5.2 SIMPLIFY NOTATION, LET $\Upsilon = \frac{2}{3(1+w)}$

(5.51) $1+z = \left(\frac{t_0}{t_e}\right)^{\Upsilon} \longrightarrow \frac{dt_e}{dt_0} = (1+z)^{-1/\Upsilon}$

(5.48) $t_0 = \Upsilon / H_0$

$$\Rightarrow \frac{dz}{dt_0} = \Upsilon t_0^{\Upsilon-1} t_e^{-\Upsilon} - \Upsilon t_e^{-\Upsilon-1} t_0^{\Upsilon} \frac{dt_e}{dt_0}$$

$$= H_0 (t_0/t_e)^{\Upsilon} - \Upsilon\left(\frac{1}{t_e}\right)(t_0/t_e)^{\Upsilon}(1+z)^{-1/\Upsilon}$$

(5.52) $t_e = \frac{\Upsilon}{H_0}(1+z)^{-1/\Upsilon}$

$$= H_0(1+z) - \Upsilon\left\{\frac{H_0}{\Upsilon}(1+z)^{1/\Upsilon}\right\}(1+z)(1+z)^{-1/\Upsilon}$$

$$= H_0(1+z) - H_0(1+z) \qquad = 0 \,?$$

(!!) SORRY, I COULDN'T FIGURE THIS ONE OUT!

this would work if $t_e = \frac{\Upsilon}{H_0}(1+z)^{-1}$ BUT THIS DIFFERS FROM eq. (5-52)!!

(I'LL LOOK INTO THIS. - flip)

SUPPOSE, AS RYDER SAYS,

$$\frac{dz}{dt_0} = H_0(1+z) - H_0(1+z)^{3(1+w)/2}$$

then REDSHIFT DECREASES w/ TIME FOR

$$3(1+w)/2 > 1$$
$$3 + 3w > 2$$
$$\boxed{w > -1/3}$$

INCREASES w/ TIME FOR $\boxed{w < -1/3}$

5.3

(5.81) $$\frac{dz}{dt_0} = H_0(1+z) - H_0(1+z)^{3(1+w)/2}$$

MATTER ONLY $\Rightarrow$ $w=0$

formally, we want

CHOSE SIGN OF 10^{-6} s.t. $\Delta t_0 > 0$

$$\Delta t_0 = \frac{1}{H_0}\int_1^{1-10^{-6}} \frac{dz}{(1+z)-(1+z)^{3/2}}$$

but we can just approximate this as

$$\Delta t_0 \approx \left(\frac{dz}{dt_0}\right)\Big|_{z=1} \Delta z$$

$$= 5.31 \times 10^{11} \text{ sec}$$
$$= 16{,}800 \text{ years}$$

(that's a long time for a small redshift!)

5.4

RADIATION ONLY :

(5.65) $$d_p(t_0) = \frac{c}{H_0}\frac{z}{1+z}$$

(5.66) $$d_p(t_e) = \frac{c}{H_0}\frac{z}{(1+z)^2}$$

MATTER ONLY :

(5.60) $$d_p(t_0) = \frac{2c}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right]$$

(5.61) $$d_p(t_e) = \frac{2c}{H_0(1+z)}\left[1 - \frac{1}{\sqrt{1+z}}\right]$$

Λ ONLY :

(5.79) $$d_p(t_0) = \frac{c}{H_0} z$$

(5.80) $$d_p(t_e) = \frac{c}{H_0}\frac{z}{1+z}$$

6.1 MATTER ONLY, $\Omega_0 > 1$, $K = +1$

(6.16) $$H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_0/a + (1-\Omega_0)}}$$

$a_0 = 1$ BY OUR CHOICE OF NORMALIZATION

SOLUTION IS PARAMETERIZED BY (6.17) & (6.18)

$$\begin{cases} (1) & a(\theta) = \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}(1-\cos\theta) \\ (2) & t(\theta) = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}(\theta - \sin\theta) \end{cases}$$

$\theta \in (0, 2\pi)$

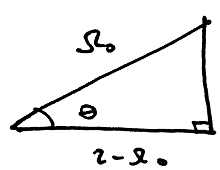
USE (1) TO DETERMINE θ

$$a_0 = 1 = \frac{1}{2}\frac{\Omega_0}{\Omega_0-1}(1-\cos\theta)$$

$$\Rightarrow \frac{\Omega_0}{2(\Omega_0-1)}\cos\theta = \frac{\Omega_0}{2(\Omega_0-1)} - 1$$

$$\cos\theta = 1 - \frac{2\Omega_0 - 2}{\Omega_0}$$

$$\boxed{\cos\theta = \frac{2-\Omega_0}{\Omega_0}}$$



$\leftarrow 2 \Rightarrow \sqrt{\Omega_0^2 - (2-\Omega_0)^2} = \sqrt{4\Omega_0 - 4}$

DETERMINE $\sin\theta$

$$\sin\theta = \frac{1}{\Omega_0}\cdot 2\sqrt{\Omega_0 - 1}$$

PLUG INTO (2)

$$H_0 t_0 = \frac{1}{2}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}\left\{\cos^{-1}\left(\frac{2-\Omega_0}{\Omega_0}\right) - \frac{2}{\Omega}\sqrt{\Omega_0 - 1}\right\}$$

$$\boxed{H_0 t_0 = \frac{\Omega_0}{2(\Omega_0-1)^{3/2}}\cos^{-1}\left(\frac{2-\Omega_0}{\Omega_0}\right) - \frac{1}{\Omega_0 - 1}}$$

6.3 (5.9) $\varepsilon_w(a) = \varepsilon_{w,0}\, a^{-3(1+w)}$

$$\Omega_w \doteq \frac{\varepsilon_w(a)}{\varepsilon_c}$$

EQUALITY WHEN $1 = \frac{\Omega_Q}{\Omega_m}$

$$= \frac{\Omega_{Q,0}\, a^{-3/2}}{\Omega_{m,0}\, a^{-3}}$$

$$= \frac{\Omega_{Q,0}}{\Omega_{m,0}}\, a^{3/2}$$

ie. $a_{mQ} = \left(\frac{\Omega_{m,0}}{\Omega_{Q,0}}\right)^{2/3}$

$$\boxed{a_{mQ} = \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{2/3}}$$

THE FRIEDMANN EQUATION (6.6) TAKES THE FORM

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{Q,0}}{a^{3/2}}$$

FOLLOWING THE STEPS ON P. 83

$$\frac{\dot{a}^2}{H_0^2} = \frac{\Omega_{m,0}}{a} + \Omega_{Q,0}\, a^{1/2}$$

$$\boxed{H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_{m,0}/a + \Omega_{Q,0}\, a^{1/2}}}} \quad \Omega_{Q,0} = 1 - \Omega_{m,0}$$

IF YOU ARE ENTERPRISING YOU CAN SOLVE THIS (PERHAPS W/ MATHEMATICA) BUT ALL OF THE <u>PHYSICAL</u> THINKING WAS UP TO HERE.

FOR $a \ll a_{mQ}$, $(\Omega_{m,0}/a) \gg (\Omega_{Q,0}\, a^{1/2})$

$$\Rightarrow H_0 t \approx \int_0^a da \sqrt{\frac{a}{\Omega_{m,0}}} = \frac{2}{3}(\Omega_{m,0})^{-1/2} a^{3/2}$$

$$\boxed{a(t) \approx \left(\frac{9}{4} H_0^2\, \Omega_{m,0}\, t^2\right)^{1/3}}$$

FOR $a \gg a_{mQ}$, $(\Omega_{Q,0}\, a^{1/2}) \gg (\Omega_{m,0}/a)$

$$\Rightarrow H_0 t \approx \int_0^a da\, (\Omega_{Q,0}\, a^{1/2})^{-1/2} = \frac{4}{3}(1-\Omega_{m,0})^{-1/2} a^{3/4}$$

$$\boxed{a(t) \approx \left(\frac{3}{4} H_0 (1-\Omega_{m,0})^{1/2} t\right)^{4/3}}$$

6.4 STATIC UNIVERSE : $\dot{a}=0, \ddot{a}=0$

ACCELERATION EQUATION (4.44) $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P)$

EQUATION OF STATE (4.50) $P = w\varepsilon$

WANT Q TO BE REPULSIVE ⇒ $\ddot{a} > 0$
SINCE $a > 0$, this means:

$(\varepsilon + 3P) < 0$

$\Rightarrow \quad 1 + 3w_Q < 0$

$\boxed{w_Q < -\frac{1}{3}}$

NOW INCLUDE MATTER AND IMPOSE $\ddot{a} = 0$

$\Rightarrow \quad 0 = \varepsilon_Q + \varepsilon_M + 3w_Q\varepsilon_Q$

$\Rightarrow \quad \boxed{\varepsilon_M = -(1+3w_Q)\varepsilon_Q}$

NOW USE FRIEDMANN EQUATION (4.13) w/ $\dot{a}=0$

$$0 = \frac{8\pi G}{3c^2}(\varepsilon_M + \varepsilon_Q) - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2}$$

$$\kappa = (+\text{ number})(\varepsilon_M + \varepsilon_Q)$$
$$= (\text{ —''— })(-3w_Q\varepsilon_Q)$$

SINCE $w_Q < -\frac{1}{3}$ ⇒ $\boxed{\kappa \text{ IS POSITIVE}}$

set $\kappa = +1$

$$\frac{c^2}{R_0^2 a^2} = -\frac{8\pi G}{3c^2}(3w_Q\varepsilon_Q) \quad \Big| \quad a_0 = 1$$

$$\boxed{R_0^2 = \frac{-c^4}{8\pi G w_Q \varepsilon_Q}}$$

6.6 $\Omega_0 = \Omega_{\Lambda,0} > 1$, $\dot{a} > 0$, $K = +1$ } ASSUMPTIONS

USE eq. (6.34), FRIEDMANN EQ. FOR MATTER + CURVATURE + Λ AND SET $\Omega_{m,0} = 0$

$$\frac{H^2}{H_0^2} = \frac{1-\Omega_{\Lambda,0}}{a^2} + \Omega_{\Lambda,0}$$

$H = \dot{a}/a$, SO THERE IS A CRITICAL POINT WHEN $H=0$

$$\Rightarrow \boxed{a_{crit} = \sqrt{\frac{\Omega_0 - 1}{\Omega_0}}}$$

is THIS A "BOUNCE POINT"?
THE ACCELERATION EQUATION (4.44) TELLS US:

$$\frac{\ddot{a}}{a} = -(\text{POSITIVE } \#)(\varepsilon + 3P)$$

for Λ, HOWEVER, $P = -\varepsilon_\Lambda$ (4.66)
$\Rightarrow \ddot{a} > 0$

thus a is CONCAVE UP, & SO $a_{crit} = a_{bounce}$. ✓

THE FRIEDMANN EQ GIVES US

$$H_0(t - t_{bounce}) = \int_{a_{bounce}}^{a} \frac{da}{\sqrt{a^2\Omega_0 + (1-\Omega_0)}} = \frac{1}{\sqrt{\Omega_0}} \int_{a_b}^{a} (a^2 - a_b^2)^{-1/2}\, da$$

this is a tricky integral
... in particular, MATHEMATICA isn't helpful!

RECALL THAT $\frac{d}{dx}(\cosh^{-1} x) = (x^2-1)^{-1/2}$
SO LET'S MANIPULATE THE ABOVE INTEGRAL TO THIS FORM

$$H_0(t-t_b) = \frac{1}{\sqrt{\Omega_0}\, a_b} \int_{a_b}^{a} \left((a/a_b)^2 - 1\right)^{-1/2} da \qquad \text{LET } b = a/a_b$$

$$= \frac{a_b}{\sqrt{\Omega_0}\, a_b} \int_{1}^{a/a_b} (b^2-1)^{-1/2}\, db$$

$$= \frac{1}{\sqrt{\Omega_0}} \left[\cosh^{-1}(a/a_b) - \underbrace{\cancel{\cosh^{-1}(1)}}_{=0}\right]$$

$$\Rightarrow \cosh(a/a_b) = \sqrt{\Omega_0} H_0 (t - t_b)$$

$$\boxed{a(t) = a_{bounce} \cosh\left(\sqrt{\Omega_0} H_0 (t-t_b)\right)}$$

$$\boxed{t_0 - t_b = \frac{1}{H_0\sqrt{\Omega_0}} \cosh^{-1}\left(a(t_0)/a_b\right)}$$

6.8 FROM TABLE 6.2

$$\left.\begin{aligned}\Omega_{m,0} &= 0.3\\ \Omega_{\gamma,0} &= 5\times10^{-5}\\ \Omega_{bary,0} &= 0.04\end{aligned}\right\} \qquad \Omega_i = \frac{\varepsilon_i}{\varepsilon_c}$$

(6.42) $d_{horizon}(t_0) = 14{,}000$ Mpc

$$\varepsilon_{i,0} = \varepsilon_{c,0}\,\Omega_{i,0} \qquad\qquad \varepsilon_c = \frac{3c^2}{8\pi G}H(t)^2$$

$$\Rightarrow \varepsilon_{m,0} = \frac{3c^2}{8\pi G}H_0^2\,\Omega_{m,0}$$

total MASS $= \varepsilon_{m,0} \times$ VOLUME

$= \varepsilon_{m,0} \times \left(\frac{4}{3}\pi\,(d_{horizon}(t_0))^3\right)$

$= \frac{3c^2}{8\pi G}H_0^2\,\Omega_{m,0}\cdot\frac{4}{3}\pi\,d_h(t_0)^3$

$\approx \boxed{9\times10^{53}\ \text{kg}}$

$\Rightarrow$ total RADIATION $= \varepsilon_{\gamma,0} \times$ VOLUME

$= \frac{3c^2}{8\pi G}H_0^2\,\Omega_{\gamma,0}\cdot\frac{4}{3}\pi\,d_h(t_0)^3$

$\approx \boxed{1\times10^{69}\ \text{J}}$

$\Rightarrow$ ASSUME $m_{baryon} = m_{proton}$

total BARYONIC MASS $= \varepsilon_{b,0} \times$ VOLUME

$= \frac{3c^2}{8\pi G}H_0^2\,\Omega_{bary,0}\times\frac{4}{3}\pi\,d_h(t_0)^3$

number of baryons $=$ (total baryonic mass) $/\,m_p$

$\approx \boxed{7\times10^{79}\ \text{BARYONS}}$

LAST PROBLEM

$$\begin{cases} \Omega_{m,0} = .26 \\ \Omega_{\Lambda,0} = .74 \\ H_0 = 70 \text{ km/s}\cdot\text{Mpc} \end{cases}$$

a) CONDITION FOR m-Λ EQUALITY:

$$\Omega_{m,0}\, a_{m\Lambda}^{-3} = \Omega_{\Lambda,0}$$

$$\Rightarrow \quad a_{m\Lambda} = \left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)^{1/3} \approx .71$$

$$z_{m\Lambda} = a_{m\Lambda}^{-1} - 1 = \boxed{0.42}$$

b) IN THE EXPONENTIAL EXPANSION ERA, Ω_m IS NEGLIGIBLE SO FRIEDMANN EQ:

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon_\Lambda$$

$$\dot a = \underbrace{\sqrt{\frac{8\pi G}{3c^2}\varepsilon_\Lambda}}_{H_0 \text{ for } \Lambda\text{-only universe (eq. 5.77)}}\, a(t)$$

$$\Rightarrow a(t) = e^{H_0(t-t_0)}$$

e-folding time: $\boxed{\Delta t = \frac{1}{H_0} \approx 5.1\times10^{17}\ s}$

c) FRIEDMANN EQ: $$H_0 t = \int_0^a \frac{da}{[\Omega_{m,0}a^{-1} + \Omega_{\Lambda,0}a^2]^{1/2}}$$

$$\Rightarrow dt = H_0^{-1}\{\Omega_{m,0}a^{-1} + \Omega_{\Lambda,0}a^2\}^{-1/2}\, da$$

(5.35) $$d_{horizon}(t_0) = c\int_0^t dt\, a(t)^{-1}$$
$$= \frac{c}{H_0}\int_0^a \frac{da}{\sqrt{\Omega_{m,0}/a + \Omega_{\Lambda,0}/a^2}}$$

max @ $a \to \infty$
(can take derivative w/rt a to confirm, or just observe)

$$\boxed{d_{horiz,max} = \frac{c}{H_0}\int_0^\infty \frac{da}{\sqrt{\Omega_{m,0}/a + \Omega_{\Lambda,0}/a^2}} \approx 19.8\times10^3 \text{ Mpc}}$$

$$d_{horiz,present} = \frac{c}{H_0}\int_0^1 \left(\sqrt{\Omega_{m,0}/a + \Omega_{\Lambda,0}/a^2}\right)^{-1} da$$
$$\approx 15000 \text{ Mpc}$$