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# The quartet state of the two-dimensional Heisenberg model with the spin- $\frac{1}{2}$ on a square lattice.

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## Abstract

The low-energy properties of the two-dimensional Heisenberg model with spin- $\frac{1}{2}$  on a square lattice are investigated on the basis of the local dimer order. The lattice is divided into square blocks consisting of the quartet of spins. The spin variables and the Heisenberg Hamiltonian are expressed in terms of the low-energy quartet variables. On the basis of the Dyson-Maleev representation the spin-wave theory of the quartet state is developed. The spectrum of the lower magnon excitations consists of three degenerate modes with the energy gap  $\Delta = 0.17J$ . The ground state energy per spin  $E/N = -0.6J$ . This preprint repeats in the main the previous one but it contains calculations of the basic corrections and therefore has complete character.

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# I. Introduction

Unusual states of the two-dimensional Heisenberg antiferromagnet attract attention in connection with the problem of the magnetic state of the cuprate superconductors [1] and very complete references in this review. This activity was stimulated by the experiments which demonstrated that superconductivity in cuprates is realized in the strongly correlated paramagnetic spin state which was named the spin-liquid state.

Many different models of the spin-liquid state were proposed [1] - [8]. The spin-liquid state as a linear combination of different dimer states was considered in the works [3, 4] and was named the RVB-state. The dimer represents the state of the two spins- $\frac{1}{2}$  with the total spin equal to zero. The dimer can be formed from two neighboring spins- $\frac{1}{2}$  as well as from two separated spins- $\frac{1}{2}$ . Numerical simulation of the RVB-state was produced in the work [4] and it was demonstrated that this state for the two-dimensional Heisenberg antiferromagnet with the nearest neighbor exchange had very low energy. This energy is higher but very close to the energy of the Neel state. Of course, the main problem is the transformation of the magnetic state with doping. But for progress in this problem we must have sufficiently simple model of the spin-liquid state because the original RVB-state permits only numerical consideration.

Such simple model of the spin-liquid state was proposed in the works [9, 10]. The plane is split into square blocks containing four spins. The block size is  $2a$ , where  $a$  is the lattice constant. The complete set of eigenstates for such block or quartet consists of 16 states which can be easily found for the exchange interaction  $J_1$  between the first and  $J_2$  between the second neighbors. For  $J_2 < J_1$  the ground state  $|\varphi\rangle$  of such quartet is the state with the total momentum equal to zero. This state can be presented as a sum of the two possible dimer states for the nearest spins. Its energy is  $-2J_1 + J_2/2$ .

As it was proposed in [9, 10] that one can get good approximation for the spin-liquid state if  $|\varphi\rangle$  state is chosen as ground state in each quartet and the quantum fluctuations are taken into account. At  $J_2 = 0$  the energy per site was in [10]  $E_0 = -0.655J$  that is very close to the numerical results [4] for  $E_0 = -0.668J$ . In other paper [9] based on the quartet approach some sort of the Schwinger boson representation for the Hubbard operators of a quartet was used. In this work the energy was sufficiently high  $E_0 = -0.57J$ . The correspondence between the numerical simulation and the quartet model is not so obvious, because in the quartet approach the quantum fluctuations are taken into account. But approximations of [9, 10] were sufficiently crude. The contribution of the higher-energy quartet states to the energy of the ground state and to the spin dynamics was omitted. In the work [10] uncontrollable approximation based on some decoupling scheme for the Green's functions was used. In the work [9] the Schwinger boson method is applicable only over parameter  $1/N$  and it is also sufficiently crude.

In this work, which is based on the main idea of [9, 10] about the quartet ground state for the Heisenberg antiferromagnet, we use spin-wave approach to description of the triplet excitations. All approximations of this work are controllable. In this work we restrict our consideration to the case of the unfrustrated Heisenberg antiferromagnet with  $J_2 = 0$ . We consider contributions of the lower singlet state and all triplet states into statistics and dynamics of the quartet state of the Heisenberg antiferromagnet. The role of the additional singlet state and the quintet state is essentially less because direct transitions from the lower singlet state into these states are absent. We do not consider these states and it will be evident that their contribution into the

ground state energy is sufficiently less.

## II. The effective Hamiltonian of the model

The Hamiltonian of the Heisenberg antiferromagnet has a well-known form:

$$H = J \sum_{\langle l,l' \rangle} (\mathbf{s}_l \mathbf{s}_{l'}) \quad (2.1)$$

where  $\langle ll' \rangle$  notes summation over the nearest neighbors. For construction of the quartet magnetic state let us divide the square lattice into four-spins blocks [9, 10]

$$\hat{Q} = \begin{pmatrix} \mathbf{s}_3 & \mathbf{s}_2 \\ \mathbf{s}_4 & \mathbf{s}_1 \end{pmatrix} \quad (2.2)$$

and find all eigenstates of the quartet of spins (see Appendix A). Every quartet has 16 states: two singlets  $|\varphi\rangle$ ,  $|\psi\rangle$ , three triplets  $|tm\rangle$ ,  $|xm\rangle$ ,  $|ym\rangle$  and quintet  $|qm\rangle$ . If we restrict our consideration to the low-energy singlet  $\varphi$  and triplet  $t$  then the spins  $\mathbf{s}_i$  consisting a quartet can be expressed in terms of the Hubbard operators acting in  $|\varphi\rangle$ ,  $|tm\rangle$ ,  $m = \pm 1, 0$  subspace at the quartet  $n$ :

$$\begin{aligned} Z_n^{\varphi\varphi} &= |\varphi n\rangle\langle n\varphi|, & Z_n^{tm,tm'} &= |tmn\rangle\langle nm't|, \\ Z_n^{\varphi,tm} &= |\varphi n\rangle\langle nmt|, & Z_n^{tm,\varphi} &= |tmn\rangle\langle n\varphi|, \end{aligned} \quad (2.3)$$

The spin operator  $\mathbf{s}_{ni}$  consisting of the quartet with the number  $n$  at the doubling square lattice may be represented as a sum of two vectors:

$$\mathbf{s}_{ni} = (-)^{i+1} \mathbf{L}_{tn} + \frac{1}{4} \mathbf{S}_{ttn} \quad (2.4)$$

where  $\mathbf{L}_{tn}$  mixes  $|\varphi\rangle$  and  $|tm\rangle$  states

$$L_{pn}^z = \frac{1}{\sqrt{6}}(Z_n^{\varphi,p0} + Z_n^{p0,\varphi}), \quad L_{pn}^+ = \frac{1}{\sqrt{3}}(Z_n^{\varphi,p-1} - Z_n^{p1,\varphi}), \quad L_{pn}^- = \frac{1}{\sqrt{3}}(Z_n^{p-1,\varphi} - Z_n^{\varphi,p1}) \quad (2.5)$$

for  $p = t$  and  $\mathbf{S}_{tt,n}$  are operators of the spin 1 acting in t-subspace

$$S_{pqn}^z = Z_n^{p1,q1} - Z_n^{p-1,q-1}, \quad S_{pqn}^+ = \sqrt{2}(Z_n^{p1,q0} + Z_n^{p0,q-1}), \quad S_{pqn}^- = \sqrt{2}(Z_n^{p0,q1} + Z_n^{p-1,q0}), \quad (2.6)$$

for  $p, q = t$ , where  $A^\pm = A^x \pm iA^y$  are the spherical components of a vector  $\mathbf{A}$ . Using expression of (A.3) in Appendix A for the energies of  $|\varphi\rangle$  one can get the Hamiltonian of the problem

$$\begin{aligned} H &= -2J \sum_n Z_n^{\varphi,\varphi} - J \sum_{nm} Z_n^{tm,tm} \\ &\quad - (J/3) \sum_{\langle nn' \rangle > m} \left( 2Z_n^{tm,\varphi} Z_{n'}^{\varphi,tm} + (-)^m (Z_n^{tm,\varphi} Z_{n'}^{t-m,\varphi} + Z_n^{\varphi,tm} Z_{n'}^{\varphi,t-m}) \right) \\ &\quad + (J/8) \sum_{\langle nn' \rangle > mm'} Z_n^{tm,tm'} \left( Z_{n'}^{tm',tm} - (-)^{m+m'} Z_{n'}^{t-m,t-m'} \right). \end{aligned} \quad (2.7)$$

The Hubbard operators  $Z_n^{a,b}$  for  $a, b = \varphi, tm$  are not convenient for the further analysis of the Hamiltonian (2.7). Therefore we map the Gilbert space consisting of  $|\varphi\rangle$  and  $|tm\rangle$  states into the Gilbert space of the three-dimensional Heisenberg algebra  $t_m^+, t_m$ . Where  $t_m^+ (t_m)$  are creation (annihilation) Bose operators. The general state of such algebra is  $|n_+, n_-, n_0\rangle$  and mapping has simple form

$$\begin{aligned} |\varphi\rangle &\rightarrow |0, 0, 0\rangle, & |t1\rangle &\rightarrow |1, 0, 0\rangle, \\ |t-1\rangle &\rightarrow |0, 1, 0\rangle, & |t0\rangle &\rightarrow |0, 0, 1\rangle \end{aligned} \quad (2.8)$$

This mapping generates the Holstein-Primakoff representation [11] for our Hubbard operators

$$\begin{aligned} (Z_n^{\varphi,\varphi})_{HP} &= P(1 - \hat{N}_n)P, & (Z_n^{\varphi,tm})_{HP} &= P\sqrt{1 - \hat{N}_n}t_{nm}P, \\ (Z_n^{tm,\varphi})_{HP} &= Pt_{nm}^+\sqrt{1 - \hat{N}_n}P, & (Z_n^{tm,tm'})_{HP} &= Pt_{nm}^+t_{nm'}P, \\ \hat{N}_n &= \sum_m t_{nm}^+t_{nm}, \end{aligned} \quad (2.9)$$

where  $P$  is a projector on the lower four states (2.8) of the Heisenberg algebra  $t_{nm}^+, t_{nm}$ . At the low temperatures  $T \ll J$  or a small excitation of oscillators  $|nm\rangle$  one can produce an isometric transformation at each site

$$(Z^{a,b})_{DM} = V^{-1}(Z^{a,b})_{HP}V \quad (2.10)$$

and omit the projector  $P$ . One can choose the operator  $V$  from the condition of elimination of roots in (2.10), and get the Dyson-Maleev representation for the Hubbard operators [12, 13]

$$\begin{aligned} (Z_n^{\varphi,\varphi})_{DM} &= (1 - \hat{N}_n), & (Z_n^{\varphi,tm})_{DM} &= t_{nm}, \\ (Z_n^{tm,\varphi})_{DM} &= t_{nm}^+(1 - \hat{N}_n), & (Z_n^{tm,tm'})_{DM} &= t_{nm}^+t_{nm'}, \\ \hat{N}_n &= \sum_m t_{nm}^+t_{nm}. \end{aligned} \quad (2.11)$$

The Dyson-Maleev representation for  $Z$ -operators is not a direct operator identity but it can be used at low temperatures and a small level of excitations. Substituting the Dyson-Maleev representation (2.11) in the Hamiltonian (2.7) we get the following effective Hamiltonian

$$\begin{aligned} H &= -(NJ/2) + J \sum_n \hat{N}_n - (J/3) \sum_{\langle nm' \rangle_m} \left( 2t_{nm}^+(1 - \hat{N}_n)t_{n'm} \right. \\ &\quad \left. + (-)^m \left( t_{nm}^+(1 - \hat{N}_n)t_{n'-m}^+(1 - \hat{N}_{n'}) + t_{nm}t_{n'-m} \right) \right) \\ &\quad + (J/8) \sum_{\langle nn' \rangle_{mm'}} t_{nm}^+t_{nm'} \left( t_{n'm'}^+t_{n'm} - (-)^{m+m'} t_{n'-m}^+t_{n'-m'} \right), \end{aligned} \quad (2.12)$$

where  $N$  is the number of spins in the plane. This Hamiltonian is nonhermitian due to properties of the Dyson-Maleev representation (2.11) but if we use it for the computation of Green's functions any contradictions do not appear.

### III. The approximate solution of the effective Hamiltonian

If we consider only the quadratic terms in the Hamiltonian (2.12) we can easily diagonalize them with the help of  $u - v$  transformation (these quadratic terms are Hermitian) and get the square of the magnon energy in the momentum representation

$$E_{\mathbf{k}}^2 = J^2(1 - 8\gamma_{\mathbf{k}}/3), \quad \gamma_{\mathbf{k}} = (\cos(2k_x a) + \cos(2k_y a))/2, \quad (3.1)$$

where  $a$  is the distance between spins,  $\mathbf{k}$  is the quasimomentum. One can easily see that  $E_{\mathbf{k}}^2$  is negative at  $\gamma_{\mathbf{k}} > 3/8$ . This means that our system is unstable in the quadratic approximation and the higher terms in the Hamiltonian (2.12) must be taken into consideration for solving the problem of stability. We will treat the Hamiltonian (2.12) by the mean-field method. The same can be done by the Green's function method. The results are identical. We seek the solution which is invariant over the rotation and consider the following normal and abnormal averages:

$$\begin{aligned} \langle t_{\mathbf{k}m}^+ t_{\mathbf{k}'m'} \rangle &= \delta_{m,m'} \delta(\mathbf{k} - \mathbf{k}') N_{\mathbf{k}}, \\ \langle t_{\mathbf{k}m}^+ t_{\mathbf{k}'m'}^+ \rangle &= (-)^m \delta_{m,-m'} \delta(\mathbf{k} + \mathbf{k}') F_{\mathbf{k}}^+, \\ \langle t_{\mathbf{k}m} t_{\mathbf{k}'m'} \rangle &= (-)^m \delta_{m,-m'} \delta(\mathbf{k} + \mathbf{k}') F_{\mathbf{k}}^-. \end{aligned} \quad (3.2)$$

Our approach based on the Dyson-Maleev representation (2.11) is correct if the contribution of the higher states at every lattice sites is small. Because of that the following conditions must be fulfilled

$$(\bar{N}_p, \bar{F}_p^\pm) = \sum_{\mathbf{k}} (\gamma_{\mathbf{k}})^p (N_{\mathbf{k}}, F_{\mathbf{k}}^\pm) \ll 1, \quad (3.3)$$

where the sum over  $\mathbf{k}$  notes the normalized integral over the Brillouin zone. In this approximation we can omit the contribution of the members of the six order and get the following mean-field Hamiltonian:

$$H_{mf} = E_{mf}^0 + J \sum_{\mathbf{k}m} \left( \alpha_{\mathbf{k}} t_{\mathbf{k}m}^+ t_{\mathbf{k}m} + (-)^m (\beta_{\mathbf{k}} t_{\mathbf{k}m}^+ t_{-\mathbf{k}-m}^+ + \delta_{\mathbf{k}} t_{\mathbf{k}m} t_{-\mathbf{k}-m}) / 2 \right), \quad (3.4)$$

where  $E_{mf}^0$  is the initial ground state energy in the mean-field approximation:

$$\begin{aligned} E_{mf}^0 &= (NJ/2) \left( -1 + (3\bar{N}_0 - 4\bar{N}_1 - 2\bar{F}_1^- - 2\bar{F}_1^+) / 4 \right. \\ &\quad \left. - (3/8) \sum_{\mathbf{k}} (2\alpha_{\mathbf{k}} N_{\mathbf{k}} + \beta_{\mathbf{k}} F_{\mathbf{k}}^+ + \gamma_{\mathbf{k}} F_{\mathbf{k}}^-) \right) \end{aligned} \quad (3.5)$$

and the coefficients  $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}, \delta_{\mathbf{k}}$  are

$$\begin{aligned} \alpha_{\mathbf{k}} &= 4(3/4 + 4(\bar{N}_1 + \bar{F}_1^+) + (4\bar{N}_0 + \bar{F}_0^+ - 1 + 3\bar{N}_1/4)\gamma_{\mathbf{k}}) / 3, \\ \beta_{\mathbf{k}} &= 4(2(\bar{N}_1 + \bar{F}_1^-) + (8\bar{N}_0 - 1 - 3\bar{F}_1^-/4)\gamma_{\mathbf{k}}) / 3, \\ \delta_{\mathbf{k}} &= 4(2\bar{F}_0^+ - 1 - 3\bar{F}_1^+/4)\gamma_{\mathbf{k}} / 3. \end{aligned} \quad (3.6)$$

The quadratic part of the Hamiltonian  $H_{mf}$  (3.4) was obtained from  $H$  (2.12) by computing all possible paired averages in the fourth order terms of  $H$  (2.12). The initial energy  $E_{mf}^0$  is

determined from the condition of coincidence of the average values of the Hamiltonian (2.12) and (3.4). This procedure corresponds to determination of the one-particle energy by the Green's function method in the one-loop approximation. The Hamiltonian (3.4) can be diagonalized with the help of the  $u - v - w$  transformation (this procedure is equivalent of solving the Dyson equation for normal and abnormal Green's functions):

$$\begin{aligned} t_{\mathbf{k}m} &= u_{\mathbf{k}}b_{\mathbf{k}m} + (-)^m v_{\mathbf{k}}b_{-\mathbf{k}-m}^+, \\ t_{\mathbf{k}m}^+ &= u_{\mathbf{k}}b_{\mathbf{k}m}^+ + (-)^m w_{\mathbf{k}}b_{-\mathbf{k}-m}, \\ u_{\mathbf{k}}^2 - v_{\mathbf{k}}w_{\mathbf{k}} &= 1. \end{aligned} \quad (3.7)$$

After this transformation the Hamiltonian (3.4) takes the form:

$$H_{mf} = E_{mf} + \sum_{\mathbf{k}m} E_{\mathbf{k}}b_{\mathbf{k}m}^+ b_{\mathbf{k}m} \quad (3.8)$$

with the ground state energy

$$E_{mf} = E_{mf}^0 + (3/4)N \sum_{\mathbf{k}} (E_{\mathbf{k}} - J\alpha_{\mathbf{k}}) \quad (3.9)$$

and

$$E_{\mathbf{k}} = JR_{\mathbf{k}}, \quad R_{\mathbf{k}} = \sqrt{\alpha_{\mathbf{k}}^2 - \beta_{\mathbf{k}}\delta_{\mathbf{k}}}. \quad (3.10)$$

The coefficients  $u_{\mathbf{k}}, v_{\mathbf{k}}, w_{\mathbf{k}}$  have form:

$$\begin{aligned} u_{\mathbf{k}} &= \sqrt{(\alpha_{\mathbf{k}} + R_{\mathbf{k}})/2R_{\mathbf{k}}}, & v_{\mathbf{k}} &= z_{\mathbf{k}}\beta_{\mathbf{k}}, \\ z_{\mathbf{k}} &= -\sqrt{|\alpha_{\mathbf{k}} - R_{\mathbf{k}}|/2|\beta_{\mathbf{k}}\delta_{\mathbf{k}}|R_{\mathbf{k}}}, & w_{\mathbf{k}} &= z_{\mathbf{k}}\delta_{\mathbf{k}}. \end{aligned} \quad (3.11)$$

So far as the coefficients  $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}, \delta_{\mathbf{k}}$  depend on the averages  $\bar{N}_p, \bar{F}_p^{\pm}$  and consequently the coefficients  $u_{\mathbf{k}}, v_{\mathbf{k}}, w_{\mathbf{k}}$  also depend on these parameters we can get the closed system of equations if we substitute Eqs. (3.7) for  $t_{\mathbf{k}m}, t_{\mathbf{k}m}^+$  into Eqs. (3.2), (3.3) which determine  $\bar{N}_p, \bar{F}_p^{\pm}$ .

$$\begin{aligned} N_{\mathbf{k}} &= (\alpha_{\mathbf{k}}/R_{\mathbf{k}})(n_{\mathbf{k}} + 1/2) - 1/2, \\ F_{\mathbf{k}}^{\pm} &= -[(\delta_{\mathbf{k}}, \beta_{\mathbf{k}})/R_{\mathbf{k}}](n_{\mathbf{k}} + 1/2), \end{aligned} \quad (3.12)$$

where  $n_{\mathbf{k}}$  is the Planck function:  $n_{\mathbf{k}} = (\exp(\beta E_{\mathbf{k}}) - 1)^{-1}$  and  $\beta = 1/T$  is the inverse temperature. The system of equations of (3.12) was solved numerically (see Appendix B for details) and we get the averages  $\bar{N}_p, \bar{F}_p^{\pm}$  as a function of the temperature. We also computed the free energy of the quartet state from (3.8) and determined the magnon energy  $E_{\mathbf{k}}$

$$\begin{aligned} E_{\mathbf{k}} &= \sqrt{A + 2B\epsilon + C\epsilon^2}, & \epsilon &= \gamma_{\mathbf{k}} \\ E_{\mathbf{k}} &\approx \Delta(1 + \mathbf{k}^2\xi^2), & \text{at } ka &\ll 1 \\ \Delta &= \sqrt{A + 2B + C}, & \xi^2 &= -(B + 2C)a^2/8\Delta^2 \end{aligned} \quad (3.13)$$

All these parameters are presented in Table 1.

$T/J$	$F/NJ$	$\Delta/J$	$A/J^2$	$B/J^2$	$C/J^2$
0	-0.567	0.174	3.474	-1.595	-0.254
0.1	-0.567	0.205	3.484	-1.592	-0.258
0.3	-0.570	0.416	3.525	-1.562	-0.229
0.5	-0.585	0.704	3.452	-1.412	-0.133

$T/J$	$N_0$	$N_1$	$F_0^+$	$F_1^+$	$F_0^-$	$F_1^-$
0	0.045	0.025	0.087	0.137	0.035	0.083
0.1	0.046	0.026	0.087	0.137	0.034	0.083
0.3	0.050	0.029	0.086	0.136	0.028	0.076
0.5	0.068	0.030	0.079	0.131	0.011	0.055

TABLE 1. The result of calculation of the free energy  $F$ , the energy gap  $\Delta$ , the parameters of the magnon energy  $A, B, C$  and the normal and abnormal averages  $\bar{N}_p, \bar{F}_p^\pm$ .

We can see that all normal and abnormal averages are small and do not exceed 0.14. This property of our solution justifies dropping of the six-order term in the effective Hamiltonian (2.12). The smallness of these averages justifies also using of the Dyson-Maleev representation for the Hubbard operators. Really, one can check that the occupancy  $W_{n_+n_-n_0}$  of the state  $|n_+, n_-, n_0\rangle$  of the Heisenberg algebra  $t_m^+, t_m$  for our  $u - v - w$  transformation (3.7) is determined by the formula

$$W_{n_+n_-n_0} = (1 + N_0)^{-3} \left( \frac{N_0}{1 + N_0} \right)^{n_++n_-+n_0} \quad (3.14)$$

and the total occupancy of the unphysical states  $W_{unp}$  is equal to

$$W_{unp} \approx (6N_0^2 + 4N_0^3 + N_0^4)/(1 + N_0)^4 = 0.044. \quad (3.15)$$

All this prove that we found correctly the rotationally invariant ground state of the Hamiltonian (2.7).

## IV. Corrections to the mean-field energy of the ground state.

For checking accuracy of our solution we compute the second order correction to the ground state energy in the mean-field approximation (3.9). This correction is presented by the Feynman diagram in Fig.1.

Fig.1 The Feynman diagram for the second order correction to the energy of the ground state.

We restrict our consideration to the case of the zero temperature. The part of the Hamiltonian (2.12) essential for computation of the correction of Fig.1 after u-v-w transformation has form

$$\begin{aligned}
H_{int} = & (J/12) \sum_{\mathbf{k}_i, m, n} \{ [16u_{\mathbf{k}_1} u_{\mathbf{k}_2} v_{\mathbf{k}_3} \gamma_{\mathbf{k}_4} (u_{\mathbf{k}_4} + w_{\mathbf{k}_4}) + \\
& 3\gamma_{\mathbf{k}_1+\mathbf{k}_2} u_{\mathbf{k}_1} v_{\mathbf{k}_2} (u_{\mathbf{k}_3} v_{\mathbf{k}_4} - u_{\mathbf{k}_4} v_{\mathbf{k}_3}) (-)^{n+m} b_{m\mathbf{k}_1}^+ b_{n\mathbf{k}_2}^+ b_{-n\mathbf{k}_3}^+ b_{-m\mathbf{k}_4}^+ ) + \\
& (b_{m\mathbf{k}}^+ \rightarrow b_{m\mathbf{k}}, u_{\mathbf{k}} \rightarrow w_{\mathbf{k}}, v_{\mathbf{k}} \rightarrow u_{\mathbf{k}}) \}. \tag{4.1}
\end{aligned}$$

Here summation  $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4$  is produced over the surface  $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4 = 0$ .

The first term in the square brackets in (4.1) follows from the  $L - L$  interaction and the second term follows from the  $S - S$  interaction. The last term in the figured brackets is the conjugated to the first term. On the base of the Hamiltonian one can easily compute the second order correction of Fig.1 to the ground state energy. We compute the six-dimensional integral over  $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$  by the Monte-Carlo method and getting the following result  $\delta E = 0.005NJ$ .

This correction to the ground state energy is sufficiently small that verifies correctness of our mean-field solution of the Hamiltonian (2.12). The validity of the perturbation theory is based on the smallness of angle integrals over  $\mathbf{k}$  because another numerical small parameters are absent in our theory. In fact, the small parameter is  $1/z$  where  $z$  is the number of the neighboring spin to our quartet of spins.

## V. Influence of higher energy levels of the quartet on the ground state energy

It is clear that transitions at the higher energy levels of the quartet reduce the ground state energy of the quartet state of the two-dimensional Heisenberg antiferromagnets. As was noticed above the most essential contributions originated from the higher triplet states  $|xm\rangle, |ym\rangle$  (see (A.5)). In order to consider this additional to  $|\varphi\rangle$  and  $|tm\rangle$  states we must modify Eqs.(2.4)-(2.6) for the spin operators  $\mathbf{s}_i$  taking into account contribution of  $|xm\rangle, |ym\rangle$  states. On the base of the quartet wave function (A.5) we get the following representation for the spin operators  $\mathbf{s}_i$

$$\begin{aligned}
4\mathbf{s}_i = & 4(-)^{i+1}\mathbf{L}_t - 2(\mathbf{L}_x - (-)^i\mathbf{L}_y) + \mathbf{S}_{tt} + \mathbf{S}_{xx} + \mathbf{S}_{yy} \\
& + z_i(\mathbf{S}_{tx} + \mathbf{S}_{xt} - (-)^i(\mathbf{S}_{ty} + \mathbf{S}_{yt})) + (-)^i(\mathbf{S}_{xy} + \mathbf{S}_{yx}) \tag{5.1}
\end{aligned}$$



where  $z_i = (1, 1, -1, -1)$  for  $i = 1, 2, 3, 4$ . The operator of the triplet excitations  $\mathbf{L}_p$  and the spin-1 operator  $\mathbf{S}_{pq}$  for  $p, q = t, x, y$ ;  $m = 0, \pm 1$  are expressed by the same manner as in Eqs.(2.5), (2.6). For the further analysis we replace the Hubbard projection operators  $Z_n^{\varphi\varphi}, Z_n^{\varphi,pm}, Z_n^{pm,\varphi}, Z_n^{pm,qm'}$  by the more convenient Bose operators  $x_{nm}^+, y_{nm}^+, t_{nm}^+$  ( $x_{nm}, y_{nm}, t_{nm}$ ) creating (annihilating) bosons of  $x, y, t$  type with projection  $m$  at site  $n$ . The corresponding Dyson-Maleev representation for the Hubbard operators is of the form (2.11).

$$\begin{aligned} (Z_n^{\varphi\varphi})_{DM} &= (1 - \hat{N}_n), & (Z_n^{\varphi,pm})_{DM} &= p_{nm}, \\ (Z_n^{pm,\varphi})_{DM} &= p_{nm}^+(1 - \hat{N}_n), & (Z_n^{pm,qm'})_{DM} &= p_{nm}^+ q_{nm'}, \\ \hat{N}_n &= \sum_{p,m} p_{nm}^+ p_{nm}. \end{aligned} \quad (5.2)$$

Hamiltonian (2.1) can be represented as

$$\begin{aligned} H = & J \sum_n (\mathbf{s}_{1n} + \mathbf{s}_{3n})(\mathbf{s}_{2n} + \mathbf{s}_{4n}) + J \sum_{\langle n, n'_x \rangle} (\mathbf{s}_{2n} \mathbf{s}_{3n'_x} + \mathbf{s}_{1n} \mathbf{s}_{4n'_x}) + \\ & J \sum_{\langle n, n'_y \rangle} (\mathbf{s}_{2n} \mathbf{s}_{1n'_y} + \mathbf{s}_{3n} \mathbf{s}_{4n'_y}) \end{aligned} \quad (5.3)$$

where  $\langle n, n'_x \rangle$  ( $\langle n, n'_y \rangle$ ) means summation over the nearest neighbor of block  $n$  in  $\vec{x}$  ( $\vec{y}$ ) direction. Using the representations (5.1),(5.2) for the spin operators  $\mathbf{s}_i$  one can rewrite the Hamiltonian (5.3) in the momentum representation:

$$\begin{aligned} H &= H_t + H_x + H_y + H_{tx} + H_{ty}, \\ H_x &= J \sum_{\mathbf{km}} \left( (2 + \eta_{\mathbf{k}}/3) x_{\mathbf{km}}^+ x_{\mathbf{km}} + (-)^m \eta_{\mathbf{k}} (x_{\mathbf{km}}^+ x_{-\mathbf{k}-m}^+ + x_{\mathbf{km}} x_{-\mathbf{k}-m}) / 6 \right), \\ H_y &= J \sum_{\mathbf{km}} \left( (2 - \eta_{\mathbf{k}}/3) y_{\mathbf{km}}^+ y_{\mathbf{km}} - (-)^m \eta_{\mathbf{k}} (y_{\mathbf{km}}^+ y_{-\mathbf{k}-m}^+ + y_{\mathbf{km}} y_{-\mathbf{k}-m}) / 6 \right), \\ H_{tx} &= (iJ/3) \sum_{\mathbf{km}} (-)^m \sin(2k_x a) \left( t_{\mathbf{km}} + (-)^m t_{-\mathbf{k}-m}^+ \right) \left( x_{-\mathbf{k}-m} + (-)^m x_{\mathbf{km}}^+ \right), \\ H_{ty} &= -(iJ/3) \sum_{\mathbf{km}} (-)^m \sin(2k_y a) \left( t_{\mathbf{km}} + (-)^m t_{-\mathbf{k}-m}^+ \right) \left( y_{-\mathbf{k}-m} + (-)^m y_{\mathbf{km}}^+ \right). \end{aligned} \quad (5.4)$$

where  $\eta_{\mathbf{k}} = (\cos(2k_x a) - \cos(2k_y a))/2$  and  $H_t$  is the Hamiltonian (2.12), which was discussed previously. In  $H_x, H_y, H_{tx}, H_{ty}$  we excluded from our consideration all terms more than the second order by creation and annihilation operators. These terms lead in the mean-field approximation to some renormalization of coefficients in the quadratic form (5.4) which does not effect essentially on results obtained below.

The further plan is as follows. We take into consideration the Hamiltonians  $H_x$  and  $H_y$  precisely and consider the Hamiltonians  $H_{tx}$  and  $H_{ty}$  as perturbation. We will check that such approach is suitable for the computation of corrections to the ground state energy.

The Hamiltonians  $H_x$  and  $H_y$  can be easily diagonalized with help of the  $u-v$  transformation (2.7). The coefficients  $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}$  and  $\delta_{\mathbf{k}}$  in this case are

$$\alpha_{\mathbf{k}} = J(2 \pm \eta_{\mathbf{k}}/3), \quad \beta_{\mathbf{k}} = \delta_{\mathbf{k}} = \pm J\eta_{\mathbf{k}}/3, \quad (5.5)$$

where the upper sign "+" corresponds to  $H_x$  and the lower sign "-"  $H_y$ . As a result we get the energy for the  $x, y$  quasiparticles in a form

$$\varepsilon_{\mathbf{k}}^{x,y} = \sqrt{\alpha_{\mathbf{k}}^2 - \beta_{\mathbf{k}}^2} = 2J\sqrt{1 \pm \eta_{\mathbf{k}}/3} \simeq J(2 \pm \eta_{\mathbf{k}}/3 - \eta_{\mathbf{k}}^2/36). \quad (5.6)$$

The average number of quasiparticles  $x, y$  in the ground state is sufficiently small

$$\langle n_x \rangle = \langle n_y \rangle = \sum_{\mathbf{k}} v_{\mathbf{k}}^2 \approx \frac{1}{144} \sum_{\mathbf{k}} \eta_{\mathbf{k}}^2 = \frac{1}{576} \ll N_0. \quad (5.7)$$

Hence, the contribution of  $H_x$  and  $H_y$  in the ground state energy can be estimated as  $10^{-3}J$  per spin.

Notice, that dependence of the energies  $\varepsilon_{\mathbf{k}}^{x,y}$  on  $\mathbf{k}$  as it follows from Eq.(5.7) is sufficiently weak. This leads to the strong interaction inside the  $x - y$  sector of the quartet Gilbert space. As it follows from Eq.(5.1) there are nonlinear terms in the total Hamiltonian of the spin-spin interaction which lead to the strong interaction inside of the  $x - y$  sector. These terms in the Hamiltonians have a characteristic form:

$$H_{int} = (2J/\sqrt{3}) \sum_{\mathbf{k}} \gamma_{\mathbf{k}} (\mathbf{L}_{t,\mathbf{k}} (\mathbf{S}_{xy,-\mathbf{k}} + \mathbf{S}_{yx,-\mathbf{k}})) \quad (5.8)$$

This means that, in fact, we do not know the structure of excitations in the  $x - y$  sector. But for computation of corrections to the ground state energy we can use the zero-dimensional approximation for the Green's function of the  $x, y$  quasiparticles [14]. In this approximation the energy of  $x$  and  $y$  particles is simply  $2J$  (5.6) and their Green's function can be easily found. One can show that in result of the interaction (5.8) a form of the Green's function in the energetic space became the Gauss form instead the Lorentz form as in the case of the weak interaction. But what is the most important for us - a position of the center of peak in the energetic space of the imaginary part of the Green's function does not change at  $\varepsilon = 2J$ . The corrections to the ground state energy due to the Hamiltonian (5.8) are equal to zero in this approximation, because a contribution of the condensate the  $x - y$  particles are small Eq.(5.7).

The Hamiltonians  $H_{tx}$  and  $H_{ty}$  lead to an equal contributions to the ground state energy, which can be calculated in the second order of the perturbation theory:

$$\begin{aligned} \delta E_{mf} &= -(N/2) \sum_{\mathbf{q}l} \frac{\langle 0 | H_{tx} | 1t_{\mathbf{q},l}, 1x_{-\mathbf{q},-l} \rangle \langle 1t_{\mathbf{q},l}, 1x_{-\mathbf{q},-l} | H_{tx} | 0 \rangle}{\varepsilon_{\mathbf{q}}^x + E_{\mathbf{q}}} \\ &= -(J^2 N/6) \sum_{\mathbf{q}} \sin^2(2q_x a) \frac{(u_{\mathbf{q}} + w_{\mathbf{q}})(u_{\mathbf{q}} + v_{\mathbf{q}})}{\varepsilon_{\mathbf{q}}^x + E_{\mathbf{q}}} = -0.024JN, \end{aligned} \quad (5.9)$$

where  $u, v, w$  are defined in (3.11) and  $N$  is the number of spins in a plane. We see that the interaction of  $x$  and  $y$  particles with  $t$  particle decreases the ground state energy by  $0.024J$  per spin. This correction will be reduced if we take into consideration excluded from Hamiltonian (5.4) terms of third and fourth order over the creation and annihilation operators.

## VI. Properties of the solution and discussion

The energy per spin of the quartet state at the temperature equal to zero is sufficiently high:  $E/N = -0.6J$ . The coefficients  $B$  and  $C$  in the magnon energy (3.14) are negative and the energy has minimum at  $\mathbf{k} = 0$  and possesses a gap. This gap is equal to  $0.174J$  at  $T = 0$ . Because our excitations have a gap, any long-range order is absent in the quartet magnetic state. We can easily calculate the simultaneous spin correlation function using representation (2.4) for spin operators and the representation (2.11) for the Hubbard operators:

$$K_{ni,n'i'}^{\alpha\alpha'} = \langle s_{ni}^\alpha s_{n'i'}^{\alpha'} \rangle = \delta_{\alpha\alpha'} K_{ii'}(\mathbf{r}), \quad \mathbf{r} = \mathbf{r}_n - \mathbf{r}_{n'}, \quad (6.1)$$

where  $i, i' = 1, 2, 3, 4$  numerate spins inside quartet,  $\alpha, \alpha' = 1, 2, 3$  are vector indices. According to (2.4) we have

$$K_{ii'}(\mathbf{r}) = (-)^{i+i'} \langle L_n^z L_{n'}^z \rangle + \frac{1}{16} \langle S_n^z S_{n'}^z \rangle. \quad (6.2)$$

Substituting (2.5), (2.6) in (6.2) and using (2.11) one can get

$$K_{ii'}(\mathbf{r}) = \frac{1}{6} (-)^{i+i'} \left( 2(1 - F_0^+ - 4N_0)N(\mathbf{r}) + (1 - 2F_0^+)F^-(\mathbf{r}) + (1 - 8N_0)F^+(\mathbf{r}) \right) + \frac{1}{8} \left( N^2(\mathbf{r}) - F^-(\mathbf{r})F^+(\mathbf{r}) \right), \quad (6.3)$$

where  $N(\mathbf{r}), F^\pm(\mathbf{r})$  are the Fourier images of correlators  $N_{\mathbf{k}}, F_{\mathbf{k}}^\pm$ :

$$\left( N(\mathbf{r}), F^\pm(\mathbf{r}) \right) = \sum_{\mathbf{k}} \exp(-i\mathbf{k}\mathbf{r}) (N_{\mathbf{k}}, F_{\mathbf{k}}^\pm). \quad (6.4)$$

When we derived (6.3) we take into consideration the second and fourth order over  $t^+, t$  terms. The contribution of the six-order terms is small over  $\bar{N}_p, \bar{F}_p^\pm$ . The long-range order is absent in our quartet state and asymptotic behavior is determined by the gap in the magnon spectrum  $\Delta$  and the effective magnon mass. It can be easily found from (6.3), (3.12), (3.6) and (3.14) :

$$K_{ij}(r) = D(-)^{i+j} (a/\xi)^2 (2\pi r \xi)^{-1/2} \exp(-r/\xi), \quad r \gg \xi \quad (6.5)$$

The constant  $D$  is a function  $\bar{N}_p, \bar{F}_p^\pm$  and the correlation length  $\xi$  is determined by the magnon energy. Taking into account interaction of  $x$  and  $y$  particles with  $t$  particle does not lead to the efficient modification of the correlation function since this interaction is strongly reduced at small  $\mathbf{k}$  due to the structure of contributions of the  $x, y$  particles into the correlation function which are proportional to  $\sin(2k_x a)$  and  $\sin(2k_y a)$ . Therefore, a corrections to  $K_{ij}$  is of the order of  $\delta K_{ij} \sim (a^2/\xi^2) K_{ij} \ll K_{ij}$ . The behavior of the correlation function at the large distance is similar to an antiferromagnet with short-range order [1, 7, 8] and contains the "stagger" factor  $(-)^{i+j}$ .

Our results essentially differ from results obtained in works [9, 10] where the quartet state was proposed. We considered only the two-dimensional unfrustrated Heisenberg antiferromagnets. Our consideration is quantitative and does not contain unjustified assumptions. We did not find any gapless excitations as in [6, 9, 10]. Moreover our magnetic excitations have the

similar structure as in [10], but the energy is quite different. The ground state energy in [10] is sufficiently low  $E_0/N = -0.655J$ .

We connect these contradictions with the crude method of solution of the Hamiltonian (2.7). The method of decoupling of Green's functions used in [10] is rather indefinite and result depends on the method of decoupling. It is reasonable to suppose that low energy obtained in [10] as well as in the Schwinger boson method [9] is an accidental result of approximation. The corrections to these approximations are not small and can destroy this low energy. We believe that application of the Schwinger boson method to the Hamiltonian (2.7) [9] has only qualitative character as well.

## VII. Conclusion

Our consideration of the quartet state of the two-dimensional antiferromagnetic Heisenberg model was sufficiently consistent. Following to [9] and [10] we have solved exactly the problem for four spins and determined the initial ground state and structure of higher excitations. At the next step we produced mapping of our quartet state into the Bose system with many degrees of freedom. At the low temperatures and level of excitations this Bose system is reduced to the system with the nonhermitian polynomial Hamiltonian. We constructed the ground state of this Hamiltonian in the mean-field approximation and found a nontrivial solution of the self-consistent equations for the normal and abnormal averages. We calculated the basic corrections to the mean-field approximation and showed that they are small. We took into consideration the most essential higher triplet excitations and calculated their contribution to the ground state energy. We also found that these contribution are small. We did not consider the higher singlet and quintet excitations but their contribution to the ground state energy definitely small because the direct transitions from the initial ground state into these excited states are absent. We did not find an unstable excitations in the quartet state.

As a result we found that the ground state energy is sufficiently high  $E_0/N = -0.6J$  per spin. We believe that this result has a quantitative character due to the effective small parameter of the perturbation theory. It is in our case  $1/z$  where  $z$  is the number of spins neighbors to the given quartet. In practice, the corrections are small due to smallness of the angle integrals over the Brillouin zone.

Therefore, it is followed from our result, that the quartet state of the two-dimensional antiferromagnetic Heisenberg model can not be in competition with the Neel state as it was suggested in [10]. Is the quartet short range order in competition with the Neel short range order for the doped antiferromagnet? It is an open question at present.

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## Appendix A: Structure of a single quartet

Every quartet consists of the four spins and has the Hamiltonian  $H_q$  :

$$H_q = J(\mathbf{s}_a \mathbf{s}_b) = (J/2) \left( (\mathbf{s}_a + \mathbf{s}_b)^2 - \mathbf{s}_a^2 - \mathbf{s}_b^2 \right), \quad (\text{A.1})$$

where  $\mathbf{s}_a = \mathbf{s}_1 + \mathbf{s}_3$ ,  $\mathbf{s}_b = \mathbf{s}_2 + \mathbf{s}_4$ . The states of a quartet can be numbered by the four quantum numbers:  $S$  is the total spin of a quartet  $\mathbf{S} = \mathbf{s}_a + \mathbf{s}_b$ ,  $s_a, s_b$  are the spins of  $a$  and  $b$  subsystems and  $m$  is the projection of the total spin. The general state is  $|s_a s_b S m\rangle$  and we have the following states:

$$\begin{aligned} |\varphi\rangle &= |1, 1, 0, 0\rangle, & |tm\rangle &= |1, 1, 1, m\rangle, & |\psi\rangle &= |0, 0, 0, 0\rangle, \\ |qm\rangle &= |1, 1, 2, m\rangle, & |am\rangle &= |1, 0, 1, m\rangle, & |bm\rangle &= |0, 1, 1, m\rangle. \end{aligned} \quad (\text{A.2})$$

The energies of these states are following:

$$E_\varphi = -2J, \quad E_t = -J, \quad E_a = E_b = E_\psi = 0, \quad E_q = J. \quad (\text{A.3})$$

It is convenient for our purpose to use linear combinations

$$|xm\rangle = (|am\rangle - |bm\rangle)/\sqrt{2}, \quad |ym\rangle = (|am\rangle + |bm\rangle)/\sqrt{2} \quad (\text{A.4})$$

instead the states  $|am\rangle$  and  $|bm\rangle$ . In this work we restrict our consideration by the  $\varphi$ -singlet and the  $a, x, y$ -triplet. The wave functions of this states can be presented in a form:

$$\begin{aligned} |\varphi\rangle &= \frac{1}{\sqrt{12}} \left[ \begin{pmatrix} + & + \\ - & - \end{pmatrix} + \begin{pmatrix} - & - \\ + & + \end{pmatrix} + \begin{pmatrix} - & + \\ - & + \end{pmatrix} + \begin{pmatrix} + & - \\ + & - \end{pmatrix} \right. \\ &\quad \left. - 2 \begin{pmatrix} + & - \\ - & + \end{pmatrix} - 2 \begin{pmatrix} - & + \\ + & - \end{pmatrix} \right], \\ |t1\rangle &= \frac{1}{2} \left[ \begin{pmatrix} + & + \\ - & + \end{pmatrix} + \begin{pmatrix} + & - \\ + & + \end{pmatrix} - \begin{pmatrix} + & + \\ + & - \end{pmatrix} - \begin{pmatrix} - & + \\ + & + \end{pmatrix} \right], \\ |t0\rangle &= \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} - & + \\ + & - \end{pmatrix} + \begin{pmatrix} + & - \\ - & + \end{pmatrix} \right], \\ |t-1\rangle &= \frac{1}{2} \left[ \begin{pmatrix} - & - \\ - & + \end{pmatrix} + \begin{pmatrix} + & - \\ - & - \end{pmatrix} - \begin{pmatrix} - & + \\ - & - \end{pmatrix} - \begin{pmatrix} - & - \\ + & - \end{pmatrix} \right], \\ |x1\rangle &= \frac{1}{2} \left[ \begin{pmatrix} + & + \\ - & + \end{pmatrix} - \begin{pmatrix} + & - \\ + & + \end{pmatrix} + \begin{pmatrix} + & + \\ + & - \end{pmatrix} - \begin{pmatrix} - & + \\ + & + \end{pmatrix} \right], \\ |x0\rangle &= \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} + & + \\ - & - \end{pmatrix} - \begin{pmatrix} - & - \\ + & + \end{pmatrix} \right], \end{aligned}$$

$$\begin{aligned}
|x-1\rangle &= -\frac{1}{2}\left[\begin{pmatrix} - & - \\ - & + \end{pmatrix} - \begin{pmatrix} + & - \\ - & - \end{pmatrix} - \begin{pmatrix} - & + \\ - & - \end{pmatrix} + \begin{pmatrix} - & - \\ + & - \end{pmatrix}\right] \\
|y1\rangle &= \frac{1}{2}\left[-\begin{pmatrix} + & + \\ - & + \end{pmatrix} + \begin{pmatrix} + & - \\ + & + \end{pmatrix} + \begin{pmatrix} + & + \\ + & - \end{pmatrix} - \begin{pmatrix} - & + \\ + & + \end{pmatrix}\right], \\
|y0\rangle &= \frac{1}{\sqrt{2}}\left[-\begin{pmatrix} - & + \\ - & + \end{pmatrix} + \begin{pmatrix} + & - \\ + & - \end{pmatrix}\right], \\
|y-1\rangle &= -\frac{1}{2}\left[\begin{pmatrix} - & - \\ - & + \end{pmatrix} - \begin{pmatrix} + & - \\ - & - \end{pmatrix} + \begin{pmatrix} - & + \\ - & - \end{pmatrix} - \begin{pmatrix} - & - \\ + & - \end{pmatrix}\right], \tag{A.5}
\end{aligned}$$

where signs  $\pm$  note the direction of corresponding spins. The state  $|\varphi\rangle$  can be also presented as a sum of two possible dimer configurations:

$$\begin{aligned}
|\varphi\rangle &= (|d12\rangle \otimes |d34\rangle + |d14\rangle \otimes |d23\rangle)/\sqrt{3}, \\
|dab\rangle &= (|a+\rangle \otimes |b-\rangle - |a-\rangle \otimes |b+\rangle)/\sqrt{2}. \tag{A.6}
\end{aligned}$$

This representation shows that our primary state sufficiently close to the RVB-state.

## Appendix B: Structure of selfconsistent equations

When solving the system of equations (3.12) we transform the two-dimensional over  $\mathbf{k}$  integral into the one-dimensional over the energetic variable  $\epsilon$  one. This is possible because the dependence over  $\mathbf{k}$  is not explicit but only via  $\gamma_{\mathbf{k}}$ :

$$\sum_{\mathbf{k}} f(\gamma_{\mathbf{k}}) = \int_{-1}^1 \rho(\epsilon) f(\epsilon) d\epsilon, \quad \rho(\epsilon) = \sum_{\mathbf{k}} \delta(\epsilon - \gamma_{\mathbf{k}}), \quad \int_{-1}^1 \rho(\epsilon) d\epsilon = 1. \tag{B.1}$$

One can easily get for  $\rho(\epsilon)$

$$\rho(\epsilon) = \frac{2}{\pi^2} K(\xi'), \quad \xi' = \sqrt{1 - \xi^2}, \tag{B.2}$$

where  $K(\xi')$  is a complete elliptic integral of the first kind. For the computation of integrals over  $\epsilon$  we have used the Gauss type formula:

$$\int_{-1}^1 \rho(\epsilon) f(\epsilon) d\epsilon = \sum_{i=1}^n c_i f(\epsilon_i). \tag{B.3}$$

Which is precise for only polynomial of the order  $2n - 1$ . The values  $\epsilon_i$  are determined by zeros of the orthonormalized polynomials  $n$ -th order on the segment  $(-1,1)$  with the weight  $\rho(\epsilon)$ . The coefficients  $c_i$  also can be expressed in terms of these polynomials [15]. Substituting the expressions (3.6) for  $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}}, \delta_{\mathbf{k}}$  in the terms  $\bar{N}_p, \bar{F}_p$  into Eqs. (3.12) we get the following selfconsistent system of the equations

$$\begin{aligned}
\bar{N}_p &= (3/4 + 4\bar{N}_1 + 4\bar{F}_1^+) I_p + (4\bar{N}_0 + \bar{F}_0^+ - 1 + 3\bar{N}_1/4) I_{p+1} - \delta_{p0}/2, \\
\bar{F}_p^- &= -(2\bar{N}_1 + 2\bar{F}_1^-) I_p - (8\bar{N}_0 - 1 - 3\bar{F}_1^-/4) I_{p+1}, \\
\bar{F}_p^+ &= -(2\bar{F}_0^+ - 1 - 3\bar{F}_1^+/4) I_{p+1}. \tag{B.4}
\end{aligned}$$

Here the integrals  $I_p$  are determined by the equations

$$f_p = I_p - \sum_{\mathbf{k}} (\gamma_{\mathbf{k}})^p R_{\mathbf{k}}^{-1} (n_{\mathbf{k}} + 1/2) = 0, \quad p = 0, 1, 2. \quad (\text{B.5})$$

As we can see from (B4) the five unknown variables  $\bar{N}_0, \bar{N}_1, \bar{F}_0^+, \bar{F}_1^+, \bar{F}_1^-$  can be expressed in terms of the three integrals  $I_p$  for  $p = 0, 1, 2$  which are determined by Eqs.(B5). These variables  $I_p$  were found by searching zeroes of the function

$$\Phi = (f_0)^2 + (f_1)^2 + (f_2)^2. \quad (\text{B.6})$$

This method of solving of Eqs.(B4) is more effective than the direct solution of these equations.

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