J. Phys. A: Math. Gen. **30** (1997) 5633–5643. Printed in the UK PII: S0305-4470(97)83019-3

Instanton picture of the spin tunnelling in the Lipkin–Meshkov–Glick model

V I Belinicher†‡, C Providencia† and J da Providencia†

† University of Coimbra, 3000, Coimbra, Portugal

‡ Institute of Semiconductor Physics, 630090, Novosibirsk, Russia

Received 3 April 1997

Abstract. A consistent theory of the ground-state energy and its splitting due to the process of tunnelling for the Lipkin–Meshkov–Glick (LMG) model is presented. We calculate accurately the trivial and the instanton saddle-point contributions to the functional integral for the partition function of the model, in terms of the spin coherent states. We show that such a calculation has to be performed very accurately taking into account the discrete nature of the functional integral. This accurate consideration leads to the replacement of the magnitude of the spin s by $s+1/2$, in the formula for the ground-state splitting obtained by a naive continuous method. We compare the numerical calculation of the ground-state energy and the splitting due to tunnelling with the results obtained by the quasiclassical method and obtain excellent agreement.

1. Introduction

Lipkin, Meshkov and Glick [1] proposed in 1965 an exactly solvable two-level many-fermion model (the LMG model) which has been used to test various kinds of many-body theories. The model Hamiltonian reads

$$H = \tilde{\omega}\hat{S}_0 - \tfrac{1}{2} f(\hat{S}_+^2 + \hat{S}_-^2) \tag{1}$$

where f is the coupling constant, $\tilde{\omega}$ is the energy difference between the two levels,

$$\hat{S}_0 = \frac{1}{2}\sum_{m=1}^{\Omega}(c_{+m}^\dagger c_{+m} - c_{-m}^\dagger c_{-m})$$

$$\hat{S}_+ = \sum_{m=1}^{\Omega} c_{+m}^\dagger c_{-m} \qquad \hat{S}_- = \hat{S}_+^\dagger \tag{2}$$

and Ω is the degree of degeneracy of each level. The single-particle states are labelled by the quantum numbers $\pm m$. The operators $\hat{S}_0$ and $\hat{S}_\pm$ satisfy the commutation relations of the $su(2)$ algebra.

The transformation associated with the unitary operator $e^{i\pi\hat{S}_0}$ leaves the Hamiltonian invariant. If $\chi = f\Omega/\tilde{\omega} < 1$, the mean-field ground state of the model is classified according to the trivial representation of the symmetry group. The symmetry is spontaneously broken if $\chi > 1$.

A splitting of the levels is then observed, which is analogous to the occurrence of rotational bands in deformed nuclei. The model is also of great interest in condensed matter physics, since it is related to the anisotropic Heisenberg model. In the strong coupling limit, $\chi \gg 1$, the level splitting vanishes for odd Ω. This behaviour is easily understood in the

0305-4470/97/165633+11\$19.50

framework of Kramers theorem and is related to the well known phenomenon of tunnelling suppression for half-integer spin anisotropic Heisenberg ferromagnets.

In this paper we give a short review of the calculations of the ground-state energy and its splitting by the instanton method in the continuous approach. Such a method has been applied to the anisotropic Heisenberg model for the description of the tunnelling of the magnetic moment of small magnetic particles with large spin [2–7] and to the LMG model [8]. Although the physical picture of the spin tunnelling was formulated in these papers the results of the papers [3–7] are not quantitatively correct.

We show that special care is required when computing the instanton contributions in order to also take correctly into account the small amplitude quantum fluctuations. On more technical language the functional determinants have to be calculated very accurately taking proper account of the operator ordering generating the functional integral for the partition sum. We show that it is essential to keep in mind that the functional integral is generated by the spin coherent state method. Taking accurate account of the discrete nature in time of the functional leads to essential corrections to the ground-state energy and to its tunnelling splitting. After taking into account all these contributions our result completely coincides with that of [2] obtained by the semiclassical method and the contradiction existing in the literature is resolved.

The structure of the paper is as follows. In section 2 we present the functional integral for the LMG model. In section 3 we review the simple results for the tunnelling in the framework of the continuous approximation for the functional integral. In section 4 we calculate the contribution of the trivial saddle point to the partition function and determine the ground-state energy. In section 5 we calculate accurately the functional determinant for the instanton saddle point taking into account the corrections due to discretization. In section 6 we present the comparison between the analytical theory and the exact numerical results for the LMG model and find an excellent agreement when the additional contributions discovered in this paper are taken into account.

2. Functional integral for the LMG model

It is convenient to rewrite the Hamiltonian of the model in the rotated reference frame when the first term in (1) is absent, i.e. $\tilde{\omega} = 0$:

$$\hat{H} = f(\hat{S}_z^2 - \hat{S}_x^2). \tag{3}$$

Here we assume that the magnitude of spin $s \gg 1$. This Hamiltonian (3) describes schematically the interaction between nucleons when the energy of splitting, regarded as an external magnetic field, is omitted. The choice of the coordinate axis is determined by the simplification of further calculations. Note that the LMG model can be reduced to the anisotropic Heisenberg model [3]. We concentrate our attention on the accurate treatment of the ground-state energy of the model and its splitting due to tunnelling.

In this special case the LMG model is invariant under time reflection. In the case of half-integer spin s this symmetry leads to the twofold Kramer's degeneracy of the ground state [3–6]. In the case of integer spin s instead of degeneracy we have the splitting of the ground state. This splitting has to be small because for large s the difference between integer and half-integer spin has to be small. In fact this splitting is exponentially small.

To solve this problem we have used the instanton method [3–6, 9] applied to the functional integral for the partition function of our spin system in terms of spin coherent states $|z\rangle$ [10]. They possess many remarkable properties and with their help the partition function can be represented in the form [10]

$$Z = \mathrm{Tr}[\exp(-T\hat{H})]$$

$$Z = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{n=0}^{N-1} \frac{(2s+1)\,\mathrm{d}z_n\,\mathrm{d}z_n^*}{\pi(1+|z_n|^2)^2} \exp(A(z)). \tag{4}$$

Here the interval of the imaginary time T is split into N parts, $T = N\Delta$, and in every section an integration over $z_n' = \mathrm{Re}\, z_n$ and $z_n'' = \mathrm{Im}\, z_n$ is performed. The action of the system $A(z)$ has the form

$$A(z) = \sum_{n=0}^{N-1} [2s \ln(1 + z_{n+1}^* z_n) - 2s \ln(1 + |z_n|^2) - \Delta H(z_{n+1}^*, z_n)] \tag{5}$$

where the variables z_n satisfy the periodic boundary conditions, $z_N = z_0$ or in the continuum limit $z(T) = z(0)$, and the Hamiltonian

$$H(z_{n+1}^*, z_n) = \frac{\langle z_{n+1}|\hat{H}|z_n\rangle}{\langle z_{n+1}|z_n\rangle} = \frac{s^2 g}{2}\left[1 - \frac{6z_{n+1}^* z_n + z_n^2 + z_{n+1}^{*2}}{(1 + z_{n+1}^* z_n)^2}\right] \tag{6}$$

where $g = f(2s-1)/s$, $g > 0$ is the coupling constant.

3. Naive classical picture for the ground-state energy and its instanton splitting

In many cases (but not in all!) one can take the continuum limit when z_{n+1} is close to z_n. In this continuum limit it is convenient to introduce the canonically conjugated variables φ and $p = \cos\theta$:

$$z = \rho \exp(\mathrm{i}\varphi) \qquad \rho^2 = \frac{s-p}{s+p} \qquad 0 \leqslant \varphi \leqslant 2\pi \qquad -s \leqslant p \leqslant s. \tag{7}$$

The action of the system $A(z)$ in the continuum limit is

$$A(\varphi, p) = \int_{-T/2}^{T/2} [\mathrm{i}(p-s)\dot{\varphi} - H(p,\varphi)]\,\mathrm{d}t$$

$$H(p,\varphi) = (g/2)(p^2(1+\cos^2\varphi) - s^2\cos^2\varphi) \tag{8}$$

where $H(p,\varphi)$ is the Hamiltonian of the problem. This Hamiltonian is peculiar because the mass depends on the coordinate φ. The second essential peculiarity is the presence in the action of the term $\mathrm{i}s\dot{\varphi}$ (the Berry phase) which separates integer and half-integer spins [5, 6].

One can easily check that the Hamiltonian has two minima at the points $\varphi = 0, \pi$ and $p = 0$. These minima are deep for $s \gg 1$ and the Hamiltonian in the neighbourhood of the minima has the simple form of a harmonic oscillator

$$H(p,\varphi) = E_{\min} + gp^2 + gs^2\varphi^2/2 \qquad E_{\min} = -s^2 g/2 \tag{9}$$

and the vibration frequency is $\omega = \sqrt{2}gs$.

The ground-state energy and the splitting can be found [3–6] by applying the instanton method [9] to the Hamiltonian (8). The result of summation of all saddle-point contributions to the partition function can be presented in the following form:

$$Z = Z_0 \cosh(|A+B|T) \tag{10}$$

where Z_0 is the contribution of the trivial saddle points and A and B are two different types of the instanton contributions excluding the determinant of the trivial saddle points.

The dependence on the Berry phase can be easily found [4]:

$$A = \exp(\mathrm{i}\pi s)I \qquad B = \exp(-\mathrm{i}\pi s)I. \tag{11}$$

Application of the continuous method of [9] to the Hamiltonian (8) gives the following expression for Z_0:

$$Z_0 = 2\exp(-E_{\min}T)[\det(L_0)]^{-1/2} = 2\exp(-(E_{\min} + \omega/2)T) \tag{12}$$

where L_0 is the Schrödinger operator for the quadratic form (9), and the factor 2 is due to the presence of two minima. For the instanton contribution I in the framework of the continuous method we have [9]

$$I = \sqrt{\frac{A_0}{2\pi}}X\exp(-A_0) \qquad X = \frac{[\det(L_0)]^{1/2}}{[\det(L_I)']^{1/2}} \tag{13}$$

where A_0 is the instanton action, L_I is the Schrödinger operator for the quadratic form describing fluctuations around the instanton solution, the prime in the determinant L_I means that the contribution of the zero mode has to be excluded and the first factor in (13) represents the measure of integration over the zero mode.

The ratio of determinants X can be expressed in the continuum limit through the asymptotic behaviour for time $t \Rightarrow \infty$ of the normalized zero-mode excitation:

$$\chi(t) = (A_0 g(1 + \cos^2\varphi_t))^{-1/2}\dot{\varphi}_0(t) \tag{14}$$

where $\varphi_0(t)$ is the instanton solution for the action (8). We denote by $p_0(t)$ the momentum conjugated to φ. We have

$$\cos\varphi_0(t) = \pm\frac{\sinh(\tau_0)}{\sqrt{2 + \sinh^2(\tau_0)}} \qquad \mathrm{i}p_0(t) = su_t^0 = \pm\frac{s}{\cosh(\tau_0)} \tag{15}$$

where $\tau_0 = \omega(t - t_0)$, and t_0 is the centre in time of an instanton. The asymptotic behaviour of the zero mode is of the form

$$\lim|\chi(t)| \Rightarrow A_\tau\exp(-|\tau_0|) \qquad A_\tau = 2(2)^{1/4}\sqrt{s\omega/A_0}. \tag{16}$$

On the basis of this solution one can prove [9] that the ratio of determinants is

$$X = \sqrt{2}A_\tau = 2^{7/4}\sqrt{s\omega/A_0} \tag{17}$$

and does not depend on T. Substituting the magnitude X (16) in the expression for I (13) we obtain the expression for the partition function in the form

$$Z = Z_0\cosh(4\times 2^{1/4}\omega\sqrt{s/\pi}\cos(\pi s)T\,\mathrm{e}^{-A_0}). \tag{18}$$

Using the expression $A_0 = 2s\ln(1+\sqrt{2})$ which follows from the action (8) for the instanton solution (15) we obtain the energy of the ground state and its splitting in the form

$$E_\pm = E_0 \pm \delta E/2 \qquad E_0 = E_{\min} + \omega/2$$
$$\delta E = 8\times 2^{1/4}\omega\sqrt{s/\pi}\exp[-2s\ln(1+\sqrt{2})]\cos(\pi s). \tag{19}$$

We can see that this splitting is exponentially small for large s and equals zero in the case of half-integer spins in full agreement with Kramers theorem [5, 6]. The cancellation of the splitting for the case of the half-integer spin takes place due to the compensation of the contributions of the instantons of A and B type to the partition function Z.

However, unfortunately, expressions (19) for the ground state energy E_0 and the instanton splitting δE are *valid only qualitatively*. *Quantitatively they are wrong*. The origin of the mistake is connected with the subtle nature of the functional integral which demands *accurate treatments when we calculate functional determinants*. In particularly the expressions for Z_0 (12) and for X (17) are wrong.

4. Energy of the ground state or the trivial saddle-point contribution to the partition function

The expression E_0 for the ground-state energy is wrong due to an incorrect definition of the product of operators at the same point in terms of the variables φ and p. The natural variables for the functional integral (4) are the z-variables. The trivial saddle points in terms of these variables are $z = \pm 1$. In the neighbourhood of the saddle points we can represent the z-variables in the form

$$z = \pm 1 + \sqrt{2/s}\, a \qquad z^* = \pm 1 + \sqrt{2/s}\, a^*. \tag{20}$$

The quadratic part of the action close to the saddle point is

$$A_q(a) = \sum_{n=0}^{N-1} [(a^*_{n+1} a_n - a^*_n a_n) - \alpha a^*_{n+1} a_n - (\beta/2)(a^{*2}_{n+1} + a^2_n)] \tag{21}$$

where $\alpha = 3Tsg/N$, $\beta = Tsg/N$. The quadratic form $A_q(a)$ can be partly diagonalized if we pass to the ω-representation for the variables a_n, a^*_n:

$$(a_n, a^*_n) = N^{-1/2} \sum_{m=0}^{N-1} \mathrm{e}^{\pm \mathrm{i}\omega_m n} (a_m, a^*_m) \tag{22}$$

where $\omega_m = 2\pi \mathrm{i} m/N$. The action in the ω-representation has the form

$$A_q(a) = \sum_{m=0}^{N-1} [((1-\alpha)\exp(-\mathrm{i}\omega_m) - 1) a^*_m a_m - \beta(a^*_m a^*_{N-m} + a_m a_{N-m})]. \tag{23}$$

The Gaussian integrals with the quadratic form (23) can be easily calculated and the inverse determinant D^{-1} can be obtained. The determinant D is equal to the product of the eigenvalues λ_m of the quadratic form (23)

$$\lambda_m = \alpha^2 - \beta^2 + 2(1-\alpha)(1 - \cos\omega_m). \tag{24}$$

The partition function Z_0 has the following expression in terms of λ_m:

$$Z_0 = 2\exp(E_{\min} T) Y \qquad Y = \prod_{m=0}^{N-1} \lambda_m^{-1/2}. \tag{25}$$

The quantity Y can be calculated with help of some simple tricks and in the limit $N \to \infty$ the partition function and the ground-state energy are

$$\begin{aligned} Z_0 &= 2\exp[-E_0 T][1 - \exp(-\omega T)]^{-1} \\ E_0 &= E_{\min} + (\omega - \omega_0)/2 \end{aligned} \tag{26}$$

where $\omega = \sqrt{2} sg$, $\omega_0 = 3sg/2$.

We can see that the result (26) differs from the continuous result E_0 obtained previously (19). From the point of view of the functional integral this difference is due to the contribution of the large eigenvalues of the quadratic form (21) which correspond to the large ω in the partition function (25). From the operator point of view the quadratic form (21) expressed in terms of operators (for example with the help of the Holstein–Primakoff representation with the quantization along x) has to be diagonalized with the help of the Bogoluibov's $u - v$ transformation. As a result of such diagonalization the ground-state energy is shifted by the amount $(\omega - \omega_0)/2$.

5. Calculation of corrections to the determinant caused by the discrete nature of the functional integral over time

In our calculation we passed carelessly from the original variables z_t^*, z_t to the natural variables φ_t, p_t. Small corrections of the order $\Delta\omega$ can lead to finite contributions to such quantities as the measure of integration and $\det(\hat{L}_0)$. They are practically the same quantities because by changing the variables of integration they can be reduced to each other. First we describe the mechanism of such corrections on a qualitative level and after that we apply it to our case.

Let us suppose that each variable c_i in the measure has a small multiplicative correction of the order $\Delta\omega$:

$$\mathrm{d}\mu \Rightarrow \prod_{i=1}^{N} \frac{\mathrm{d}c_i}{\sqrt{2\pi}}(1+\alpha_i\Delta\omega) = \mathrm{d}\mu_0 R$$

$$R = \prod_{i=1}^{N}(1+\alpha_i\Delta\omega). \tag{27}$$

This factor R to $\mathrm{d}\mu_0$ can be expressed in an exponential form and is different from unity

$$R = \exp(\delta A_0) \qquad \delta A_0 = \sum_{i=1}^{N}\ln(1+\alpha_i\Delta\omega) \simeq \omega\int_0^T \alpha_t\,\mathrm{d}t \tag{28}$$

where δA_0 is the correction to the classical action. If the function α_t is asymptotically constant for $0 \leqslant t \leqslant T$, then $\delta A_0 \sim$ constant $\times\,\omega T$. If the function α is localized over time within a region order ω^{-1} then $\delta A_0 \sim$ constant.

If we calculate the determinant, the mechanism of influence of corrections is more subtle. Let us suppose that the matrix $\hat{L}$ can be presented in a form

$$\hat{L} = \hat{L}^0 + \Delta\omega\hat{Q} \tag{29}$$

where the operator $\hat{L}^0$ represents some discrete version of the operator $\hat{L}_I$ (13) and the operator $\hat{Q}$ represents some discrete version of the second-order differential operator

$$\hat{Q} = a_0 + a_1\omega^{-1}\frac{\partial}{\partial t} + a_2\omega^{-2}\left(\frac{\partial}{\partial t}\right)^2 \tag{30}$$

where functions a_0, a_1 and a_2 are some smooth functions of time t of the order of unity. Simple arguments show that the contribution to the determinant due to the a_0 term is always small in contrast to the contributions of the a_1 and a_2 terms which are of the order of unity. To demonstrate this statement let us represent the operator $\hat{L}$ in the form

$$\hat{L} = [1+\Delta\omega\hat{Q}\hat{G}^0]\hat{L}^0 \qquad \hat{G}^0 = (\hat{L}^0)^{-1} \tag{31}$$

and the determinant takes the form

$$\det[\hat{L}] \simeq \det[\hat{L}^0]\exp(-2\delta A_0) \qquad \delta A_0 = -(1/2)\Delta\omega\mathrm{Tr}[\hat{Q}\hat{G}^0]. \tag{32}$$

One can check that in the discrete representation the matrix elements of the Green function $(\hat{G}^0)_{nn'}$ are of the order Δ and the characteristic width over $n-n'$ is of the order of $(\omega\Delta)^{-1}$. This means that the contribution of the a_0 terms of the operator $\hat{Q}$ into δA_0 is of the order of Δ. This can be easily checked if we convert the trace in equation (32) into an integral over t.

The first impression is that the contributions of the a_1 and a_2 terms to $\det[\hat{L}]$ are also small because the Green function $(\hat{G}^0)_{nn'}$ has a width over $t-t'$ of order ω^{-1} and its derivatives have the order of magnitude ω. However, these arguments are completely wrong because the Green function is a singular function over $t - t'$. It has a jump at $t = t'$ and the time derivative of this jump is of the order of Δ^{-1}. If the functions a_1 and $a_2 \to$ constant at $t \to \pm T/2$ then $\delta A_0 \sim$ constant $\times\, \omega T$. If the functions a_1 and a_2 are localized over time in a region of order ω^{-1} then $\delta A_0 \sim$ constant.

At this point we can calculate corrections to the instanton contribution due to renormalization of the measure and the functional determinant to leading order with respect to $1/s$. There are three sources of corrections of the order of Δ to the continuous approximation. The first origin of corrections is the kinetic term. With the desired accuracy, the action contains a term of the form

$$A_B = s \sum_{n=0}^{N-1} \left[2\frac{(z_{n+1}^* - z_n^*)z_n}{1 + z_n^* z_n} - \frac{(z_{n+1}^* - z_n^*)^2 z_n^2}{(1 + z_n^* z_n)^2} \right]. \tag{33}$$

The second origin of corrections is the Hamiltonian due to its dependence on z_{n+1}^*:

$$A_{H1} = g s^2 \Delta \sum_{n=0}^{N-1} \frac{(z_{n+1}^* - z_n^*)(3z_n + z_n^* - 3z_n^* z_n^2 - z_n^3)}{(1 + z_n^* z_n)^3}. \tag{34}$$

The third origin of corrections is the difference between the average of the square of the Hamiltonian and the square of the average of the Hamiltonian:

$$A_{H2} = (1/2)\Delta^2 \sum_{n=0}^{N-1} \left[\frac{\langle z_n|\hat{H}^2|z_n\rangle}{\langle z_n|z_n\rangle} - \left(\frac{\langle z_n|\hat{H}|z_n\rangle}{\langle z_n|z_n\rangle} \right)^2 \right]. \tag{35}$$

Since this correction depends (to leading order) only on z_n^* and z_n it can contribute only to the renormalization of the measure of integration.

At this stage we can calculate the measure and the determinant for the instanton contribution with the necessary accuracy. We begin by interpreting the integration over the variables z_n' and z_n'' ($z_n = z_n' + \mathrm{i} z_n''$ and $z_n^* = z_n' - \mathrm{i} z_n''$) in expression (4) for the partition function Z as an integral over the two-dimensional surface $\mathrm{Im}(z_n') = 0$ and $\mathrm{Im}(z_n'') = 0$ of the four-dimensional space where variables z_n' and z_n'' are considered as complex variables. Because the function $(1 + z_n^* z_n)^{-2} \exp(A(z_n^*, z_n))$ is an analytic function of the variables z_n' and z_n'' it can be continued in the four-dimensional complex manifold and, in this way, we can arrive at the instanton saddle point. In the neighbourhood of the saddle point we have to integrate over the two-dimensional manifold which realizes the directions of steepest descent. The direction of steepest descent is chosen correctly if all eigenvalues of the quadratic form in the exponent of the action are real.

This program can be realized with the help of the following change of variables z_n^* and z_n in the neighbourhood of the saddle point:

$$\begin{aligned} z_n &= \bar{z}_n + (\mathrm{i}/\sqrt{s})\bar{z}_n(\psi_n - v_n/(1 + u_n^2)) \\ z_n^* &= \bar{z}_n^* - (\mathrm{i}/\sqrt{s})\bar{z}_n^*(\psi_n + v_n/(1 + u_n^2)). \end{aligned} \tag{36}$$

Here we understand $\bar{z}_n$ and $\bar{z}_n^*$ as classical (non-fluctuating) variables connected with the previously introduced variables φ_n and u_n by the relations:

$$\begin{aligned} \bar{z}_n(\varphi_n, u_n) &= \sqrt{(1 - \mathrm{i}u_n)/(1 + \mathrm{i}u_n)}\, \exp(\mathrm{i}\varphi_n) \\ \bar{z}_n^*(\varphi_n, u_n) &= \sqrt{(1 - \mathrm{i}u_n)/(1 + \mathrm{i}u_n)}\, \exp(-\mathrm{i}\varphi_n) \end{aligned} \tag{37}$$

thus the variables $\bar{z}_n$ and $\bar{z}_n^*$ are not in our case complex conjugated. We shall understand that the classical variables $\bar{z}_n$ and $\bar{z}_n^*$ or $\boldsymbol{x}_n = (\varphi_n, u_n)$ satisfy the classical equations of motion which determine the saddle point with corrections of the order of Δ taken into account. This means that only to leading order are the variables $\boldsymbol{x}$ equal to $\boldsymbol{x}_n^0 = (\varphi_n^0, u_n^0)$

determined by equation (15). The difference between the variables $\boldsymbol{x}$ and the variables $\boldsymbol{x}^0$ is of the order of Δ and is determined by small corrections of the order of Δ contained in the action A_B (33) and corrections to the action A_{H1} (34) and A_{H2} (35). It is fortunate that these corrections are non-essential to our problem.

The reason for this is the canonical form of the measure in terms of the variables $\boldsymbol{y}_n = (\psi_n, v_n)$. Note that the change of variables (36) was suggested by the formulae of differentiation over time of the quantities $\bar{z}_n(\varphi_n, u_n)$ and $\bar{z}_n^*(\varphi_n, u_n)$.

At this stage we can calculate the renormalization of the functional determinant due to the Δ-corrections. This can be done on the basis of the following formula for the decomposition of the action in the neighbourhood of the saddle point:

$$A(z^*, z) = A_0 + \frac{1}{2s} \sum_{ni,n'j} \frac{\partial^2 A(\varphi, u)}{\partial x_{ni} \partial x_{n'j}} y_{ni} y_{n'j} + \cdots. \tag{38}$$

This formula strongly simplifies the calculations and can be proved if we consider the original variables z_n and z_n^* as functions of variables $\boldsymbol{y}_n = (\psi_n, v_n)$ on the one hand and of the variables $\boldsymbol{x}_n = (\varphi_n, u_n)$ on the other.

The matrix $\hat{L}^0$ can be chosen on the basis of (38) in the form

$$\hat{L}^0 = \begin{pmatrix} a_n, & -\partial_- + b_n \\ \partial_+ + b_n, & c_n \end{pmatrix} \tag{39}$$

where the explicit form of the functions a_n, b_n and c_n is derived from the action (8)

$$\begin{aligned} &a_t = gs(1 + u_t^2)\cos(2\varphi_t) \qquad b_t = gsu_t \sin(2\varphi_t) \\ &c_t = -gs(1 + \cos^2(\varphi_t)) \end{aligned} \tag{40}$$

and the difference derivatives are determined by the following relations:

$$\begin{aligned} &\partial_- f_n = (f_n - f_{n-1})/\Delta \qquad \partial_+ f_n = (f_{n+1} - f_n)/\Delta \\ &\partial^2 f_n = (f_{n+1} + f_{n-1} - 2f_n)/\Delta^2. \end{aligned} \tag{41}$$

The Green function $\hat{G}^0$ satisfies the relation

$$\sum_{n_a} (\hat{L}^0)_{n,n_a} (\hat{G}^0)_{n_a,n'} = \delta_{n,n'} \tag{42}$$

and cannot be found in a general form. However this is unnecessary for our purposes. We are interested in the singular part of the Green function $(\hat{G}^0)_{n'n}$ at $n' \approx n$. This singular part of the Green function at $n' \approx n$ can be found in a general form

$$(\hat{G}^0)_{n',n} = \Delta \begin{pmatrix} \Delta\theta_{n',n}(n' - n)c_n, & \theta_{n',n}(1 - \Delta(n' - 1 - n)b_n) \\ -\theta_{n'+1,n}(1 + \Delta(n' + 1 - n)b_n), & \Delta\theta_{n'+1,n}(n' - n)a_n \end{pmatrix} \tag{43}$$

where $\theta_{n'n}$ is the θ-function defined in the following manner:

$$\theta_{n'n} = \begin{cases} 1 & n' \geqslant n + 1 \qquad \partial_+\theta_{n',n} = \Delta^{-1}\delta_{n',n} \\ 0 & n' \leqslant n \qquad \partial_-\theta_{n'+1,n} = \Delta^{-1}\delta_{n',n}. \end{cases} \tag{44}$$

After some tedious calculations the singular (containing essential derivatives) part of the operator $\hat{Q}$ entering in equation (31) can be presented in the form

$$(\hat{Q}f)_n = (1/2\omega) \begin{pmatrix} -(1 + u_n^2)\partial^2, & \partial^2 + (2\dot{\varphi}_n u_n - 3gs)\partial_- \\ \partial^2 - (2\dot{\varphi}_n u_n - 3gs)\partial_+, & (1 + u_n^2)^{-1}\partial^2 \end{pmatrix} f_n \tag{45}$$

where f_n is an arbitrary function. For the diagonal elements of the matrix $\hat{Q}$ it is sufficient to keep the second derivative. For non-diagonal elements we have to keep the first and second derivatives. Acting with the operator $\hat{Q}$ on the Green function $(\hat{G}^0)_{n',n}$ and calculating the trace we obtain for the correction to the action δA_0

$$\delta A_0 = -(1/2)\int (f_t - f_\infty)\,\mathrm{d}t$$
$$f_t = (1+u_t^2)^{-1}a_t - (1+u_t^2)c_t + 4\dot{\varphi}_t u_t. \tag{46}$$

Since we are interested in the ratio of determinants we subtract from the function f_t its value at the trivial saddle point f_∞. Using equation (15) for $\dot{\varphi}$ we find the final expression for the correction to the instanton action δA_0:

$$\delta A_0 = -gs\int_{-\infty}^{\infty}(1-\cos^2(\varphi_t))\,\mathrm{d}t = -\ln(1+\sqrt{2}). \tag{47}$$

The obtained result is surprisingly simple. It can be found if we change $s \Rightarrow s+1/2$ in the expression for the energy splitting (19). Such a change has a quasiclassical meaning and can be found by a simpler method than in this section. As the final result, instead of the expression (19), we have for the ground-state energy E_0 and for the instanton splitting

$$E_0 = E_{\mathrm{min}} + (\omega - \omega_0)/2$$
$$\delta E = 8\times 2^{1/4}\omega\sqrt{s/\pi}\,\exp[-(2s+1)\ln(1+\sqrt{2})]\cos(\pi s) \tag{48}$$

where in the limit $s\to\infty$ we have $E_{\mathrm{min}} = -s^2 f$, $\omega = 2\sqrt{2}sf$, $\omega_0 = 3sf$. This result completely coincides with the result of [2] for the case $A = B = f$.

6. Comparison with the numerical results

In this section we will compare the exact ground-state energy and splitting with the results obtained in the above discussion, namely equation (48). The Hamiltonian (3) is easily diagonalized on the basis of the eigenfunctions of $\hat{S}^2$ and $\hat{S}_0$. The ground state belongs to the symmetric representation and so we will just consider this representation for different values of the total spin s. In table 1 the numerical and theoretical results for the ground-state energy and its splitting (48) are presented. As expected the relative error between the calculated and the exact ground-state energy decreases with increasing s.

7. Discussion

We want to discuss two points here: the interpretation of the replacement of s by $s+1/2$ in the effective action and the procedure of the calculation of the corrections to the instanton approximation.

Let us consider the explicit form of the spin operators acting on the space of functions

$$\psi_m(\varphi) = \mathrm{e}^{\mathrm{i}m\varphi}/\sqrt{2\pi} \qquad 0\leqslant\varphi\leqslant 2\pi \qquad m = -s,\ldots,s. \tag{49}$$

The spin operators have the form [2, 11]

$$\hat{S}_+ = \sqrt{(s+1/2)^2 - (\hat{p}-1/2)^2}\,\mathrm{e}^{\mathrm{i}\varphi} \qquad \hat{S}_z = \hat{p}$$
$$\hat{S}_- = \mathrm{e}^{-\mathrm{i}\varphi}\sqrt{(s+1/2)^2 - (\hat{p}-1/2)^2} \qquad \hat{p} = -\mathrm{i}\partial/\partial\varphi. \tag{50}$$

Substituting this representation for the spin operators (50) into the Hamiltonian (3) and decomposing it in powers of $\hat{p}$ (this decomposition can be justified) we obtain, up to order $1/s$,

$$\hat{H}(\hat{p},\varphi) = (f/2)(\hat{p}^2(1+\cos^2\varphi) - (s+1/2)^2\cos^2\varphi). \tag{51}$$

Table 1. Results for the ground-state energy and its splitting. The first column shows the magnitude of spin, the second and third columns the exact and calculated ground-state energy, respectively, the fourth and the fifth columns the exact and the calculated ground-state splitting, respectively.

s	E/f	E_0/f	E_0/E	$\Delta E/f$	E_{inst}/f	$E_{\text{inst}}/\Delta E$
1	-0.1000×10^{1}	-0.5429×10^{0}	0.5429	0.1000×10^{1}	0.5395×10^{0}	0.5395
2	-0.3464×10^{1}	-0.3129×10^{1}	0.9032	0.4641×10^{0}	0.3927×10^{0}	0.8461
3	-0.7899×10^{1}	-0.7714×10^{1}	0.9766	0.1530×10^{0}	0.1375×10^{0}	0.8988
4	-0.1442×10^{2}	-0.1430×10^{2}	0.9915	0.4137×10^{-1}	0.3814×10^{-1}	0.9220
5	-0.2299×10^{2}	-0.2289×10^{2}	0.9956	0.1004×10^{-1}	0.9408×10^{-2}	0.9368
6	-0.3357×10^{2}	-0.3347×10^{2}	0.9971	0.2282×10^{-2}	0.2161×10^{-2}	0.9470
7	-0.4615×10^{2}	-0.4606×10^{2}	0.9980	0.4959×10^{-3}	0.4733×10^{-3}	0.9544
8	-0.6074×10^{2}	-0.6064×10^{2}	0.9985	0.1043×10^{-3}	0.1002×10^{-3}	0.9600
9	-0.7732×10^{2}	-0.7723×10^{2}	0.9988	0.2142×10^{-4}	0.2066×10^{-4}	0.9644
10	-0.9591×10^{2}	-0.9581×10^{2}	0.9991	0.4314×10^{-5}	0.4176×10^{-5}	0.9680
11	-0.1165×10^{3}	-0.1164×10^{3}	0.9992	0.8555×10^{-6}	0.8305×10^{-6}	0.9708
12	-0.1391×10^{3}	-0.1390×10^{3}	0.9994	0.1675×10^{-6}	0.1630×10^{-6}	0.9733
13	-0.1637×10^{3}	-0.1636×10^{3}	0.9995	0.3244×10^{-7}	0.3164×10^{-7}	0.9753
14	-0.1902×10^{3}	-0.1902×10^{3}	0.9995	0.6228×10^{-8}	0.6084×10^{-8}	0.9770
15	-0.2188×10^{3}	-0.2187×10^{3}	0.9996	0.1186×10^{-8}	0.1161×10^{-8}	0.9787
16	-0.2494×10^{3}	-0.2493×10^{3}	0.9996	0.2247×10^{-9}	0.2198×10^{-9}	0.9784
17	-0.2820×10^{3}	-0.2819×10^{3}	0.9997	0.4206×10^{-10}	0.4139×10^{-10}	0.9839

We can calculate with this Hamiltonian the partition function applying the procedure of the $p-\varphi$ construction of the functional integral. We obtain a 'functional integral' in which in each time section we have an integration over φ from 0 to 2π and a summation over m from $-s$ to s. Replacing the summation over m by an integration over p and extending it to infinity, we see that the Hamiltonian (51) coincides with the Hamiltonian (8) with one essential difference, namely, instead of s^2 in (8) we have $(s+1/2)^2$ in (51). This means that we can obtain the correct answer for the tunnelling splitting δE (48) in the continuous representation [2]. Because we know at present that corrections are small (see discussion later) there is one question remaining: Why are the corrections absent in the approach discussed in this section? The explanation lies in the difference between the coherent state construction of the functional integral and the $p-\varphi$ construction. One can check that in the calculation of the functional determinant in the $p-\varphi$ functional integral the corrections of the order of Δ are absent. These corrections are also absent in the usual problem of tunnelling in quantum mechanics which is confirmed by the coincidence of the result of the energy splitting with the quasiclassical one [9].

In conclusion, note that we can construct a perturbation expansion around the trivial saddle point (10) as well as around the instanton solution (36). Both these expansions are over the very well defined small parameter $1/s$. We want to stress one peculiarity of this perturbation theory: the presence of terms proportional to the number of time sections N. The presence of such terms is the characteristic feature of the perturbation theory when the measure of the integration is not trivial. In such theories the kinetic term is also non-trivial: the effective mass or the coefficient before a term with the time derivative is a function of the field variables. In the framework of perturbation theory such terms lead to divergencies at large frequencies ω. These divergencies have to completely cancel the N-terms which follow from the measure.

The inadequacy of the simple-minded continuous approximation in the framework of the coherent state representation, to account for the splitting in the LMG model, has been previously remarked on in [2, 12]. In [12] it is shown that the WKB approximation produces the correct splitting. However, these authors do not pinpoint the precise origin of the detected discrepancy.

Acknowledgments

We are grateful to K A Kikoin for valuable suggestions and to A Vieira. This work was supported in part by the Portuguese project PBIC/C/CEN/1130/92, VB was supported in part by the Portuguese program PRAXIS XXI/BCC/4381/94, and in part by the Russian Foundation for Fundamental Researches, grant No 94-02-03235.

References

[1] Lipkin H J, Meshkov N and Glick A J 1965 *Nucl. Phys.* **62** 188
[2] Enz M and Schilling R 1986 *J. Phys. C: Solid State Phys.* **19** 1765
[3] Chudnovsky E M and Gunter L 1988 *Phys. Rev. Lett.* **60** 661
[4] Chudnovsky E M and DiVincenzo D P 1993 *Phys. Rev.* B **48** 10 548
[5] Loss D, DiVincenzo D P and Grinstein G 1992 *Phys. Rev. Lett.* **69** 3232
[6] von Delft J and Henley C L 1992 *Phys. Rev. Lett.* **69** 3236
[7] Galyshev V Yu and Popkov A F 1995 *JETP* **81** 962
[8] Vourdas A and Bishop R F 1985 *J. Phys. G: Nucl. Phys.* **11** 95
[9] Coleman S 1985 *Aspects of Symmetry* (New York: Cambridge University Press) ch 7
Callan C G and Coleman S 1977 *Phys. Rev.* D **16** 1762
[10] Klauder J R and Skagerstam Bo-S 1985 *Coherent States* (Singapore: World Scientific)
[11] Klein A and Marshalek E R 1991 *Rev. Mod. Phys.* **63** 375
[12] van Hemmen J L and Sütö A 1986 *Europhys. Lett.* **1** 481
van Hemmen J L and Sütö A 1986 *Physica* **141B** 37