

The representations of the Hubbard algebra in terms of spin-fermion operators and motion of a hole in an antiferromagnetic state

V.I.Belinicher and A.L.Chernyshev

Institute of Semiconductor Physics, 630090 Novosibirsk, Russia

Abstract

The representation of the Hubbard operators in terms of the $\text{spin}-\frac{1}{2}$ operators and the fermion operator with $\text{spin}-\frac{1}{2}$ is proposed. In the low-energy limit this representation is reduced to the representation following from the Hubbard diagramm technique. In framework of this approach motion of a hole in an antiferromagnetic state of the t-J model is considered. It is shown that the primary hole energy is strongly renormalized and the band width has an order of J rather than t. The functional integral for the strongly correlated model induced by the obtained representation is formulated. The representation of the total Hubbard algebra for states in the lower and the upper Hubbard bands is formulated in terms of the $\text{spin}-\frac{1}{2}$ and two fermion fields with $\text{spin}-\frac{1}{2}$ is formulated.

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I. Introduction

The strong correlation in the electron system can be described by the restriction of the number of states at every lattice site. Such restriction can be described by the introduction of the Hubbard operators. But the Hubbard operators are not convenient for description of the statistical and dynamical properties of the strongly correlated electron system. That is why some representation of the Hubbard operators in terms of more convenient operators are used. The well known representation of such type is slave-boson (fermion) representation [1, 2, 3]. These representations are based on the mapping of three electron states $|0\rangle, |\uparrow\rangle, |\downarrow\rangle$ at a lattice site into fermion and boson Gilbert space and restriction of the one-particle states.

This work has mainly methodical character. At first the simple historical review of the description of spin system on the basis of the Holstein-Primakoff and the Dyson-Maleev representation is presented. Further the mapping of three Hubbard states into the eight spin- $\frac{1}{2}$ and fermions (with a spin- $\frac{1}{2}$) Gilbert space is produced. Such mapping generates the representation of the Hubbard operators in terms of fermion and spin- $\frac{1}{2}$ operators. Such representation can be useful in some applications.

Renormalization of the hole energy in a antiferromagnetic state of the $t - J$ model is considered in the next section. This consideration is based on the approach to an antiferromagnetic state developed in work [7]. We will show that the method developed by Kane, Lee and Read [9] for the description of the motion of a single hole in quantum antiferromagnet is also applicable to the our approach. We will show that the hole spectrum is strongly renormalized in the limit $t \ll J$ where t and J are the hopping and exchange constants for the $t - J$ model. The quasiparticle band has the width J and the quasiparticle residue is equal to J/t .

After that two methods of obtaining the functional integral for strongly correlated electron system is discussed. In final part of the work the representation of the total Hubbard algebra is constructed and its physical interpretation in terms of the local spins, electrons in the upper Hubbard bands and holes in the lower Hubbard bands is presented.

II. Spin systems

For the clarification of the structure of representation for the Hubbard model we want to remember connection between the Holstein-Primakoff (HP) [4] and the Dyson-Maleev (DM) [5, 6] representations. They based on mapping of the $2S+1$ spin states into the lowest $2S+1$ states of the Heisenberg algebra:

$$\begin{aligned} S_{HP}^- &= P(a^\dagger a) \sqrt{2S - a^\dagger a} a P(a^\dagger a) \\ S_{HP}^+ &= P(a^\dagger a) a^\dagger \sqrt{2S - a^\dagger a} P(a^\dagger a) \\ S_{HP}^z &= P(a^\dagger a) (-S + a^\dagger a) P(a^\dagger a) \end{aligned} \quad (2.1)$$

here S^+, S^-, S^z are spin operators, a^\dagger, a are creation and annihilation Bose operators,

$P(a^+a)$ is the projector operator on the lower $2S+1$ states: $|0\rangle, |\uparrow\rangle, \dots, |2S\rangle$. The representation (2.1) is the direct operator identity: the matrix elements of the original spin operator \mathbf{S} and the HP spin operators \mathbf{S}_{HP} are the same on the lowest $2S+1$ states of the Heisenberg algebra. The DM representation for spin operators has the form:

$$\begin{aligned} S_{DM}^- &= a \\ S_{DM}^+ &= a^+(2S - a^+a) \\ S_{DM}^z &= -S + a^+a \end{aligned} \quad (2.2)$$

This representation does not represent a direct operator identity but the spin algebra is valid for DM spin operators (2.2) also. One can check that there exists the operator $V(a^+a)$ with property

$$\mathbf{S}_{HP} = PV^{-1}\mathbf{S}_{DM}VP \quad (2.3)$$

here $\mathbf{S}_{HP}, \mathbf{S}_{DM}$ are the HP (2.1) and the DM (2.2) spin operators. The identity (2.3) means that matrix elements of both parts of Eq.(2.3) between any states $\langle n|$ and $|n'\rangle$ are equal to each other. If $|n, n'| > 2S$ these matrix elements are equal to zero. The explicit form of the matrix elements V is following

$$\langle n'|V|n\rangle = \delta_{n',n} \prod_{m=0}^n \sqrt{2S+1-m}, \quad n, n' \leq 2S \quad (2.4)$$

If we want to calculate the partition function

$$Z = \text{Tr}(\exp(-\beta H(\mathbf{S}))) = \text{Tr}(P \exp(-\beta H(\mathbf{S}_{HP}))) \quad (2.5)$$

we can use the identity (2.3) and the commutation of the operators V and P . As a result, we have for Z

$$Z = \text{Tr}(P \exp(-\beta H(\mathbf{S}_{DM}))). \quad (2.6)$$

Eq.(2.6) is valid because the action of \mathbf{S}_{DM} on the state $|n\rangle$ for $n = 0, 1, \dots, 2S$ leads only to the same states, i.e. the lower $2S+1$ states form an invariant subspace with respect to the DM spin operators \mathbf{S}_{DM} .

If we are interested in the properties of the partition function (2.5) at the low temperatures in the ferromagnetic or antiferromagnetic state, we can omit the projector P in formula (2.6) because the contribution to the higher states $|n\rangle$ with $n > 2S$ into the trace (2.6) is exponentially small over the parameter βJ where J is an exchange integral. This discussion explains the correctness of using the DM representation for the description of the low-energy processes at the low temperatures in ferromagnets and antiferromagnets.

III. Strongly correlated systems.

A similar interpretation can be given for the representation for the Hubbard operators X^{ab} . Let us consider the Gilbert space at every lattice site representing the direct product of a Gilbert space of a spin $-\frac{1}{2}$ and fermion with a spin $-\frac{1}{2}$. The total number of states in that spin-hole Gilbert space is equal to eight: $|0, \sigma\rangle$, $|1, \sigma', \sigma\rangle$, $|2, \sigma\rangle$, here the first index represents the number of fermions; $\sigma, \sigma' = \uparrow, \downarrow$ are spin projections. One can introduce the Fermi operators h_σ^+, h_σ then

$$|1, \sigma', \sigma\rangle = h_{\sigma'}^+ |0, \sigma\rangle, \quad |0, \sigma\rangle = h_{\sigma'} |1, \sigma', \sigma\rangle. \quad (3.1)$$

The singlet and triplet states can be formed from one-fermion states:

$$\begin{aligned} |s\rangle &= (1/\sqrt{2})(|1\uparrow, \downarrow\rangle - |1\downarrow, \uparrow\rangle), & |t1\rangle &= |1\uparrow, \uparrow\rangle \\ |t0\rangle &= (1/\sqrt{2})(|1\uparrow, \downarrow\rangle + |1\downarrow, \uparrow\rangle), & |t-1\rangle &= |1\downarrow, \downarrow\rangle. \end{aligned} \quad (3.2)$$

We can map the initial Gilbert space of the Hubbard model without two-fermion states into the spin-hole Gilbert space:

$$|0\rangle \Rightarrow |s\rangle, \quad |1, \sigma\rangle \Rightarrow |0, \sigma\rangle, \quad (3.3)$$

and obtain the following representation of the Hubbard operators X^{ab} in terms of the holes h_σ^+, h_σ and the spin $-\frac{1}{2}$ operators \mathbf{s} :

$$\begin{aligned} X_{HP}^{0\sigma} &= 2\sigma\sqrt{2}(h^+\hat{S})_{-\sigma}(1 - \hat{n}) \\ X_{HP}^{\sigma 0} &= 2\sigma\sqrt{2}(1 - \hat{n})(\hat{S}h)_{-\sigma} \\ \hat{N}_{HP} &= X_{HP}^{\uparrow\uparrow} + X_{HP}^{\downarrow\downarrow} = 1 - \hat{n} + \hat{d} \\ X_{HP}^{00} &= (1/4)(\hat{n} - 2\hat{d} - 2(h^+\sigma h)\mathbf{s}) \\ \mathbf{S} &= \mathbf{s}(1 - \hat{n} + \hat{d}) \end{aligned} \quad (3.4)$$

here

$$\hat{S} = (1/4)(1 - 2\mathbf{s}\sigma), \quad \hat{n} = (h^+h), \quad \hat{d} = h_\uparrow^+ h_\uparrow h_\downarrow^+ h_\downarrow. \quad (3.5)$$

This representation is the direct operator identity: (1) the matrix elements X_{HP}^{ab} operators (3.4) between the physical states $|s\rangle$, $|0\sigma\rangle$ are the same as for the initial Hubbard operators X^{ab} ; (2) the matrix elements between the unphysical states $|tm\rangle$, $|2\sigma\rangle$ are equal to zero; (3) the matrix elements between physical states $|s\rangle$, $|0\sigma\rangle$ and unphysical states $|tm\rangle$, $|2\sigma\rangle$ are equal to zero. The representation (3.4) for X_{HP}^{ab} certainly is hermitian and does not require any constraints. If we compare the representation (3.4) with initial expression of the Hubbard operators X^{ab} in terms of the physical Fermi operators of electrons we can see that the number of electrons is close to unity while the number of Fermi holes is small at filling closed to unity. The empty space without

electrons can be imaged as the bound singlet state of the Fermi hole and the spin $-\frac{1}{2}$ with the total spin equal to zero.

In addition to representation (3.4) in work [7] was proposed the nonhermitian representation for the Hubbard operators

$$\begin{aligned} X_{DM}^{0\sigma} &= 2\sigma h^+(1/2 - (\sigma\mathbf{s}) + (h^+h))_{-\sigma}, & X_{DM}^{\sigma 0} &= 2\sigma h_{-\sigma}, \\ \hat{N}_{DM} &= X_{DM}^{\uparrow\uparrow} + X_{DM}^{\downarrow\downarrow} = 1 - \hat{n}, & X_{DM}^{00} &= \hat{n}, & \mathbf{S} &= \mathbf{s} + (1/2)(h^+\sigma h). \end{aligned} \quad (3.6)$$

This representation does not present direct operator identity but the Hubbard algebra is valid for this representation. The representation (3.6) can be named as the Dyson-Maleev representation for the Hubbard operators.

What is the connection between representations (3.4) and (3.6) for the Hubbard operators ? One can check that if we produce canonical transformation of the hole Fermi operators in (3.6) $h_\sigma \rightarrow \sqrt{2}h_\sigma$, $h_\sigma^+ \rightarrow (1/\sqrt{2})h_\sigma^+$ then the matrix elements of the Hubbard operators in the representation (3.4) and (3.6) are the same between the physical states $|s\rangle, |0\sigma\rangle$. Moreover the action of the Hubbard operators in the form (3.6) on the physical states $|s\rangle, |0\sigma\rangle$ does not lead to the unphysical states $|tm\rangle, |2\sigma\rangle$, i.e., the physical states form invariant subspace over the algebra (3.6). The relation between the Hubbard operators X_{HP}^{ab} (3.4) and X_{DM}^{ab} (3.6) can be represented in a form similar to (2.3)

$$X_{HP}^{ab} = PV^{-1}X_{DM}^{ab}VP \quad (3.7)$$

here V is the generator of the canonical transformation determined in the physical subspace

$$V_{ss} = \sqrt{2}, \quad V_{\sigma,\sigma'} = \delta_{\sigma,\sigma'}, \quad V_{s,\sigma} = V_{\sigma,s} = 0 \quad (3.8)$$

and P is the projector on physical subspace

$$P = (1/4)(2 - \hat{n})[2 - \hat{n} - 2\hat{n}(h^+\sigma h)\mathbf{s}]. \quad (3.9)$$

Naturally the operators P and V commute. The partition function of the Hubbard model can be presented in the form similar to (2.5), (2.6)

$$Z = Tr(P \exp(-\beta H(X_{HP}^{ab})) = Tr(P \exp(-\beta H(X_{DM}^{ab}))) \quad (3.10)$$

Since the energies of the two-hole states and the triplet states are essentially higher, one can omit the projector P in (3.10) at the low temperatures and work with the transformed Hamiltonian. This approximation is valid for small number of holes so far as the energy of the two-hole states at the lattice site and the triplet state are situated approximately at the center of the singlet hole bands.

IV. Renormalization of the hole energy in an antiferromagnetic state of the $t - J$ model

In this part of the work we will concern ourselves with an antiferromagnetic state for the $t - J$ model [7, 9, 10] and will discuss the hole energy.

As was recently proposed by Kane, Lee and Read [9] (KLR hereafter) in the limit that the exchange energy J is much less than the hopping matrix element t , the hole spectrum is strongly renormalized by the interaction with spin excitations. It was established that the hole can be described by a narrow quasiparticle band with the quasiparticle residue of an order J/t and a bandwidth of an order J , both for the Neel and RVB groundstates. Here we consider the groundstate of the local spins to be a quantum Neel state. By applying KLR approach we will show that our case is rather more complicated than in "holon" model, but the results are the same.

An antiferromagnetic state for the Hubbard model was considered in Ref. [7]. The Hamiltonian of the $t - J$ model in Ref. [7] was obtained on the basis of the nonhermitian representation for the Hubbard operators (3.6)

$$H_{t-J} = t \sum_{\langle n, n' \rangle} h_n^+ [-1/2 - (h_n^+ h_n) + (\mathbf{S}_n \sigma)] h_n + J \sum_{\langle n, n' \rangle} \mathbf{S}_n \mathbf{S}_{n'} \quad (4.1)$$

Retaining the notations of [7] we will write the Hamiltonian of the model.

The Heisenberg part of the Hamiltonian (4.1) expressed through the bose-operators b and c , connected by $u - v$ transformation with the primary Dyson-Maleev bose-operators, which are associated with two Neel sublattices. It has the form:

$$\begin{aligned} H^s &= \sum_{\mathbf{k}} \Omega_{\mathbf{k}} (b_{\mathbf{k}}^+ b_{\mathbf{k}} + c_{\mathbf{k}}^+ c_{\mathbf{k}}), \\ \Omega_{\mathbf{k}} &= zJ(1 - \gamma_{\mathbf{k}}^2)^{1/2}, \\ \gamma_{\mathbf{k}} &= 1/2(\cos(k_x a) + \cos(k_y a)). \end{aligned} \quad (4.2)$$

Next we consider the addition of holes. The hopping Hamiltonian (4.1) expressed in terms of two-sublattice excitations and holes in lower band is

$$\begin{aligned} H^{sp1} &= 1/2 \sum_{\mathbf{k}, \mathbf{q}} |t_{\mathbf{k}}| (U_{\mathbf{q}} + V_{\mathbf{q}}) [(b_{\mathbf{q}}^+ + c_{-\mathbf{q}}) p_{\downarrow \mathbf{k}-\mathbf{q}}^+ p_{\uparrow \mathbf{k}} + (b_{\mathbf{q}} + c_{-\mathbf{q}}^+) p_{\uparrow \mathbf{k}+\mathbf{q}}^+ p_{\downarrow \mathbf{k}}] \\ H^{sp2} &= 1/2 \eta^2 \sum_{\mathbf{k}, \mathbf{q}, \mathbf{k}'} |t_{\mathbf{k}}| [p_{\uparrow \mathbf{k}-\mathbf{q}}^+ p_{\uparrow \mathbf{k}} (-1/2 + V_{\mathbf{k}'}^2 + \\ &U_{\mathbf{k}'} U_{\mathbf{k}'+\mathbf{q}} b_{\mathbf{k}'+\mathbf{q}}^+ b_{\mathbf{k}'} + V_{\mathbf{k}'} V_{\mathbf{k}'+\mathbf{q}} c_{\mathbf{k}'+\mathbf{q}}^+ c_{\mathbf{k}'} + U_{\mathbf{q}-\mathbf{k}'} V_{\mathbf{k}'} b_{-\mathbf{k}'+\mathbf{q}}^+ c_{\mathbf{k}'}^+ + \\ &V_{-\mathbf{k}'} U_{\mathbf{k}'-\mathbf{q}} c_{-\mathbf{k}'} b_{\mathbf{k}'-\mathbf{q}} + p_{\downarrow \mathbf{k}-\mathbf{q}}^+ p_{\downarrow \mathbf{k}} (b \leftrightarrow c)], \end{aligned} \quad (4.3)$$

where $\eta^2 = 1/\sqrt{4z}$; $U_{\mathbf{q}} = ((1 + \nu_{\mathbf{q}})/(2\nu_{\mathbf{q}}))^{1/2}$, $V_{\mathbf{q}} = ((1 - \nu_{\mathbf{q}})/(2\nu_{\mathbf{q}}))^{1/2}$, $\nu_{\mathbf{q}} = (1 - \gamma_{\mathbf{q}}^2)^{1/2}$, $p_{\alpha \mathbf{k}}^+, p_{\alpha \mathbf{k}}$ are holes operators in the lower hole band. The first two terms in the round brackets in H^{sp2} gives in mean-field approximation the primary dispersion of the holes: $\omega_{\mathbf{k}} = |t_{\mathbf{k}}|/\sqrt{4z}$, where z - number of the nearest neighbours, $t_{\mathbf{k}} \equiv zt\gamma_{\mathbf{k}}$.

The terms in (4.3) with unconserving of the number of the particles give the two self-energy diagrams at zero temperature corresponding to emission and absorption one and two magnons.

So the differences between the "holons", or spinless holes approach [9, 10] and this model are: (1) the existence of the primary mean-field dispersion of the hole, (2) two types of the self-energy diagrams which strongly renormalized the hole spectrum.

One can obtain from the Hamiltonian (4.3) the expression of the self-energy for the hole with particular spin projection.

It consists of the two contributions containing emission and absorption one and two magnons. Since the model is symmetric over the spin projections the self-energy in the non-crossing approximation (KLR) has the form:

$$\begin{aligned} \Sigma(\mathbf{k}, \omega) = & \omega_{\mathbf{k}}^0 + \sum_{\mathbf{q}} f_1(\mathbf{k}, \mathbf{q}) G(\mathbf{k} - \mathbf{q}, \omega - E_{\mathbf{q}}) + \\ & \sum_{\mathbf{q}, \mathbf{k}'} f_2(\mathbf{k}, \mathbf{q}, \mathbf{k}') G(\mathbf{k} - \mathbf{q}, \omega - E_{\mathbf{k}'} - E_{\mathbf{k}' - \mathbf{q}}), \end{aligned} \quad (4.4)$$

Where f_1 and f_2 - "two-hoppes" functions, which contain the information about different types of coupling of a hole with spin excitations. One can easily get from (4.3)

$$\begin{aligned} f_1(\mathbf{k}, \mathbf{q}) &= (1/4) t_{\mathbf{k}} t_{\mathbf{k} - \mathbf{q}} (U_{\mathbf{q}} + V_{\mathbf{q}})^2 \\ f_2(\mathbf{k}, \mathbf{k}', \mathbf{q}) &= z t_{\mathbf{k}} t_{\mathbf{k} - \mathbf{q}} V_{\mathbf{k}' - \mathbf{q}}^2 U_{\mathbf{k}'}^2. \end{aligned} \quad (4.5)$$

In the KLR method we have the selfconsistent integral equation for the hole propagator:

$$G(\mathbf{k}, \omega) = \frac{1}{\omega - \Sigma(\mathbf{k}, \omega)}. \quad (4.6)$$

Using the dominant pole approximation we write the hole propagator as,

$$G(\mathbf{k}, \omega) = \frac{a_{\mathbf{k}}}{\omega - \omega_{\mathbf{k}} + i\Gamma_{\mathbf{k}}} + G_{inc}, \quad (4.7)$$

where $\omega_{\mathbf{k}} = \text{Re}\Sigma(\mathbf{k}, \omega_{\mathbf{k}})$, and $\Gamma_{\mathbf{k}} = \text{Im}\Sigma(\mathbf{k}, \omega_{\mathbf{k}})$. As pointed out in Ref.[9] there is a general statement that $\Gamma_{\mathbf{k}} = 0$ for all of the low-energy poles. Since we are interested in the low energies, we can set $\Gamma_{\mathbf{k}} = 0$.

The quasiparticle residue in (4.7)

$$a_{\mathbf{k}} = \frac{1}{1 - \frac{\partial \Sigma}{\partial \omega}(\mathbf{k}, \omega_{\mathbf{k}})} \quad (4.8)$$

may be estimated by omitting the incoherent part in (4.7)

$$\begin{aligned} a_{\mathbf{k}} \leq & [1 + \sum_{\mathbf{q}} f_1(\mathbf{k}, \mathbf{q}) \frac{a_{\mathbf{k} - \mathbf{q}}}{(\omega_{\mathbf{k}} - \omega_{\mathbf{k} - \mathbf{q}} - E_{\mathbf{q}})^2} + \\ & \sum_{\mathbf{q}, \mathbf{k}'} f_2(\mathbf{k}, \mathbf{q}, \mathbf{k}') \frac{a_{\mathbf{k} - \mathbf{q}}}{(\omega_{\mathbf{k}} - \omega_{\mathbf{k} - \mathbf{q}} - E_{\mathbf{k}'} - E_{\mathbf{k}' - \mathbf{q}})^2}]^{-1}. \end{aligned} \quad (4.9)$$

By applying the scheme of [9] we will evaluate the integrals in the denominator of (4.9) and will show that both integrals are of the order $t/J \gg 1$.

Near the bottom of the spectrum, where $\omega_k = 0$ and in the case $J = 0$ we get the divergence of the integrals at the small $|\mathbf{q}|$:

$$\begin{aligned} I_1 &= (t^2 z^2 / 8\sqrt{2}) \gamma_{\mathbf{k}^*}^2 \sum_{\mathbf{q}} |\mathbf{q}| \frac{a_{\mathbf{k}^*}}{\left[\frac{q^2}{2m^*}\right]^2} \\ I_2 &= (t^2 z^3 / 2) \gamma_{\mathbf{k}^*}^2 \sum_{\mathbf{q}, \mathbf{k}'} \frac{1}{|\mathbf{k}'| |\mathbf{k}' - \mathbf{q}|} \frac{a_{\mathbf{k}^*}}{\left[\frac{q^2}{2m^*}\right]^2}, \end{aligned} \quad (4.10)$$

where m^* is the mass at the bottom of the band. Throughout this work the distance between lattice sites $a = 1$. The both integrals (4.10) are diverged in the dimensions two and three. Here we will be concerned with only in the case of $d = 2$. In the case of small but finite J we have

$$I_1' \sim t^2 \sum_{\mathbf{q}} |\mathbf{q}| \frac{a_{\mathbf{k}^*}}{\left[\frac{q^2}{2m^*} + J|\mathbf{q}|\right]^2} \approx t^2 \frac{m^* a_{\mathbf{k}^*}}{J}. \quad (4.11)$$

Furthermore, as shown in [9] $m^* = m/a_{\mathbf{k}^*}$ where m depends only on t and $m \sim 1/t$. So

$$I_1' \sim \frac{t}{J}. \quad (4.12)$$

The integral I_2' has the more complicated form. In the case $d = 2$

$$I_2' \sim t^2 \int \int \int d|\mathbf{q}| d|\mathbf{k}'| d\varphi \frac{|\mathbf{q}|}{|\mathbf{k}' - \mathbf{q}|} \frac{a_{\mathbf{k}^*}}{\left[\frac{q^2}{2m^*} + J|\mathbf{k}'| + J|\mathbf{k}' - \mathbf{q}|\right]^2}, \quad (4.13)$$

and its mathematical treatment is unwieldy, but the result is the same

$$I_2' \sim t^2 \frac{m^* a_{\mathbf{k}^*}}{J} \sim \frac{t}{J}. \quad (4.14)$$

thus, the quasiparticle residue $a_{\mathbf{k}} \leq J/t$.

By making physically justified assumption about the behavior of the imaginary part of the self-energy, one can obtain a rough estimation for $a_{\mathbf{k}}$ which is in coincides with (4.14). Now this assumption may be checked.

Using the Kramers-Kronig relation we have for the quasiparticle residue

$$a_{\mathbf{k}} = \left(1 + \int \Gamma(\mathbf{k}, y) / (y - \omega_{\mathbf{k}})^2 dy\right)^{-1}. \quad (4.15)$$

As it was argued by KLR, $\Gamma(\mathbf{k}, \omega)$ vanishes like a power of $(\omega - \omega_{\mathbf{k}})$ at $(\omega - \omega_{\mathbf{k}}) \ll J$. At $(\omega - \omega_{\mathbf{k}}) \geq J$ we expect that the scattering dominates and $\Gamma(\mathbf{k}, \omega) \simeq t$. Therefore, the integral in the denominator of (4.15) may be cut at $y = J$

$$a_{\mathbf{k}} = \left(1 + \int_J^\infty y^{-2} dy\right)^{-1}. \quad (4.16)$$

The behavior of the imaginary part of the self-energy in non-crossing approximation may be obtained from expressions

$$\begin{aligned}\Gamma(\mathbf{k}, \omega) &= \frac{1}{\pi} \text{Im} \Sigma(\mathbf{k}, \omega) = \Gamma_1(\mathbf{k}, \omega) + \Gamma_2(\mathbf{k}, \omega), \\ \Gamma_1(\mathbf{k}, \omega) &= \sum_{\mathbf{q}} f_1(\mathbf{k}, \mathbf{q}) A(\mathbf{k} - \mathbf{q}, \omega - E_{\mathbf{q}}), \\ \Gamma_2(\mathbf{k}, \omega) &= \sum_{\mathbf{q}, \mathbf{k}'} f_2(\mathbf{k}, \mathbf{q}, \mathbf{k}') A(\mathbf{k} - \mathbf{q}, \omega - E_{\mathbf{k}'} - E_{\mathbf{k}' - \mathbf{q}}),\end{aligned}\quad (4.17)$$

where $A(\mathbf{k}, \omega) = \frac{1}{\pi} \text{Im} \Gamma(\mathbf{k}, \omega)$ is the spectral function.

For $(\omega - \omega_{\mathbf{k}}) \ll J$, the dominant contribution to $\Gamma(\mathbf{k}, \omega)$ will come from the pole of $A(\mathbf{k}, \omega)$. Furthermore, the leading contribution will be given by the very small $|\mathbf{q}|$ and $|\mathbf{k}'|$. For $\Gamma_1(\mathbf{k}, \omega)$ we may write

$$\Gamma_1(\mathbf{k}, \omega) \approx (t^2 z^2 / 8\sqrt{2}) \gamma_{\mathbf{k}}^2 a_{\mathbf{k}} \sum_{\mathbf{q}} |\mathbf{q}| \delta(\omega - \omega_{\mathbf{k}} - J|\mathbf{q}|) \approx \gamma_{\mathbf{k}}^2 \frac{t^2 a_{\mathbf{k}}}{J} \left(\frac{\omega - \omega_{\mathbf{k}}}{J} \right)^2. \quad (4.18)$$

So, $\Gamma_1(\mathbf{k}, \omega) \sim (\omega - \omega_{\mathbf{k}})^2$ very near to the pole. Extrapolating Eqs. (4.17) in the region $(\omega - \omega_{\mathbf{k}}) \sim J$ we get $\Gamma_1(\mathbf{k}, \omega) \sim t$. For $\Gamma_2(\mathbf{k}, \omega)$ we obtain

$$\Gamma_2(\mathbf{k}, \omega) = (t^2 z^3 / 2) \gamma_{\mathbf{k}}^2 \frac{a_{\mathbf{k}}}{J} \sum_{\mathbf{q}, \mathbf{k}'} \frac{1}{|\mathbf{k}'| |\mathbf{k}' - \mathbf{q}|} \delta\left(\frac{\omega - \omega_{\mathbf{k}}}{J} - |\mathbf{k}' - \mathbf{q}| - |\mathbf{k}'|\right). \quad (4.19)$$

After some mathematical treatment one can get

$$\Gamma_2(\mathbf{k}, \omega) \approx \frac{t^2 a_{\mathbf{k}}}{J} \left(\frac{\omega - \omega_{\mathbf{k}}}{J} \right)^2, \quad (4.20)$$

and at $(\omega - \omega_{\mathbf{k}}) \sim J \Rightarrow \Gamma(\mathbf{k}, \omega) \sim t$.

Thus, we have demonstrated the application of the approach of KLR method to this model and have obtained the similar results, i.e. strong renormalization of a hole spectrum, narrow quasiparticle band with the quasiparticle residue J/t and the bandwidth J .

V. The functional integral for the strongly correlated system

Several types of functional integrals can be generated by the representation (3.4) for the Hubbard operators. We shall consider two types of such representation. The first can be named "constraint" representation. The second can be named as "compensation" representation.

For the first type of a functional integral we shall use some representation for spin $-\frac{1}{2}$ operators

$$\mathbf{S} = 1/2(b^+ \sigma b), \quad A_s = (b^+ b) - 1 = 0 \quad (5.1)$$

here b_α^+, b_α for $\alpha = \uparrow, \downarrow$ are creation and annihilation operators which can be boson or fermion operators, A_s is the spin antiprojector operator which is equal to zero on the physical subspace and it is equal to $n = -1, 1, 2, 3$ on the unphysical subspace. The condition $A_s = 0$ is usually named a constraint, but it is very special type of a constraint. The antiprojector operator A_h can also be introduced for the representation (3.4) for the Hubbard operators:

$$A_h = (1/4)(3\hat{n}^h - 2\hat{d}^h + 2\mathbf{S}h^+ \sigma h). \quad (5.2)$$

This antiprojector is equal to unity on the two-hole state and on the one-hole triplet states. It is equal to zero on the hole vacuum state and on the one-hole singlet state. The introduction of the antiprojector A is motivated by an identity connecting it with the projector P :

$$P = (1/2\pi) \int_0^{2\pi} \exp(i\lambda A) d\lambda. \quad (5.3)$$

The identity (5.3) gives us possibility to construct the functional integral for any dynamical system which can be expressed in terms of canonical variables and act on a limiting part of the Gilbert space which is defined by the condition $A = 0$. In our case of a strongly correlated system we have the following representation for the partition function or the generating functional of the Green functions

$$\begin{aligned} Z(y_l^{ab}) = & \int \int \exp(S + i \int_0^\beta d\tau \sum_l (\lambda_{sl}((b_l^+ b_l) - 1) + \\ & \lambda_{Hl}(3\hat{n}_l^h - 2\hat{d}_l^h + 2\mathbf{S}_l h_l^+ \sigma h_l)) \delta\left(\frac{\partial \lambda_{sl}(\tau)}{\partial \tau}\right) \delta\left(\frac{\partial \lambda_{Hl}(\tau)}{\partial \tau}\right) \\ & \prod_{l, \tau, \alpha} D h_{l\alpha}^+(\tau) D h_{l\alpha}(\tau) D b_{l\alpha}^+(\tau) D b_{l\alpha}(\tau) D \lambda_{sl}(\tau) D \lambda_{Hl}(\tau), \\ & S = \int_0^\beta d\tau \sum_l (h_l^+ \dot{h}_l + b_l^+ \dot{b}_l - H(h_l^+ h_l, \mathbf{S}_l(b_l^+, b_l)) + \\ & X_l^{ab}(h_l^+, h_l, \mathbf{S}_l(b_l^+, b_l)) y_l^{ab}) \end{aligned} \quad (5.4)$$

Here S is the action of a system; $H(h^+, h, \mathbf{S}(b^+, b))$ is the Hamiltonian of a strongly correlated system; l is the lattice site index; λ_s, λ_H are the Lagrange multipliers which do not depend on the temperature time τ what is ensured due to the presence of δ - functions. Because the spin subsystem possess the gauge invariance the $\delta(\partial \lambda_s(\tau)/\partial \tau)$ fixes the gauge.

Another possibility to construct the functional integral is connected with the idea of compensation of an unphysical contributions suggested in the work by Porov and Fedotov

[8]. They use the equality to zero of the Hamiltonian on the unphysical states. Let us add to the Hamiltonian some addition δH which is equal to zero on the physical states and possess the properties

$$Tr(\exp(-\beta\delta H))_{HP} = 0, \quad [H, \delta H] = 0 \quad (5.5)$$

Summation in (5.5) is produced over the unphysical states only. Then the total Hamiltonian $H_t = H + \delta H$ can be used in the total Gilbert space without any constraints for computation of the partition function. The functional in this case has simple form:

$$Z(y_{ab}) = \int \exp(S_\tau) \prod_{l,\tau,\alpha} Dh_{l\alpha}^+(\tau) Dh_{l\alpha}(\tau) Db_{l\alpha}^+(\tau) Db_{l\alpha}(\tau), \quad (5.6)$$

here $b_{l\alpha}^+, b_{l\alpha}$ are the Fermi fields and the addition to the action comparing with (5.4) has form:

$$\delta S = -\beta\delta H = 1/2 \sum_l [i\pi(b_l^+ b_l - 1) - (i\pi + \ln(3/2))(3h_l^+ h_l + 2\mathbf{S}_l h_l^+ \sigma h_l)] \quad (5.7)$$

This addition to the action is nonhermitian. In fact we can easily construct a two-parametric form of δS .

VI. The representation of the total Hubbard algebra

In some physical situations for example when the metal-insulator transition is studied it can be necessary to describe holes in the lower Hubbard band and electrons in the upper Hubbard band simultaneously. In this case it will be convenient to have representation that describes these electrons and holes as the different degrees of freedom simultaneously. For realization of this possibility we shall consider the additional Gilbert space consisting of the states of spin $-\frac{1}{2}$ and two types of fermion with spin $-\frac{1}{2}$ for describing electrons in the upper band and holes in the lower bands. The mapping of the physical electrons states $|0\rangle, |\uparrow\rangle, |\downarrow\rangle, |2\rangle$ into states of the additional Gilbert space is following:

$$\begin{aligned} |\uparrow\rangle &\Rightarrow |s\uparrow\rangle, & |\downarrow\rangle &\Rightarrow |s\downarrow\rangle \\ |0\rangle &\Rightarrow (1/\sqrt{2})[h_\uparrow^+ |s\downarrow\rangle - h_\downarrow^+ |s\uparrow\rangle], \\ |2\rangle &\Rightarrow (1/\sqrt{2})[e_\uparrow^+ |s\downarrow\rangle - e_\downarrow^+ |s\uparrow\rangle]. \end{aligned} \quad (6.1)$$

Here $|s\alpha\rangle$ for $\alpha = \uparrow, \downarrow$ are the states of the spin $-\frac{1}{2}$; $h_\alpha^+, h_\alpha, e_\alpha^+, e_\alpha$ are the creation and annihilation the Fermi operators of holes in the lower band and electrons in the upper band. The physical states are singlets formed from the local spin $-\frac{1}{2}$ and the hole (the electron) spin $-\frac{1}{2}$. One can get a representation of the total Hubbard algebra based on the described mapping. The representation of the lower subalgebra has form:

$$\begin{aligned} X^{0\alpha} &= 2\alpha\sqrt{2}(h^+ \hat{S})_{-\alpha}(1 - n^h)P_e \\ X^{\alpha 0} &= 2\alpha\sqrt{2}P_e(1 - n^h)(\hat{S}h)_{-\alpha} \\ X^{00} &= (1/4)(n^h - 2d^h - 2t^h)P_e. \end{aligned} \quad (6.2)$$

The representation of the upper subalgebra is quite similar

$$\begin{aligned}
X^{2\alpha} &= 2\alpha\sqrt{2}(e^+\hat{S})_{-\alpha}(1-n^e)P_h \\
X^{2\alpha} &= 2\alpha\sqrt{2}P_h(1-n^e)(\hat{S}e)_{-\alpha} \\
X^{22} &= (1/4)(n^e-2d^e-2t^e)P_h.
\end{aligned} \tag{6.3}$$

Here we use the following short notation:

$$\begin{aligned}
\hat{S} &= (1/4)(1-2\mathbf{S}\sigma), \quad P_c = 1 - n^c + d^c, \\
n^c &= (c^+c), \quad d^c = c_{\uparrow}^+c_{\uparrow}c_{\downarrow}^+c_{\downarrow}, \\
t^c &= \mathbf{S}c^+\sigma c, \quad \text{for } c := h, e.
\end{aligned} \tag{6.4}$$

The representation of the middle subalgebra has form:

$$\begin{aligned}
N &= X^{\uparrow\uparrow} + X^{\downarrow\downarrow} = P_e P_h \\
\mathbf{S} &= (1/2)\sigma_{\alpha\beta}X^{\beta\alpha} = \mathbf{s}P_e P_h,
\end{aligned} \tag{6.5}$$

here \mathbf{s} is operator of the local spin. Using this representation for the Hubbard operators we can easily construct the functional integral and the diagram technique. Such type representation can be also useful in variational approach for the construction of the explicit form of interaction the charge carriers with the local spins.

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