P318 : SECTION 2

FEB 1

- TOTAL & PARTIAL DERIVATIVES
- CONFORMAL CORNELL COMMENTARY
- HW2, ELLIPTIC INTEGRAL
- INDICES
- FORCES THAT DEPEND ON VELOCITY
- E&M, GAUGE TRANSFORMATIONS, (mention tangent space)
  ↳ if time
- comment on holonomic vs. nonholonomic

⇒ any q. re: PROB 7?

<u>Remark for hw 2</u>

IN GENERAL, $T \sim \dot{q}^2$

SO EULER-LAGRANGE EQNS $\sim \quad \ddot{q} + f(q) = 0$

↳ in general, this is a hard 2ND O ODE

USE CONSERVATION LAWS!

$$E = T + V \sim (\dot{q})^2 + g(q) \Rightarrow \frac{dq}{dt} = \sqrt{-g(q)}$$

$$\Rightarrow \int_{t_0}^{t} d\tilde{t} = \int_{q(t_0)}^{q(t)} \frac{dq}{\sqrt{-g(q)}}$$

THIS IS STILL HARD TO INTEGRATE, BUT IT IS ~~AND~~ AN INTEGRAL OF A CLOSED FORM EXPRESSION. TYPICALLY THE INTEGRAL IS NASTY, BUT CARRIES ALL INFO. EG CAN WRITE $t(q)$ FUNC. IN Mathematica. $t(q)$ CARRIES SAME INFO AS $q(t)$.

⇒ ON HW: $t(q)$ SUFFICIENT IF INTEGRAL IS NASTY.

Remark 19

WHAT IF MANY (>1) COORDINATES, q_k?

IDEALLY THERE ARE OTHER CONSERVED QUANTITIES

eg $$P_1 = \frac{\partial L}{\partial \dot{q}_1}$$

$\uparrow$ ~ ODE of DEGREE 1

MAYBE USE THIS TO WRITE $\dot{q}_1(\dot{q}_2, \cdots, \dot{q}_N, q_1, q_2, \cdots)$

then plug into E to get ODE in one variable.

BUT: this is a red herring for HW #2!
stick falling problem - USE YOUR 1116 INTUITION.

MT p.240 EXAMPLE: the pendulum

$x = l \sin\theta$ $\qquad V = mgy$

$y = -l \cos\theta$

$\uparrow$ note sign!

things to note: cancel the m across the board

$\rightarrow$ physical interp: $F = ma \sim -\nabla mgy$

$\rightarrow$ note elliptic integral

EXAMPLE TO TRY ON YOUR OWN: DOUBLE PENDULUM

E&M

INDICES: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0$

btw: DIM ANALYSIS RUBRIC

eg when no constraints, $q = \vec{r}$
SO WE WRITE THINGS LIKE $\partial/\partial\vec{r}$
(short hand for each index)

but sometimes this can confuse us

REVIEW: $\vec{r}_i = x_i \hat{e}_i = (x_1, 0, 0) + (0, x_2, 0) + (0, 0, x_3)$

component; basis vector

NOTE: IMPLIED SUM OF REPEATED INDICES

THEN WE CAN WORK COMPONENT-WISE:

$(\vec{A}\cdot\vec{B}) = A_j B_j = A_i \delta_{ij} B_j$

"dot product" → KRONECKER DELTA

$(\vec{A}\times\vec{B})_i = \varepsilon_{ijk} A_j B_k$

"cross product" → LEVI-CIVITA TENSOR
totally antisymmetric
w/ $\varepsilon_{123} = +1$ (convention)

remark: ε's USUALLY INDICATE THE GEOMETRIC STRUCTURE OF THE THEORY! (eg. $A\times B \sim$ AREA)
[cf DIFF. FORMS, STOKES' THM]

in homework there's a problem where you are faced w/ $\vec{\nabla}\times(\vec{A}\times\vec{B})$. HOW TO DO?

1. look up / rederive vector identities
2. INDICES

$$(\vec{\nabla}\times(\vec{A}\times\vec{B}))_i = \varepsilon_{ijk}\overset{\partial/\partial x_j}{\partial_j}(\vec{A}\times\vec{B})_k$$
$$= \varepsilon_{ijk}\partial_j(\varepsilon_{klm}A_l B_m)$$
$$= \underline{\underline{\varepsilon_{ijk}\varepsilon_{klm}}}\ \underbrace{\partial_j(A_l B_m)}$$

$$= (\partial_j A_l)B_m + A_l\partial_j B_m$$

NOW SUPPOSE $A_l = (\vec{r})_l = x_l$

WHAT is $\partial_j x_l$?

$\to \delta_{jl}$

[ANYWAY, YOU GET SOMETHING HERE]

can we simplify? $\varepsilon_{ijk}\varepsilon_{klm}$ IS A 4 INDEX OBJECT. IT IS ROTATIONALLY COVARIANT (eg $\varepsilon_{ijk}A_j B_k$ IS A PSEUDOVECTOR IF $\vec{A},\vec{B}$ ARE VECTORS). ONLY OTHER OBJECTS THAT WE HAVE ARE δ_{ij}.

so expect $\varepsilon_{ijk}\varepsilon_{klm} \sim (\delta\delta)_{ijlm}$

$\varepsilon_{ijk}\varepsilon_{klm} \sim \cancel{\delta_{ij}\delta_{lm}} \quad \delta_{il}\delta_{jm} \ominus \delta_{im}\delta_{jl}$

not possible by symmetry

RELATED BY, eg $i \leftrightarrow j$
$\Rightarrow$ MUST HAVE A RELATIVE SIGN !

$$\Rightarrow \varepsilon_{ijk}\varepsilon_{klm} = A(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})$$

NOW CAN DETERMINE A :

$$\underset{(+1)}{\varepsilon_{123}}\underset{(+1)}{\varepsilon_{312}} = A(\delta_{11}\delta_{22} - \cancel{\delta_{12}\delta_{21}}^{\,0}) \Rightarrow A = 1.$$

must be 3.
all other terms vanish

& similarly for, say, $\varepsilon_{ijk}\varepsilon_{kjm}$.

VELOCITY-DEP FORCES

recall generalized EOM

$$F = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q}$$

$$F_{cons} + F_v \quad \text{eg } F_{FRIC} = -KX$$

$$\Rightarrow F_v = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$$

$$\cancel{F_v = F_v(\dot{q})} = \cancel{\partial/\partial\dot{q}\, G(\dot{q})}$$

$$\vec{F}_v(v) = -\partial/\partial\vec{v}\, G(v)$$

COMPONENT OF F (generalized) COMING FROM v-dep force:

$$F_v = \vec{F}_v \cdot \partial\vec{r}/\partial q = -\vec{\nabla}_v G \cdot \partial\vec{r}/\partial q$$

$$= -\vec{\nabla}_v G \cdot \partial\dot{\vec{r}}/\partial\dot{q} \quad \text{DOT RULE}$$

$$= -\partial G/\partial\dot{q}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial G}{\partial \dot{q}} = 0$$

★ SEE EXAMPLE

Another velocity-dep force

$$F_{mag} = e\vec{v}\times\vec{B}$$ ← see HW

$$L = \tfrac{1}{2}mv^2 + e\vec{v}\cdot\vec{A}$$

↑ vector potential!

remark: describes a theory of a particle interacting w/ a [BACKGROUND] field

↳ does not describe backreaction on field

... how to describe?

⇒ PROBLEM of ∞ DoF

... also: how to make relativistic?

↳ can you think of a relativistic quantity to use for L?

GAUGE SYM

full em L: $\tfrac{1}{2}mv^2 - e\phi(q) + e\vec{A}(q)\cdot\vec{v}$

$$\vec{A} \to \vec{A} + \nabla\lambda$$
$$\phi \to \phi - \dot{\lambda}$$

Gold (2) 1.22

PARTICLE falling w/ $G = \frac{1}{2}kv^2$

$$L = \frac{1}{2}m\dot{y}^2 - mgy$$

EOM: $\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \frac{\partial G}{\partial \dot{y}} = 0$

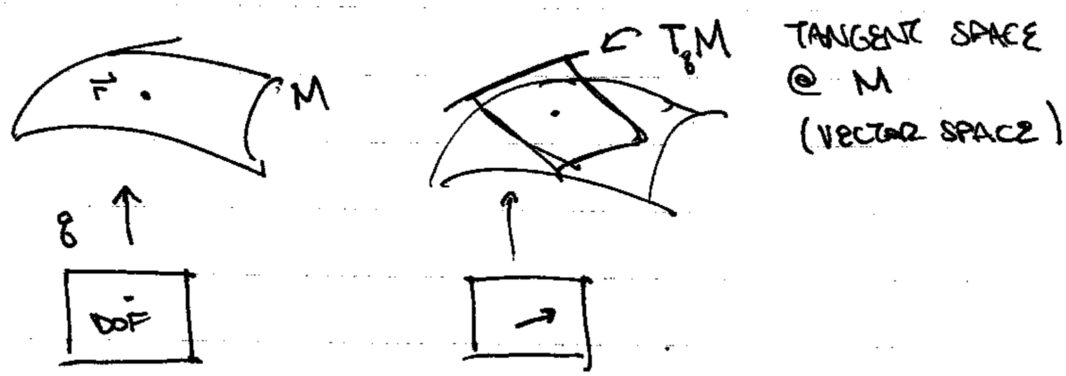
$$\Rightarrow \boxed{m\ddot{y} = mg - k\dot{y}}$$

when $\dot{y} = mg/k$, $\ddot{y} = 0 \rightarrow$ terminal vel.

Geometry remarks

- CONFORMAL CORNELL
- $L: TM \to \mathbb{R}$

  ↑ tangent BUNDLE



the collection of points on M & their tangent spaces is called a tangent bundle.