Lecture 8

- extensions of variational calculus
  - more dependent var's (DoF)
  - more independent var's (Continuum systems)
  - variations with constraints

When our system has more than one DoF, the Lagrangian has the form

$$L = L(q_1, q_2, \ldots, \dot{q}_1, \dot{q}_2, \ldots, t)$$

As with 1 DoF the action is just the time integral of L:

$$S[q_1(t), q_2(t), \ldots] = \int_0^T L\, dt$$

The variational derivatives are exactly as one would expect:

$$\frac{\delta S}{\delta q_k(t)} = \frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right), \quad k = 1, 2, \dots$$

So the condition that the action is extremal with respect to variations in <u>any</u> of the coordinates at any time, is just the set of E-L equations for the system.

Continuum systems are systems ~~with~~ whose degrees of freedom (huge in number!) are naturally expressed in terms of a continuous variable.

Example: elastic string

x = ~~the~~ continuum label of mass point

Here are expressions for the string's kinetic and potential energies:

$$T = \int \tfrac{1}{2}\dot{y}^2 \mu \, dx \qquad \mu = \text{mass per unit length}$$

$$V = \int \tfrac{1}{2} t \, y'^2 \, dx \qquad t = \text{string tension}$$

$$\frac{dy}{dx}$$

$$S[y(t,x)] = \int (T - V)\, dt$$

$$= \iint \underbrace{\left(\tfrac{1}{2}\mu \dot{y}^2 - \tfrac{1}{2} t \, y'^2\right)}_{L} dx \, dt$$

In the calculation of the variation in S we will need to perform an integration by parts in the x variable in addition to t:

$$\delta S = \iint \underbrace{\frac{\delta S}{\delta y(t,x)}}_{0} \cdot \delta y(t,x) \; dx \, dt$$

$$\frac{\delta S}{\delta y(t,x)} = \frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)$$

elastic string:

$$\frac{\partial L}{\partial y} = 0 \qquad \frac{\partial L}{\partial \dot{y}} = \mu \dot{y} \qquad \frac{\partial L}{\partial y'} = -t y'$$

$$\frac{\delta S}{\delta y} = 0 \implies -\mu \ddot{y} + t y'' = 0$$

$$\ddot{y} - (t/\mu)\, y'' = 0$$

wave equation, velocity $v = \sqrt{t/\mu}$.

Finding extrema subject to constraints

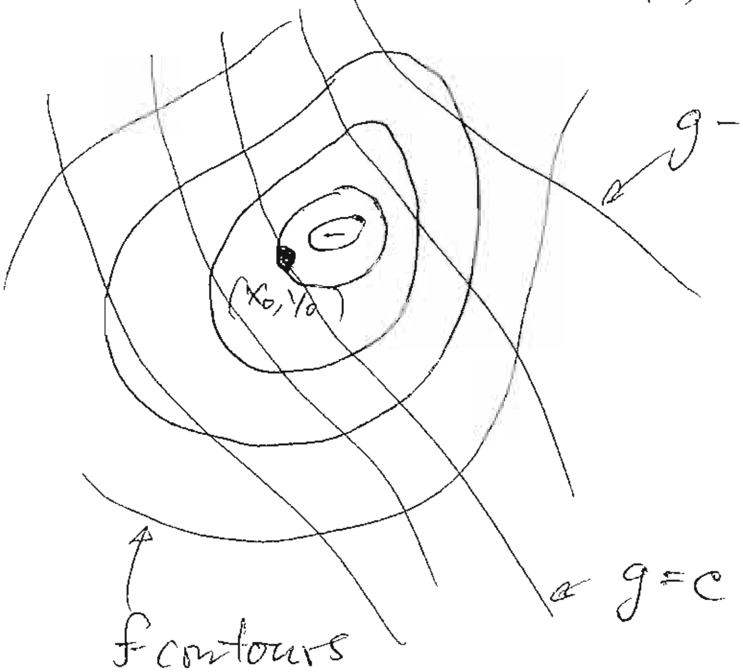
This is something that comes up in mechanics when we use Hamilton's principle to find the equations of motion but the Lagrangian contains more variables than the number of DoF because

- we are too lazy to use the minimum number of variables,
- the constraints are non-holonomic and this is not possible.

The method of Lagrange multipliers is a general calculus procedure for dealing with these situations

2 variable example:

minimize $f(x,y)$ such that $g(x,y)=c$



at minimum x_0, y_0, the gradient of f is parallel to gradient of g

parallel gradients: $\vec{\nabla} f = \lambda \vec{\nabla} g$
(2 equations)

constraint eqn: $g = c$
(1 equation)

3 equations, 3 unknowns (x, y, λ)

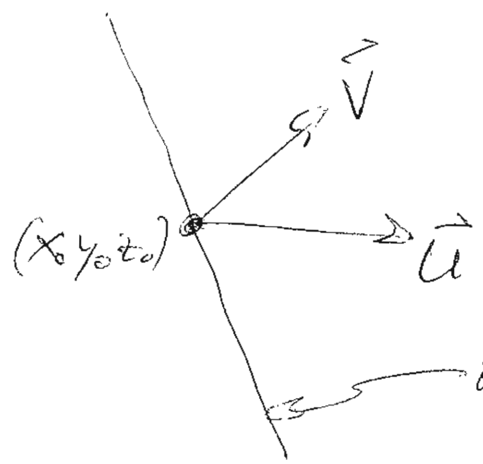
3 variables, 2 constraints:

min $f(x,y,z)$ such that $g(x,y,z)=C_1$

$h(x,y,z)=C_2$

Consider the gradients of the constraints at the solution point

$$\vec{u} = \vec{\nabla} g \Big|_{x_0 y_0 z_0} \qquad \vec{v} = \vec{\nabla} h \Big|_{x_0 y_0 z_0}$$



~~the~~ axis $\perp$ to both $\vec{u}$ & $\vec{v}$

f ~~may~~ better not change along this, otherwise $(x_0 y_0 z_0) \neq$ minimum

$\Rightarrow \vec{\nabla} f \Big|_{x_0 y_0 z_0}$ must be in plane spanned by $\vec{u}, \vec{v}$

$$\Rightarrow \vec{\nabla} f = \lambda_1 \vec{\nabla} g + \lambda_2 \vec{\nabla} h$$

gradient condition: 3 equations

constraints : 2 equations

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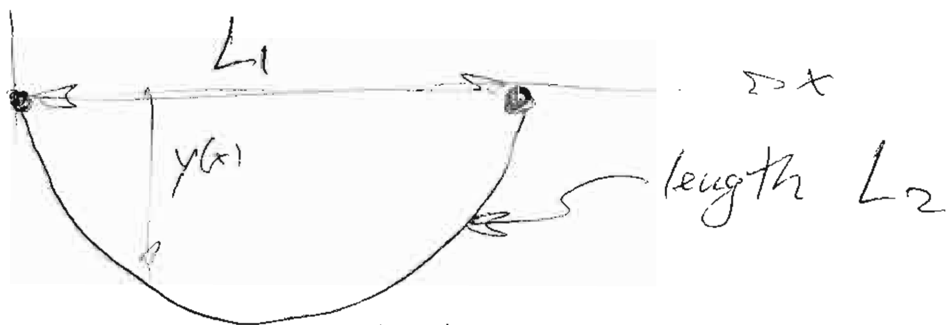
unknowns : $x, y, z, \lambda_1, \lambda_2$

Let's apply this approach to the hanging chain problem:



A fine chain of length L_2 is hung between two supports at the same height and horizontal separation L_1. For which shape $y(x)$ does the chain achieve the minimum gravitational energy?

We need: $\ell[y(x)]$ = length of chain (for constraint)

$V[y(x)]$ = potential energy (to be minimized)

$$\ell[y(x)] = \int \sqrt{dx^2+dy^2} = \int_0^{L_1} \sqrt{1+y'^2}\,dx$$

$$V[y(x)] = \int \underbrace{\mu\sqrt{dx^2+dy^2}}_{dm}\, g\, y = \int_0^{L_1} \mu g y \sqrt{1+y'^2}\,dx$$

μ = mass/length

gradient condition:

$$\frac{\delta V}{\delta y(x)} = \lambda \frac{\delta \ell}{\delta y(x)} \quad \text{or} \quad \frac{\delta}{\delta y(x)}\overbrace{\left(V-\lambda \ell\right)}^{L}=0$$

$$\left(\frac{\delta(V-\lambda\ell)}{\delta y(x)} = \mu g\sqrt{1+y'^2} - \frac{d}{dx}\left(\underbrace{\mu g y}_{(\mu g y-\lambda)}\frac{y'}{\sqrt{1+y'^2}}\right)\right)$$

This will be a second order diff. eqn. for $y(x)$. To get a 1st order eqn. think of $V - \lambda l$ as a "Lagrangian" and determine the "conserved" "Hamiltonian" (since x is absent):

$$H = y' \frac{\partial L}{\partial y'} - L$$

$$= y'(\mu g y - \lambda) \frac{y'}{\sqrt{1+y'^2}} - (\mu g y - \lambda)\sqrt{1+y'^2}$$

$$= E \text{ (constant)}$$

$$\Rightarrow (\mu g y - \lambda)(\cancel{y'^2} - (1 + \cancel{y'^2})) = E\sqrt{1+y'^2}$$

$$\Rightarrow \left(\frac{\mu g y - \lambda}{E}\right)^2 = 1 + y'^2$$

define: $\dfrac{\mu g y(x) - \lambda}{E} = \tilde{y}(\underbrace{\tfrac{\mu g}{E} x}_{\tilde{x}})$

$$\Longrightarrow \quad \frac{\mu g}{E} y' = \frac{\mu g}{E} \tilde{y}'$$

$$\Longrightarrow \quad \tilde{y}^2 - 1 = \tilde{y}'^2$$

$$\tilde{y} = ch(\tilde{x} - x_0) = ch\left(\frac{\mu g}{E} x - x_0\right)$$

$$= \frac{\mu g}{E} y(x) - \frac{\lambda}{E}$$

$$y(x) = \frac{\lambda}{\mu g} + \frac{E}{\mu g} ch\left(\frac{\mu g}{E} x - x_0\right)$$

determine λ, E, x_0 from three equations:

$$y(0) = Y_1$$

$$y(L_1) = Y_2$$

$$\int_0^{L_1} \sqrt{1 + y'^2}\, dx = L_2$$