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Chapter 5. MAXWELL'S EQUATIONS IN UNIFORM RIGID MEDIA

This chapter studies the properties of electromagnetic fields in uniform (and sectionally uniform) bulk solid media, where propagating electromagnetic plane waves constitute the most common (but by no means the only) situation (see Sec. 12 for a lossless dielectric insulator and Sec. 13 for the quite different situation of a bulk conductor). These model systems are of interest both for the detailed behavior of the electric and magnetic fields *and* for the electromagnetic properties of the medium itself. For example, waves in a nonconducting dielectric are remarkably different from those in a conducting metal, and Sec. 14 explores in some detail the materials physics that underlies this distinction.

11. Stress tensor

The three fundamental conservation laws for energy, linear momentum, and angular momentum provide a basis for much of modern physics. The electromagnetic aspects of the first have been treated in Sec. 10, in connection with the derivation of Poynting's theorem and the electromagnetic work in Eq. (10.45) done on some general material system. In the present section, an analogous, but considerably more intricate, analysis will consider the conservation of linear momentum, along with an abbreviated treatment of angular momentum. The difference between energy on the one hand and linear and angular momentum on the other arises from the vector character of the momentum, in contrast to the energy, which is a scalar quantity.

Although it goes somewhat beyond the present context of classical electromagnetism, it is worth recalling that these three conservation laws acquire a particular significance in quantum theory, for they ultimately reflect the invariance of the vacuum with respect to infinitesimal translations in time and space, and with respect to infinitesimal rotations. Specifically, the quantum-mechanical hamiltonian operator H (which is usually related to the total energy) is the operator that generates infinitesimal translations in time; this result becomes obvious from the "formal" solution $\Psi(t) = \exp(iHt/\hbar)\,\Psi(0)$ to the Schrödinger equation $-i\hbar\partial\Psi/\partial t = H\Psi$, where $\Psi(0)$ is the initial wavefunction at $t = 0$. Similarly, the total momentum $\mathbf{P}$ is the operator that generates infinitesimal spatial displacements, and the total angular momentum $\mathbf{L}$ is the operator that generates infinitesimal rotations (for an elementary but profound discussion of these ideas, see the Feynman Lectures, Vol. I, Chap. 52, Vol. II, Chap. 27, and Vol. III, Chaps. 8, 17, and 20; for a more thorough and conventional discussion, see, for example, Schiff, Quantum Mechanics, 3rd edition, Secs. 26 and 27).

One other fundamental conservation law is that for electric charge. Its elementary form in Eq. (8.5) serves as a paradigm for all the remaining classical conservation laws. Like the three laws discussed above, it also acquires a much deeper interpretation in the context of quantum mechanics, where charge conservation and gauge invariance are related to the invariance of physical phenomena under a (*global* and/or *local*) change in the phase of the wave function. Some of these ideas will be examined in Chap. 10 in connection with the lagrangian formulation of electromagnetism (for a more comprehensive treatment, see, for example, Raymond, Field Theory. A Modern Primer, pp. 244-248, or Itzykson & Zuber).

Conservation of linear momentum

For simplicity, the conservation of linear and angular momentum will be analyzed only for particles in vacuum, where **E** and **B** constitute the complete electromagnetic fields; in this approach, *all the charges and currents are made explicit.* The alternative treatments that deal explicitly with the polarization **P** and the magnetization **M** are complicated and subtle; remarkably, even now, some fundamental questions remain unresolved, in part because of the evident difficulty in measuring the local momentum density in a material medium (for a brief discussion of the issues, see, for example, Jackson, Sec. 6.9, where additional references are given). Ultimately, the present description in terms of only **E** and **B** must be correct, for the auxiliary fields **P** and **M** merely represent suitable averages over particular forms of charges and currents—the electric and magnetic dipoles. As will be seen, the electromagnetic momentum density involves expressions that are quadratic in the electromagnetic fields, and it is by no means obvious that the average of such products are simply related to the product of the corresponding averages.

As in the beginning of Sec. 8, consider a set of point charges $\{q_i\}$ located at instantaneous positions $\{\mathbf{r}_i(t)\}$. The total electromagnetic force on the particles contained in some volume V follows from the Lorentz force in Eq. (9.3)

$$\mathbf{F} = \sum_i q_i\,(\mathbf{E} + \mathbf{v}_i \times \mathbf{B})|_{\mathbf{r}_i}, \quad \text{here written in } \textit{particle} \text{ form,} \tag{11.1}$$

where the electric and magnetic fields are evaluated at the point $\mathbf{r}_i(t)$. With the definition of the charge and current density from Eqs. (8.11) and (8.12),

$$\rho(\mathbf{r}, t) = \sum_i q_i\,\delta(\mathbf{r} - \mathbf{r}_i), \tag{11.2}$$

$$\mathbf{j}(\mathbf{r}, t) = \sum_i q_i\,\mathbf{v}_i\,\delta(\mathbf{r} - \mathbf{r}_i), \tag{11.3}$$

it is easy to verify that Eq. (11.1) has the equivalent expression

$$\mathbf{F} = \int dV\,(\rho\mathbf{E} + \mathbf{j} \times \mathbf{B}), \quad \text{here written in } \textit{continuum} \text{ form.} \tag{11.4}$$

From Newton's laws of motion, this force must equal the rate of change of the total mechanical momentum $\mathbf{P}_{\text{mech}}$ in the volume, because it is simply the sum of the change in the momentum of each particle

$$\frac{d}{dt}\mathbf{P}_{\text{mech}} = \mathbf{F} = \int dV\,(\rho\mathbf{E} + \mathbf{j}\times\mathbf{B}). \tag{11.5}$$

Both of these two expressions (11.4) and (11.5) contain field aspects (through $\mathbf{E}$ and $\mathbf{B}$) and matter aspects (through particle coordinates and velocities or charge and current densities).

The two inhomogeneous vacuum Maxwell's equations

$$\rho = \epsilon_0\,\nabla\cdot\mathbf{E}, \tag{11.6}$$

$$\mathbf{j} = \frac{1}{\mu_0}\nabla\times\mathbf{B} - \epsilon_0\,\frac{\partial\mathbf{E}}{\partial t}, \tag{11.7}$$

will now be used to eliminate the matter aspects entirely from Eq. (11.5); direct substitution immediately gives

$$\frac{d}{dt}\mathbf{P}_{\text{mech}} = \int dV\left[\epsilon_0\,(\nabla\cdot\mathbf{E})\,\mathbf{E} + \frac{1}{\mu_0}\,(\nabla\times\mathbf{B})\times\mathbf{B} - \epsilon_0\,\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}\right]. \tag{11.8}$$

Now perform the following steps:

(1) Integrate the last term by parts with the obvious relation

$$-\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B} = -\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) + \mathbf{E}\times\frac{\partial\mathbf{B}}{\partial t}; \tag{11.9a}$$

(2) Use Faraday's law

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \tag{11.9b}$$

in the last term of Eq. (11.9a), so that Eq. (11.8) becomes

$$\frac{d}{dt}\mathbf{P}_{\text{mech}} = \int dV\left[\epsilon_0\,(\nabla\cdot\mathbf{E})\,\mathbf{E} + \frac{1}{\mu_0}\,(\nabla\times\mathbf{B})\times\mathbf{B} - \epsilon_0\,\mathbf{E}\times(\nabla\times\mathbf{E}) - \frac{\partial}{\partial t}\,(\epsilon_0\mathbf{E}\times\mathbf{B})\right]. \tag{11.10}$$

(3) Move the last term to the left-hand side and add a null term involving $\nabla\cdot\mathbf{B} = 0$ to the integrand on the right-hand side; the result is

$$\frac{d}{dt}\left[\mathbf{P}_{\text{mech}} + \int dV\,\epsilon_0\,\mathbf{E}\times\mathbf{B}\right] = \int dV\left[\epsilon_0\,(\nabla\cdot\mathbf{E})\,\mathbf{E} - \epsilon_0\,\mathbf{E}\times(\nabla\times\mathbf{E}) + \frac{1}{\mu_0}\,(\nabla\cdot\mathbf{B})\,\mathbf{B} - \frac{1}{\mu_0}\,\mathbf{B}\times(\nabla\times\mathbf{B})\right]. \tag{11.11}$$

Note that all *four* of Maxwell's equations have now been used to transform the original Eq. (11.5) into Eq. (11.11).

The quantity

$$\mathbf{g} \equiv \epsilon_0\, \mathbf{E} \times \mathbf{B} \tag{11.12}$$

is interpreted as the local "momentum density" of the electromagnetic fields in vacuum—why is this so? A heuristic but plausible argument starts from the vacuum form of the energy flux vector $\mathbf{j}_E = \mathbf{E} \times \mathbf{B}/\mu_0$ in Eq. (9.16), which characterizes the flow of electromagnetic energy in the direction of the vector $\mathbf{E} \times \mathbf{B}$ at the vacuum speed of light c. Consider a region where $\mathbf{E} \times \mathbf{B}$ is nonzero, and construct a small right circular cylinder with its axis oriented along the direction of $\mathbf{E} \times \mathbf{B}$, as shown in Fig. 11.1. Let dA be the area of the base, with its perpendicular length $c\,dt$; evidently, the volume of the cylinder is $c\,dA\,dt$. In an interval of time dt, all the energy originally in cylinder flows out through the end, because dt is the time it takes the energy to travel the length $c\,dt$, and no energy is lost through other processes. Thus the *total energy* inside the cylinder at time t is simply $\mathbf{j}_E \cdot d\mathbf{A}\,dt = \mu_0^{-1}\,|\mathbf{E} \times \mathbf{B}|\,dA\,dt$. Correspondingly, the *energy density* in the cylinder becomes

$$\text{energy density} = \frac{\text{total energy}}{\text{volume}} = \frac{\mu_0^{-1}|\mathbf{E} \times \mathbf{B}|\,dA\,dt}{c\,dA\,dt} = \frac{1}{\mu_0\,c}\,|\mathbf{E} \times \mathbf{B}|. \tag{11.13}$$

To proceed, although the following discussion lies outside the strict bounds of classical electromagnetism, think of the electromagnetic field as a set of "photons," each with an energy $E = \hbar\omega = h\nu$ determined through Planck's celebrated relation, where ω is the angular frequency (units of radians/sec), ν is the conventional frequency (units of cycles/sec, or Hz) and $\hbar \equiv h/2\pi \approx 1.055 \times 10^{-34}$ J s, with h the conventional Planck's constant. Photons are known to have zero mass and to travel in vacuum at the speed of light c; for *any* massless particle, special relativity shows that the associated momentum is simply $1/c$ times its energy. Consequently, Eq. (11.13) immediately yields the magnitude of the momentum density $\mathbf{g} = \mathbf{E} \times \mathbf{B}/\mu_0 c^2$ associated with the energy density of the photons that make up the electromagnetic field, and the direction of the momentum density in vacuum is necessarily the same as that for the energy flux $\mathbf{j}_E$. Equation (9.16) and the basic relation $\epsilon_0\mu_0 c^2 = 1$ then yield the equivalent expressions

$$\mathbf{g} = \frac{1}{c^2}\,\mathbf{j}_E = \epsilon_0\,\mathbf{E} \times \mathbf{B}, \tag{11.14}$$

which is just the expression in Eq. (11.12).

The total electromagnetic momentum in some finite volume V is simply the volume integral of the electromagnetic momentum density

$$\mathbf{P}_{\text{em}} = \int dV\,\mathbf{g}. \tag{11.15}$$

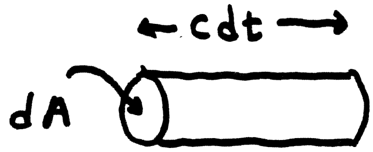

Fig. 11.1. Infinitesimal right circular cylinder with basal area dA and length $c\,dt$ used to study the energy flux and momentum density associated with electromagnetic fields in free space.

As a result, Eq. (11.11) takes the form

$$\frac{d}{dt}(\mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{em}}) = \text{rate of increase of total momentum in } V$$
$$\text{(including both mechanical and electromagnetic parts)}$$
$$= -\int dV \Big[\epsilon_0\, \mathbf{E} \times (\nabla \times \mathbf{E}) - \epsilon_0\, \mathbf{E}(\nabla \cdot \mathbf{E})$$
$$+ \frac{1}{\mu_0} \mathbf{B} \times (\nabla \times \mathbf{B}) - \frac{1}{\mu_0} \mathbf{B}(\nabla \cdot \mathbf{B})\Big]. \qquad (11.16)$$

Although it is not immediately obvious, it turns out that the ith component of the right-hand side of this vector equation can be rewritten as a surface integral $-\int dS_j\, T_{ij}$, with T_{ij} a second-rank tensor. Temporarily accepting this assertion, Eq. (11.16) then becomes

$$\frac{d}{dt}(\mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{em}})_i = -\int dS_j\, T_{ij}, \quad \text{using the familiar Einstein convention,} \qquad (11.17)$$

where T_{ij} is known as the "electromagnetic stress tensor." A simple rearrangement gives

$$\frac{d}{dt}(\mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{em}})_i + \int dS_j\, T_{ij} = 0, \qquad (11.18)$$

which is the statement of the conservation of the ith component of total linear momentum in some volume V. For comparison, recall the conservation of total charge Q; Sec. 8 showed that $dQ/dt + \int dS_j\, j_j = 0$, where j_j is the flux of charge in direction $\hat{r}_j$ (equivalently, the jth component of the current density). Apart from the extra cartesian index i, they are very similar, with the replacements $Q \to P_i$ and $j_j \to T_{ij}$. Evidently, T_{ij} has the interpretation:

$$T_{ij} = \text{the flux of the } i\text{th component of}$$
$$\text{total momentum density in the direction } \hat{r}_j, \qquad (11.19)$$

and $dS_j\, T_{ij}$ is the flux of the ith component of momentum through the oriented area $d\mathbf{S}$.

The proof that the right-hand side of Eq. (11.16) can be transformed into a surface integral is somewhat tedious, but not subtle, and it is easiest to proceed in steps. Since the electric and magnetic fields enter wholly symmetrically, it is sufficient to treat only (say) the terms involving **E**.

(1) Consider the ith component of the first term

$$[\mathbf{E} \times (\nabla \times \mathbf{E})]_i = \epsilon_{ijk}\, E_j\, \epsilon_{klm}\, \partial_l E_m$$
$$= E_j\, \partial_i E_j - E_j\, \partial_j E_i \quad \text{using Eq. (8.20)}$$
$$= \tfrac{1}{2}\, \partial_i (E_j E_j) - E_j\, \partial_j E_i, \qquad (11.20a)$$

or, returning to the more familiar vector notation,

$$\mathbf{E} \times (\nabla \times \mathbf{E}) = \tfrac{1}{2}\,\nabla E^2 - (\mathbf{E}\cdot\nabla)\,\mathbf{E} \quad [\text{note that } (\mathbf{E}\cdot\nabla)\mathbf{E} \neq \mathbf{E}(\nabla\cdot\mathbf{E})]. \tag{11.20b}$$

(2) The full electric contribution to Eq. (11.16) has the following ith component:

$$\begin{aligned}\big[\mathbf{E}\times(\nabla\times\mathbf{E}) - \mathbf{E}(\nabla\cdot\mathbf{E})\big]_i &= \tfrac{1}{2}\,\partial_i E^2 - E_j\,\partial_j E_i - E_i\,\partial_j E_j \\ &= \tfrac{1}{2}\,\partial_i E^2 - \partial_j\,(E_i E_j) \\ &= \partial_j(\tfrac{1}{2}E^2\delta_{ij} - E_i E_j),\end{aligned} \tag{11.21}$$

which is indeed the divergence of a symmetric second-rank tensor. Evidently, the terms involving $\mathbf{B}$ have an identical structure.

(3) The integral on the right-hand side of Eq. (11.16) thus has the form

$$-\int dV\,\partial_j T_{ij} = -\int dS_j\,T_{ij}, \tag{11.22}$$

where the right-hand side follows from the divergence theorem (the additional free index i plays no direct role here), and the electromagnetic stress tensor T_{ij} is given explicitly by

$$T_{ij} = \tfrac{1}{2}\,\delta_{ij}\left(\epsilon_0\,E^2 + \frac{1}{\mu_0}\,B^2\right) - \epsilon_0\,E_i E_j - \frac{1}{\mu_0}\,B_i B_j, \tag{11.23a}$$

or, equivalently

$$T_{ij} = \tfrac{1}{2}\,\delta_{ij}\epsilon_0\,(E^2 + c^2\,B^2) - \epsilon_0\,(E_i E_j + c^2\,B_i B_j). \tag{11.23b}$$

(4) This fundamental expression merits several comments.

(a) It is obvious by inspection that

$$T_{ij} = T_{ji}\,; \quad \text{the electromagnetic stress tensor } T_{ij} \text{ is symmetric.} \tag{11.24}$$

(b) To emphasize the remark after Eq. (11.19),

$dS_j\,T_{ij}$ is the flux of the ith component of momentum across the oriented area $d\mathbf{S} = dS\,\hat{n}$, along the direction $\hat{n}$. (11.25)

(c) If the total electromagnetic momentum $\mathbf{P}_{\text{em}}$ does not change in time, then Eq. (11.17) provides an *explicit* expression for the ith component of the mechanical force on the volume V surrounded by the surface S:

$$\frac{d}{dt}\,(P_{\text{mech}})_i = F_i^{\text{mech}} = -\int_S dS_j\,T_{ij}, \tag{11.26}$$

which accounts for the name "stress tensor;" remarkably, *the force on the charged particles in the interior of V is given here by a surface integral that can lie wholly in the surrounding vacuum.* If $d\mathbf{P}_{\text{em}}/dt \neq 0$, then this contribution must also be included.

(d) The signs in Eqs. (11.18) and (11.23) are chosen specifically to emphasize the parallelism with charge conservation. Unfortunately, the commonly defined "Maxwell stress tensor" $T_{ij}^{\text{Max}} \equiv -T_{ij}$ has the *opposite* sign from that in Eq. (11.23), so that it is essential to be careful in using results from other sources [see, for example, Jackson, Sec. 6.8; see, also L&L, Media, footnote to Sec. 5, where Maxwell's original convention is followed, and L&L, Classical Theory of Fields, Sec. 33, which instead uses the convention of Eq. (11.23)].

(e) The stress tensor plays a central role in many formulations of electromagnetism, especially in connection with special relativity (see Chap. 10), where T_{ij} will ultimately be recognized as the spatial part of the relativistic 4×4 energy-momentum tensor. The concept of the stress tensor is also essential in understanding much of fluid mechanics (especially compressible viscous fluid) and elasticity (see, for example, FW, Mechanics, Chaps. 9, 12, and 13), as well as magnetohydrodynamics (see Chap. 6 for a derivation of the corresponding stress tensor for nonviscous fluids). Finally, in general relativity, the 4×4 energy-momentum tensor serves as the source of the gravitational metric, in close analogy to the role played by the charge and current density in electromagnetism.

(f) To be very explicit, the electromagnetic stress tensor has the typical diagonal and off-diagonal elements

$$\begin{aligned} T_{xx} &= \tfrac{1}{2}\left(\epsilon_0\, E^2 + \frac{1}{\mu_0}\, B^2\right) - \epsilon_0\, E_x^2 - \frac{1}{\mu_0}\, B_x^2 \\ &= \tfrac{1}{2}\left[\epsilon_0(E_y^2 + E_z^2 - E_x^2) + \frac{1}{\mu_0}\,(B_y^2 + B_z^2 - B_x^2)\right] \\ &= \tfrac{1}{2}\epsilon_0\,[(E_y^2 + E_z^2 - E_x^2) + c^2\,(B_y^2 + B_z^2 - B_x^2)], \\ T_{xy} &= -\epsilon_0\, E_x E_y - \frac{1}{\mu_0}\, B_x B_y = -\epsilon_0(E_x E_y + c^2\, B_x B_y). \end{aligned} \tag{11.27}$$

Equation (11.18) constitutes the integral formulation of the conservation of linear momentum, and it is often valuable to consider the alternative differential form. Let $\mathbf{f}^{\text{mech}}$ be the mechanical force density; in the present case, Eq. (11.4) shows that this quantity is simply the Lorentz force density

$$\mathbf{f}^{\text{mech}} = \rho\mathbf{E} + \mathbf{j} \times \mathbf{B}. \tag{11.28}$$

Use of Eqs. (11.14), (11.15), and (11.18) shows that the integral formulation can be written as

$$\int dV \left(f_i^{\rm mech} + \frac{\partial g_i}{\partial t} + \partial_j T_{ij} \right) = 0. \tag{11.29}$$

Just as in the derivation of the conservation law for charge density in Eq. (8.10), this null integral vanishes for an arbitrary volume V, no matter how small, and it follows that the integrand itself must vanish, leading to the differential conservation law for linear momentum

$$f_i^{\rm mech} + \frac{\partial g_i}{\partial t} + \partial_j T_{ij} = 0, \tag{11.30}$$

which is analogous to $\partial\rho/\partial t + \nabla\cdot\mathbf{j} = 0$. Note that Eq. (11.30) now involves the "divergence of the electromagnetic stress tensor" (which is itself a vector), in contrast to the surface integral of T_{ij} that appears in Eq. (11.18). To repeat in words:

> The sum of: (1) the mechanical force density (namely,
> the rate of change of mechanical momentum density),
> (2) the rate of change of electromagnetic momentum density, and
> (3) the divergence of the electromagnetic stress tensor
> necessarily vanishes identically at every point of the system. (11.31)

Alternatively, the quantity $-\partial_j T_{ij}$ at the point $\mathbf{r}$ is the rate at which the ith component of total momentum density increases from local inflow through an infinitesimal surrounding surface.

It is instructive to consider a very simple example—a plane perfect conductor occupies the halfspace $(z < 0)$, with an external electrostatic field $\mathbf{E}$ (see Fig. 11.2). Near the surface, the electric field necessarily lies along the normal $\hat{n} = \hat{z}$ to the plane surface, so that $\mathbf{E} = E\,\hat{z} = E_z\,\hat{z}$, and $E_x = E_y = 0$. The electromagnetic stress tensor has the simple diagonal form

$$T = \tfrac{1}{2}\,\epsilon_0 \begin{pmatrix} E_z^2 & 0 & 0 \\ 0 & E_z^2 & 0 \\ 0 & 0 & -E_z^2 \end{pmatrix}. \tag{11.32}$$

Thus, T has two positive elements

$$T_{xx} = T_{yy} = \tfrac{1}{2}\,\epsilon_0\,E_z^2 > 0, \tag{11.33a}$$

and one negative element

$$T_{zz} = -\tfrac{1}{2}\,\epsilon_0\,E_z^2 < 0; \tag{11.33b}$$

both of these expressions are independent of the sign of E_z. Furthermore, since $\mathbf{B} = \mathbf{0}$, the electromagnetic momentum density $\mathbf{g} = \epsilon_0\,\mathbf{E}\times\mathbf{B}$ vanishes identically and plays no role here.

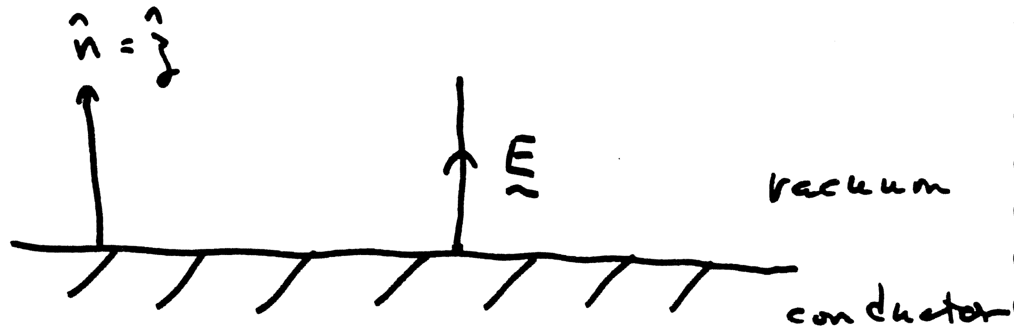

Fig. 11.2. Perfect conductor occupying the region $z < 0$, with an external electrostatic field $\mathbf{E}$.

The ith component of the total mechanical force on the conductor is simply

$$F_i = \int dV\, f_i = -\int dS_j\, T_{ij}, \tag{11.34}$$

where the volume integral is over the region $z < 0$ and the surface integral is over the plane $z = 0$. As always, the normal $\hat{n}$ points *away* from the region in question, with $d\mathbf{S} = dx\, dy\, \hat{z}$, and the total force becomes

$$F_i = -\int dx\, dy\, \hat{z}_j\, T_{ij} = -\int dx\, dy\, T_{iz}. \tag{11.35}$$

In the present case, the stress tensor T_{ij} is diagonal, so that $F_x = F_y = 0$ (as expected from the symmetry of the geometry), and

$$F_z = -\int dx\, dy\, T_{zz} = -\int dx\, dy\, \big(-\tfrac{1}{2}\epsilon_0 E_z^2\big) = \text{area} \times \tfrac{1}{2}\epsilon_0 E_z^2. \tag{11.36}$$

This quantity is positive, independent of the sign of the electric field; hence an electric field applied to the plane surface of a perfect conductor always *pulls* on the surface (this result is obvious from the picture of the lines of electric field as analogous to elastic fibers). The physically relevant quantity is the force per unit area, which is just

$$\frac{F}{\text{area}} = \tfrac{1}{2}\epsilon_0 E_z^2 = \tfrac{1}{2}\sigma E_z, \tag{11.37}$$

because Eq. (3.5) shows that $\sigma = \epsilon_0 E_z$ is the surface charge density. The factor of $\frac{1}{2}$ may initially be surprising, given that the force on an *external* point charge q at the surface would just be $qE_z\,\hat{z}$; here, however, the surface charge density also serves as the source of the electric field in the exterior (since E vanishes deep in the interior), and the factor of $\frac{1}{2}$ can be considered a screening correction that accounts for the continuous decrease of the electric field in passing from the vacuum through the surface layer to the interior of the conductor (see Purcell, Sec. 1.14 for a careful treatment). This simple example shows in detail how the stress tensor converts a volume effect into a surface one. It is somewhat analogous to Gauss's law, in which the total charge inside some surface can be determined from the electric field on a surrounding surface; here the validity of this equivalence depends on the explicit use of all four Maxwell's equations in the derivation of Eq. (11.17) from Eq. (11.5).

Conservation of angular momentum

The stress tensor facilitates the analogous discussion of the conservation of angular momentum and the associated mechanical and electromagnetic torque; most of the analysis is very similar, so that the treatment will be quite brief. For the same volume V, consider the total mechanical *torque* on the particles inside

$$\begin{aligned}\mathbf{\Gamma}^{\text{mech}} &= \int dV\, \mathbf{r} \times \mathbf{f}^{\text{mech}} \\ &= \frac{d}{dt}\mathbf{L}^{\text{mech}},\end{aligned} \tag{11.38}$$

where $\mathbf{f}^{\rm mech}$ is the Lorentz force density from Eq. (11.28) and $\mathbf{L}^{\rm mech}$ is the total (mechanical) angular momentum of the particles contained in the volume V. Note that both $\mathbf{\Gamma}$ and $\mathbf{L}$ depend on the choice of origin of coordinates.

Take the ith component of this equation and use Eq. (11.30)

$$\frac{d\,L_i^{\rm mech}}{d\,t} = \epsilon_{ijk}\int dV\,r_j\,f_k^{\rm mech} = \epsilon_{ijk}\int dV\,r_j\left(-\frac{\partial g_k}{\partial t} - \partial_l\,T_{kl}\right). \tag{11.39}$$

The electromagnetic angular momentum $\mathbf{L}^{\rm em}$ is the volume integral of $\mathbf{r}\times\mathbf{g}$, so that

$$\frac{d}{d\,t}\left(L_i^{\rm mech} + L_i^{\rm em}\right) = -\epsilon_{ijk}\int dV\,r_j\,\partial_l\,T_{kl} = -\epsilon_{ijk}\int dV\left[\partial_l\left(r_j\,T_{kl}\right) - T_{kl}\,\partial_l r_j\right]. \tag{11.40}$$

The last term is zero because $\partial_l\,r_j = \delta_{jl}$ and T is symmetric (hence this term involves $\epsilon_{ijk}\,T_{kj}$, which vanishes by symmetry); as a result, the right-hand side is again the volume integral of a "vector divergence." Define the flux of angular-momentum density (equivalently, the angular-momentum current density)

$$M_{il} \equiv \epsilon_{ijk}\,r_j\,T_{kl}\,; \tag{11.41}$$

the right-hand side of Eq. (11.40) can then be written $-\int dV\partial_l\,M_{il} = -\int dS_l\,M_{il}$, where the last form follows from the divergence theorem. Evidently, $dS_l\,M_{il}$ is the flux of the ith component of angular momentum through the oriented surface $d\mathbf{S} = dS\,\hat{n}$ and $-\int dS_l\,M_{il}$ is the net influx of the ith component of angular momentum into the volume V through the surrounding surface. In this way, Eq. (11.40) yields the integral form of the conservation of total angular momentum

$$\frac{d}{d\,t}\left(L_i^{\rm mech} + L_i^{\rm em}\right) + \int dS_l\,M_{il} = 0. \tag{11.42}$$

The corresponding local differential expression follows from the integrand of this relation

$$\left(\mathbf{r}\times\mathbf{f}^{\rm mech}\right)_i + \frac{\partial}{\partial t}\left(\mathbf{r}\times\mathbf{g}\right)_i + \partial_l\,M_{il} = 0. \tag{11.43}$$

Prob 5.? considers an application of these relations to a long charged rod placed inside a solenoid. where the electromagnetic momentum density and electromagnetic angular-momentum density play an essential role (see Feynman, Vol. II, Secs. 17-4 and 27-6, for a few elementary and brief, but very relevant, remarks).