); (*2*) $\mathbf{M} \longrightarrow \mathbf{M}_{tr} = \mathbf{M} - \mathbf{\Omega}(\mathbf{\Omega} \cdot \mathbf{M})$ in the Lagrangian (8); (*3*) an additional contribution to the action appears, the Lagrangian of which, $\mathcal{L}_{ga}$, can be chosen in the form $\mathcal{L}_{ga} = Js^2\sum_{l' \in \langle l \rangle}[(\mathbf{\Omega} \cdot \mathbf{M})^2 + (\mathbf{\Omega} \cdot \mathbf{M})((\mathbf{\Omega}' \cdot \mathbf{M}')]$, which kills the strongest interaction between the $\mathbf{\Omega}$ and $\mathbf{M}$ fields in the Lagrangian (10) that appears due to the substitution $\mathbf{M} \longrightarrow \mathbf{M}_{tr}$; and (*4*) in the measure of the integration in Eq. (9), the additional factor $(\det(\hat{B}_{ga}))^{1/2}$ arises, where the operator $\hat{B}_{ga}$ is just the operator in the quadratic form in the variable $(\mathbf{\Omega} \cdot \mathbf{M})$ for $\mathcal{L}_{ga}$. In this way, Eq. (10) for $\mathcal{L}_{quad}$ is valid in the leading order with respect to $1/2s$. The final expression for the total quadratic Lagrangian is $\mathcal{L}_{tqu} = \mathcal{L}_{quad} + \mathcal{L}_{ga} + \mathcal{L}_{\lambda quad}$.

Now, from the quadratic part of the total Lagrangian $\mathcal{L}_{tot}$ one can find the Green functions of the $\mathbf{\Omega}$ and $\mathbf{M}$ fields in $q = (\omega, \mathbf{k})$ representation:

$$\hat{G}_q\mathbf{X}_q^* \equiv \begin{pmatrix} G_q^\Omega, & G_q^d \\ G_q^u, & G_q^M \end{pmatrix}\begin{pmatrix} \mathbf{\Omega}_q^* \\ \mathbf{M}_q^* \end{pmatrix} = \frac{1}{2sL_q}\begin{pmatrix} Q_\mathbf{k}, & -\omega \\ \omega, & P_\mathbf{k}' \end{pmatrix},$$

$$L_q = \omega^2 + \omega_{0\mathbf{k}}^2,$$
$$\omega_{0\mathbf{k}}^2 = P_\mathbf{k}'Q_\mathbf{k} = (1 - \gamma_\mathbf{k}^2)\mathcal{J}^2 + (1 + \gamma_\mathbf{k})\mu_0^2/2, \quad (11)$$
$$(Q_\mathbf{k}, P_\mathbf{k}) = \mathcal{J}(1 \pm \gamma_\mathbf{k}),$$
$$\gamma_\mathbf{k} = (1/2)(\cos(k_x a) + \cos(k_y a)),$$

where the momentum $\mathbf{k}$ runs over the AF Brillouin zone, a is the lattice constant, $\omega = 2\pi jT$, and j is an integer number.

From Eq. (11), one can calculate the parameter of spin-wave nonlinearity of the theory, $\langle \mathbf{M}_{tr}^2 \rangle = (1/2s)C_M(T)$, where $C_M(T) = 0.65075$ for $T \ll \mathcal{J}$ and $C_M(T) = 1.48491T/\mathcal{J}$ for $T \geq \mathcal{J}$.

We also have the saddle point condition for the λ field $\langle \mathbf{\Omega}^2 \rangle = 1$, which is the most important constraint of the theory which determines its phase state:

$$1 = \langle \mathbf{\Omega}^2 \rangle = N\sum_q G_q^\Omega$$
$$= \frac{NT}{2s}\sum_{\omega, \mathbf{k}}\frac{Q_\mathbf{k}}{\omega^2 + \omega_{0\mathbf{k}}^2} = \frac{N}{2s}\sum_\mathbf{k}\frac{Q_\mathbf{k}}{2\omega_{0\mathbf{k}}}(1 + 2n_{0\mathbf{k}}), \quad (12)$$

where $N = 3$ and $n_{0\mathbf{k}} = (\exp(\omega_{0\mathbf{k}}/T) - 1)^{-1}$ is the Planck function. The right-hand side of Eq. (12) contains two terms. The first term $Q_\mathbf{k}/2\omega_{0\mathbf{k}}$ is responsible for the quantum fluctuations of the Ω fields. The second term $Q_\mathbf{k}n_{0\mathbf{k}}/\omega_{0\mathbf{k}}$ is responsible for the classical thermal fluctuations of the Ω fields. The role of these two terms is quite different. The quantum fluctuations are small according to the parameter of perturbation theory $1/2s$ and, for the basic approximation, they can be neglected. The thermal fluctuations can be considered in the continuum approximation, which leads to the well-known [1–3] zero-order expression for μ_0, $\mu_0 = T\exp[-2\pi Js^2/(TN)]$, and $\xi = \hbar c_s/\mu$, where ξ is the correlation length. From this expression for μ_0 an important conclusion follows: *in the regime of weak coupling the correlation length ξ is much larger than the lattice constant a.*

To close the theory, it is helpful to define the polarization operator $\Pi(q)$ of the $\mathbf{\Omega}$ field $A_{\lambda quad} = -\frac{1}{2}\sum_q \lambda^*(q)\Pi(q)\lambda(q)$, and the Green function of the λ field is $\Pi(q)^{-1}$. In the lowest approximation, $\Pi(q)$ is simply a loop from two Green functions $G^\Omega\Pi_0(q) = 2NT\sum_{q'} G^\Omega(q')G^\Omega(q - q')$. The main contribution in $1/2s$ for $\Pi_0(q)$ comes from the thermal fluctuations even at low temperatures T, because the integral strength of such fluctuations is fixed by the saddle point condition (12) and does not depend on the temperature. The explicit form for $\Pi_0(q)$ can be obtained in two limiting cases, $\hbar q \gg T$ and $\hbar q \ll T$, where $q^2 = w^2 + c_s^2 k^2$. In the first case, the momentum $q' \sim T/c_s \ll q$, and we can separate summation and integration over q' and put $q' = 0$ in $G^\Omega(q - q')$ in Eq. (14). The result is extremely simple:

$$\Pi_0(q) = 4G^\Omega(q) = \frac{2\mathcal{J}(1 + \gamma_\mathbf{k})}{s(\omega^2 + \omega_{0\mathbf{k}}^2)}, \quad (13)$$
$$q \gg k_T, \quad k_T = T/c_s.$$

Notice that it exceeds the quantum contribution in Eq. (14), $\Pi_0(q) = N/4q$, by a large parameter $16s\mathcal{J}/Nq$. For small $q \ll c_s/a$ and $q \ll k_T$, our results coincide with [3].

The dynamical spin susceptibility $\chi_{ij}(\omega, \mathbf{k})$ for all values of ω and $\mathbf{k}$ can be calculated. In the lowest order in $1/2s$, we can use the lowest order relation $\mathbf{n}(\mathbf{\Omega}(\tau, l), \mathbf{M}(\tau, l), \tau, l) \simeq \exp(ial) \cdot \mathbf{q}_{\mathbf{AF}}\mathbf{\Omega}(\tau, l) - [\mathbf{\Omega}(\tau, l) \times \mathbf{M}(\tau, l)]$, where $\mathbf{q}_{AF} = (\pi/a, \pi/a)$ is the AF vector (7). Calculating the average of two vectors $\mathbf{n}$, we get the dynamical spin susceptibility as a sum of two terms $\chi_{ij}(\omega, \mathbf{k}) = \delta_{ij}[\chi_A(\omega, \mathbf{k}) + \chi_F(\omega, \mathbf{k})]$. The spin susceptibility $\chi_A(\omega, \mathbf{k})$ is responsible for the AF fluctuations. It is proportional to the Green function G_q^Ω analytically continued to imaginary ω and shifted by the

AF vector $\mathbf{q}_{AF}$. For the ferromagnetic spin susceptibility $\chi_F(\omega, \mathbf{k})$, we have a loop expression which can be calculated on the basis of thermal fluctuation domination: $\chi_F(\omega, \mathbf{k}) \simeq -(2s^2/N)G^M(q)$ for $q \geq k_T$. As a result, we have

$$\chi_A(\omega, \mathbf{k}) = -\frac{Js^2 z(1+\gamma_{\mathbf{k}^*})}{2(\omega^2 - \omega^2_{0\mathbf{k}^*} + i\omega\delta)},$$
$$\chi_F(\omega, \mathbf{k}) = -\frac{Js^2 z(1-\gamma_{\mathbf{k}})}{N(\omega^2 - \omega^2_{0\mathbf{k}} + i\omega\delta)}, \quad (14)$$

where $\mathbf{k}^* = \mathbf{k} - \mathbf{q}_{AF}$.

The theory of spin fluctuations in the disordered QAF at sufficiently low temperature $T \ll \mathcal{J}$ allows one to perform the scale separation. In this case, $k_T \ll \pi/a$, the thermal fluctuations can be considered in a "renormalized classical" manner [2]. The magnitude of the quantum fluctuations at $q \leq k_T$ is small as compared with the classical fluctuations. In this situation, the parameters of the effective long-wavelength, low-frequency sigma model are renormalized by the quantum fluctuations. This renormalization is performed with respect to the parameter $1/2s$, but the interaction of the thermal fluctuations with the scales $|\mathbf{k}| \leq k_T$ and $\omega \leq T$ is over parameter $1/N$, where N is the number of components of the $\mathbf{n}$ field of the long-wavelength, low-frequency nonlinear sigma model. This picture follows directly from the approach of this paper.

Unfortunately, the continuum approximation in time does not work when we calculate corrections to the basic approximation. The reason for this observation is in the canonical structure of the Lagrangian (8) and the Green function (11): the sums over ω including this Green function are ambiguous and must be defined at the final time step Δ. Instead of Eq. (4) for the action $A(\mathbf{n})$, we shall use a more accurate expression for $A(\mathbf{n})$, in which the integral over τ is changed to the sum over $\tau = j\Delta$, $j = 0, 1, \ldots, N_\tau - 1$, where $\Delta N_\tau = \beta$. Now $\mathcal{L}_{kin}(j, l)$ is not Berry phase and consists of two parts, $\mathcal{L}_{kin} = \mathcal{L}_{mod} + \mathcal{L}_{pha}$. The first term is purely real and the second term is purely imaginary:

$$\mathcal{L}_{mod} = -\frac{s}{\Delta}\sum_{p=a,b} \ln[(1 + \underline{\mathbf{n}}_p \cdot \mathbf{n}_p)/2],$$
$$\mathcal{L}_{pha} = -\frac{s}{2\Delta}\sum_{p=a,b} \ln\left(\frac{R_p \underline{R}^*_p}{R^*_p \underline{R}_p}\right). \quad (15)$$

Here, the quantity $R_p = \underline{\mathbf{n}}_p \cdot (\mathbf{m}_p + i\mathbf{k}_p)$ for $p = a, b$; vectors $\mathbf{n}$, $\mathbf{m}$, $\mathbf{k}$ were defined in the introduction of the SCS; the underlined quantities $\mathbf{n}$, $\mathbf{m}$, $\mathbf{k}$ correspond to the time $\Delta(j+1)$, and the usual ones correspond to the time Δj. Notice that the Lagrangian $\mathcal{L}_{mod}$ can be expressed in terms of vectors $\mathbf{n}_{a,b}$ only, but $\mathcal{L}_{pha}$ cannot.

The Hamiltonian $\mathcal{H}(\mathbf{n})$ can be obtained on the basis of the following relation for the matrix element of the spin operator $\hat{\mathbf{S}}$: $\langle \underline{\mathbf{n}}|\hat{\mathbf{S}}|\mathbf{n}\rangle = \mathcal{S}(\underline{\mathbf{n}}, \mathbf{n})\langle \underline{\mathbf{n}}|\mathbf{n}\rangle$, where the vector $\mathcal{S}(\underline{\mathbf{n}}, \mathbf{n}) = (\underline{\mathbf{n}} + \mathbf{n} - i[\underline{\mathbf{n}} \times \mathbf{n}])/(1 + \underline{\mathbf{n}} \cdot \mathbf{n})$. If we substitute them into the matrix element of the Heisenberg Hamiltonian, we obtain

$$\mathcal{H}(\mathbf{n}) = Js^2 \sum_{l' \in \langle l \rangle} \mathcal{S}(\underline{\mathbf{n}}, \mathbf{n}) \cdot \mathcal{S}(\underline{\mathbf{n}}', \mathbf{n}'). \quad (16)$$

It was assumed that all vectors $\mathbf{n}_p$, $\mathbf{m}_a$, $\mathbf{k}_p$ for $p = a, b$ entering into Eqs. (15)–(16) are functions of the dynamical variables $\mathbf{\Omega}$ and $\mathbf{M}$ according to Eq. (7). For example, expansion of $\mathcal{L}_{pha}$ in the vector $\mathbf{M}$ has a rather complicated form, but one can prove that it is regular and contains only odd powers of $\mathbf{M}$.

By expanding the Lagrangians $\mathcal{L}_{mod}$, $\mathcal{L}_{pha}$ (15) and the Hamiltonian (16) in the vector $\mathbf{M}$ up to second order, we get $\mathcal{L}_{quad} = (\mathcal{L}_{kin} + \mathcal{H})|_{quad}$:

$$\Delta\mathcal{L}_{quad} = s[1 - \underline{\mathbf{\Omega}} \cdot \mathbf{\Omega} + \mathbf{M}^2 - \underline{\mathbf{M}} \cdot \mathbf{M}$$
$$+ i(\underline{\mathbf{\Omega}} \cdot \mathbf{M} - \mathbf{\Omega} \cdot \underline{\mathbf{M}})] + \Delta Js^2 \sum_{l' \in \langle l \rangle} [\underline{\mathbf{\Omega}} \cdot \mathbf{\Omega} - \mathbf{\Omega} \cdot \mathbf{\Omega}' \quad (17)$$
$$+ \underline{\mathbf{M}} \cdot \mathbf{M} + \mathbf{M} \cdot \mathbf{M}' - i(\underline{\mathbf{\Omega}} \cdot \mathbf{M} - \mathbf{\Omega} \cdot \underline{\mathbf{M}})].$$

According to the analysis performed above, it is necessary to add to the Lagrangian $\mathcal{L}_{quad}$ (17) the quadratic part of the Lagrangian $\mathcal{L}_\lambda$ and the gauge Lagrangian $\mathcal{L}_{ga}$ generalizing for the case of finite time step,

$$\mathcal{L}_{ga} = (s/\Delta)[(\mathbf{\Omega} \cdot \mathbf{M})^2 - (\underline{\mathbf{\Omega}} \cdot \underline{\mathbf{M}})(\mathbf{\Omega} \cdot \mathbf{M})]$$
$$+ Js^2 \sum_{l' \in \langle l \rangle} [(\underline{\mathbf{\Omega}} \cdot \underline{\mathbf{M}})(\mathbf{\Omega} \cdot \mathbf{M}) + (\mathbf{\Omega} \cdot \mathbf{M})(\mathbf{\Omega}' \cdot \mathbf{M}')],$$

which also kills the strongest interaction between the $\mathbf{\Omega}$ and $\mathbf{M}$ fields. The total quadratic Lagrangian is $\mathcal{L}_{tqu} = \mathcal{L}_{quad} + \mathcal{L}_{ga} + \mathcal{L}_{\lambda quad}$. The Green function for this case is

$$\hat{G}_q = \frac{1}{2s\bar{L}(q)}\begin{pmatrix} 1 - c_\omega + \Delta\mathcal{J}(c_\omega + \gamma_{\mathbf{k}}) & -s_\omega(1 - \Delta\mathcal{J}) \\ s_\omega(1 - \Delta\mathcal{J}) & 1 - c_\omega + \Delta\mathcal{J}(c_\omega - \gamma_{\mathbf{k}}) + \Delta\mu_0^2/2\mathcal{J} \end{pmatrix}, \quad (18)$$
$$\bar{L}(q) \simeq (1 - \Delta\mathcal{J} + \Delta\mu_0^2/4\mathcal{J})[2(1 - c_\omega) + \Delta^2\omega^2_{0\mathbf{k}}].$$

Here, $c_\omega = \cos(\omega\Delta)$ and $s_\omega = \sin(\omega\Delta)$; the quantities $Q_{\mathbf{k}}$, $P'_{\mathbf{k}}$, and the bare frequency $\omega_{0\mathbf{k}}$ were defined in Eq. (11). At $\Delta\omega \ll 1$, this Green function G_q transforms into Eq. (11), to the normalization factor $1/\Delta$. The Green function (18) is well defined in the sense that the summation over ω in the expressions including it must be performed in the limits $-\pi/\Delta \le \omega \le \pi/\Delta$. The result of such averaging depends crucially on the contribution at large $\omega \simeq \pi/\Delta$. For example, we have $\langle M_i M_j\rangle = (1/4s)\delta_{ij}(1 + c_0 - c_1)$, $\langle \underline{M}_i M_j\rangle = (1/4s)\delta_{ij}(c_0 - c_1)$. We see that the average $\langle \mathbf{M}^2\rangle$ discussed above, in fact, corresponds to the average $\langle \mathbf{M}\cdot\underline{\mathbf{M}}\rangle$ but the average $\langle \mathbf{M}^2\rangle$ is different.

The free energy of QAF in the paramagnetic state has three contributions, $F_{AF} = -T\ln(Z) = F_{\Omega M} + F_\lambda + F_{ga}$, $Z = Z_{\Omega M}Z_\lambda Z_{ga}$. In the lowest approximation in $1/2s$, $Z_{\Omega M}$, Z_λ, and Z_{ga} are powers of determinants. The explicit form of these determinants leads to

$$F_{\Omega M} = \frac{TNN_s}{2}\sum_{\omega\mathbf{k}} \ln[\bar{L}(q)],$$

$$F_{ga} = -\frac{TN_s}{2}\sum_{\omega\mathbf{k}} \ln[2s\bar{Q}(q)], \qquad (19)$$

$$F_\lambda = \frac{TN_s}{2}\sum_{\omega\mathbf{k}} \ln[s^2\Pi_0(q)].$$

One can verify that $F_{\Omega M}$ has a finite limit at $\Delta \longrightarrow 0$, $\Delta N_\tau = \beta$. F_{ga} and F_λ do not have a finite limit at $\Delta \longrightarrow 0$, $\Delta N_\tau = \beta$ separately, but their sum has a finite limit. After some transformation, the free energy F_{QAF} of QAF in the lowest order in $1/2s$ can be presented in the form $F_{QAF} = ((N-1)/N)F_{\Omega M} + F_{\lambda l}$, where

$$F_{\Omega M} = -N_s\mathcal{J} + 2N_s\sum_{\mathbf{k}}\{\omega_{0\mathbf{k}}/2 + T\ln[1 - \exp(-\omega_{0\mathbf{k}}/T)]\}, \qquad (20)$$

$$F_{\lambda l} = \frac{TN_s}{2}\sum_{\omega\mathbf{k}} \ln\left[\frac{s(\omega^2 + \omega_{0\mathbf{k}}^2)\Pi_0(q)}{2\mathcal{J}(1+\gamma_{\mathbf{k}})}\right].$$

Here, $2N_s$ is the number of lattice sites, and the polarization operator $\Pi_0(q)$was defined above. The temperature-dependent part of free energy (20) at low temperatures $T \ll \mathcal{J}$ is proportional to $F_{AF} \approx N_s T^3/\mathcal{J}$. Such a contribution has two origins: one from $F_{\Omega M}$ and another one from $F_{\lambda l}$.

Now, we present the result of the calculation of corrections to the mass operators of the $\mathbf{\Omega}$ and $\mathbf{M}$ fields. In the lowest order in $1/2s$, these corrections can be presented as renormalization of the initial quadratic Lagrangian (17). It is necessary to have the Lagrangian $\mathcal{L}_{mod}$ and the Hamiltonian $\mathcal{H}$ up to the fourth order in field $\mathbf{M}$, and the Lagrangian $\mathcal{L}_{pha}$ up to the third order.

The effective Lagrangian $\mathcal{L}_{eff}$ in the first $1/2s$ approximation is

$$\begin{aligned}\Delta\mathcal{L}_{eff} = s[&a_0(1 - \mathbf{\Omega}\cdot\underline{\mathbf{\Omega}}) + b_0(\mathbf{M}^2 - \mathbf{M}\cdot\underline{\mathbf{M}}) \\ &- ie_0(\mathbf{\Omega}\cdot\underline{\mathbf{M}} - \underline{\mathbf{\Omega}}\cdot\mathbf{M})] + \Delta Js^2\sum_{l'\in\langle l\rangle}[a_1(1 - \mathbf{\Omega}\cdot\underline{\mathbf{\Omega}}) \\ &+ a_2(1 - \mathbf{\Omega}\cdot\mathbf{\Omega}') + a_3(1 - \mathbf{\Omega}'\cdot\underline{\mathbf{\Omega}}) + b_1\mathbf{M}^2 \\ &+ b_2\mathbf{M}\cdot\underline{\mathbf{M}} + b_3\mathbf{M}\cdot\mathbf{M}' + b_4\mathbf{M}'\cdot\underline{\mathbf{M}} \\ &- ie_1(\mathbf{\Omega}\cdot\underline{\mathbf{M}} - \underline{\mathbf{\Omega}}\cdot\mathbf{M}) - ie_2(\mathbf{\Omega}'\cdot\underline{\mathbf{M}} - \underline{\mathbf{\Omega}}\cdot\mathbf{M}')],\end{aligned} \qquad (21)$$

where the constants $a_0, \ldots, e_2$ are $a_i = a_i^0 + g\alpha_i$, $b_i = b_i^0 + g\beta_i$, $e_i = e_i^0 + g\gamma_i$, where $g = (N-1)/4s$, $i = 0, 1, 2, 3$. The constants a_i^0, b_i^0, e_i^0 follow from Eq. (17). The explicit form of the constants α_i, β_i, γ_i will be presented in the complete version of this paper.

We shall give the explicit result for the correlation radius in this order in $1/2s$ on the basis of Eq. (12). The contribution of different frequencies ω and momenta $\mathbf{k}$ in this constraint relation can be separated into two parts. The first part is the high frequency and momentum part. To calculate this contribution it is sufficient to take the Green function $G^\Omega(q)$ in bare approximation (18), because this contribution, is of the order of $1/2s$. The second contribution, which is proportional to the distribution function $n_{\mathbf{k}}$, can be considered in the continuum approximation, but with $1/2s$ corrections taken into account: $G^\Omega(q) \simeq 1/[2a^2\chi_\perp\Delta(\omega^2 + \omega_{\mathbf{k}}^2)$, $\chi_\perp = \tilde{\rho}_s/c_s^2$, $\omega_{\mathbf{k}}^2 = c_s^2\mathbf{k}^2 + \mu^2$. Here, $\tilde{\rho}_s = Js^2a_{23}$; $c_s^2 = e_0^{-2}a_{23}b_{1234}\mathcal{J}^2a^2/z$, where $a_{23} = a_2 + a_3$, $b_{1234} = b_1 + b_2 + b_3 + b_4$. Now, instead of Eq. (12), we have $(N/4s\tilde{\rho}_s)\sum_{\mathbf{k}}[n_{\mathbf{k}}/\omega_{\mathbf{k}}] = R$, $R = 1 - g(1 + c_0 + c_1)$. The factor R includes in itself the direct short-wavelength renormalizations. Performing the integration, we have $\mu = T\exp[-2\pi\rho_s/TN]$, $\rho_s = \tilde{\rho}_s R$, $\xi = \hbar c_s/\mu$. The actual temperature dependence is changed in the preexponent factor ($T \longrightarrow \mathcal{J}$) if we take into account the long-wavelength fluctuations in the next order in $1/N$ approximation [3].

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