HOUSEKEEPING

- OFFICE HOURS: F (today) AFTER CLASS; M say 3-4 pm (but all morning too); Tu 2-3 pm — PSB 432
- NEXT WEEK: Mathematica practicum?
- FEEDBACK: ALWAYS APPRECIATED! thanks for all the nice things-to-do in Cornell (BUCKET LIST) items.

SOME REMARKS

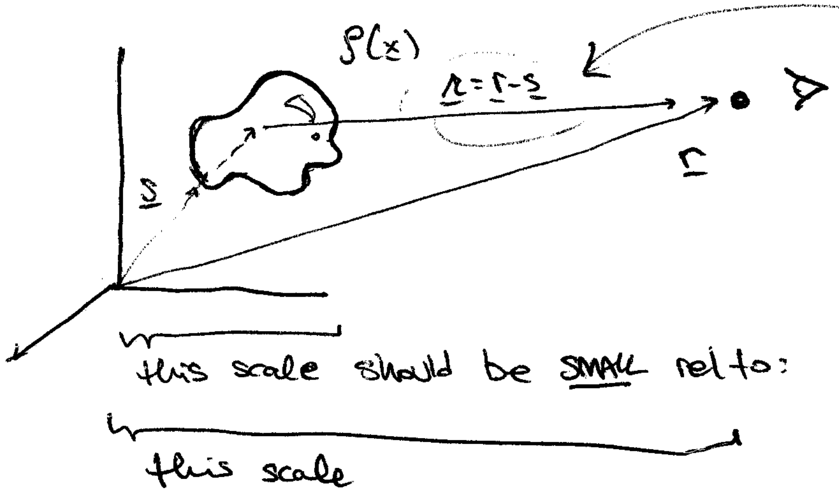
① the pedagogical use of the word 'trivial' :)

② ASK QUESTIONS
  good template: "EXCUSE ME, BUT IS IT OBVIOUS THAT..."

③ BEST COMMENT: "Mike looks like the kid from UP."

Physics: MULTIPOLE EXPANSION

RECALL INTUITION: TAYLOR EXPAND:



$$\Phi = f(R)$$

$$\sim \frac{1}{R}$$

$$\underline{E} = -\underline{\nabla}_r \Phi$$

EXPAND AS FUNCTIONS OF r

(GOOD for $s/r \ll 1$)

THERE'S A SHORTCUT: THIS TAYLOR EXPANSION IS FAMOUS

$$\frac{1}{|\underline{r}-\underline{s}|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{s}{r}\right)^l P_l(\underbrace{\hat{r}\cdot\hat{s}}_{\cos\theta})$$

Legendre polynomials

(not "Pessel functions"!)

$$\underline{\text{Def}}: \frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

$$= \frac{1}{\sqrt{\underline{r}^2 - 2rs\cos\theta + \underline{s}^2}} = \frac{1}{r\sqrt{(s/r)^2 - 2(s/r)\cos\theta + 1}}$$

Where do these Legendre guys show up?

IN SPHERICAL COORDS ... ALL THE TIME!

$$\nabla^2 = \left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{1}{r^2}\mathcal{O}^2\right)$$

radial stuff

$$\mathcal{O}^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

encodes all angular stuff

TURNS OUT: $\mathcal{O}^2 P_l(\cos\theta) = l(l+1)P_l(\cos\theta)$

$\Rightarrow$ $P_l(\cos\theta)$ IS AN EIGENVECTOR (EIGENFUNCTION) OF THE ANGULAR PART OF THE LAPLACIAN IN SPHERICAL COORDINATES.

FOR OUR PURPOSES:

$$P_0 = 1$$
$$P_1 = \cos\theta$$
$$P_2 = \frac{1}{2}(3\cos^2\theta - 1)$$

~~of "$3x_ix_j - \delta_{ij}$"~~

$$= \frac{1}{2}[3(\hat{r}\cdot\hat{s})^2 - 1]$$
$$= \frac{1}{2}[3\hat{r}_i\hat{r}_j\hat{s}_i\hat{s}_j - 1]$$
$$= \frac{1}{2}[3\hat{s}_i\hat{s}_j - \delta_{ij}]\,\hat{r}_i\hat{r}_j$$

so: $\Phi(\underline{r}) = \int d^3\underline{s} \ \frac{\rho(\underline{s})}{|\underline{r}-\underline{s}|}$

expand in Legendre polynoms.

$$\frac{1}{|\underline{r}-\underline{s}|} = \text{mono} + \text{di} + \text{quad} + \underbrace{\text{not in this class}}$$

my officemate has this

lets focus on this

quad $\leftrightarrow$ $l=2$ term

NOT REALLY USEFUL USUALLY BETTER TO DO SPHERICAL MULTIPOLE EXP. (Y^l_m)

$$= \cdots + \int d^3\underline{s}\ \rho(\underline{s}) \frac{1}{r}\left(\frac{s}{r}\right)^2 P_2(\hat{r}\cdot\hat{s})$$

$$= \cdots + \frac{\hat{r}_i\hat{r}_j}{2r^3} \int d^3\underline{s}\ \underbrace{s^2(3\hat{s}_i\hat{s}_j - \delta_{ij})}$$

PEEL OF PART WHICH ONLY HAS TO DO W/ THE OBSERVATION POINT!

$$\underbrace{(3s_is_j - s^2\delta_{ij})}$$

QUADRUPOLE TENSOR — under rotation of coords, $Q \mapsto RQR^T$.

(the derivation in Heald & Marion doesn't make sense to me — adding terms ≈ 0 to make a specific term w/o explaining the motivation)

REMARKS

- $\mathbb{R}$ SYMMETRIC TRACELESS 3x3 MATRIX (cf LAST WK)
  - $\hookrightarrow$ 5 INDEPENDENT ELEMENTS (dof)
  - $= (2l+1)$, AS EXPECTED (eg Y^l_m HAS $2l+1$ values of m for ea l)

cf: $\Phi^{(2)} = \frac{1}{2}\int d^3\underline{s}\ \underbrace{s_a s_b}\ \partial_{s_a}\partial_{s_b} \frac{1}{|\underline{r}-\underline{s}|}$

$9\times9 \rightarrow 6$ from sym

EXAMPLE PROBLEMS

① [inspired by Jackson 4.2 & of HW #2.4]

CALCULATE $\Phi(\underline{r})$ FOR $\rho(\underline{r}) = -\underline{a}\cdot\underline{\nabla}\,\delta(\underline{r})$

→ WHAT ARE THE FIRST 3 MULTIPOLE MOMENTS?
→ HOW DO YOU INTERPRET ρ? (are you familiar w/ this?)

Solution : feynman: "shut up & calculate!"

$$\Phi = \int d^3\underline{s}\;\rho(\underline{s})\,\frac{1}{|\underline{r}-\underline{s}|}$$

$$= \int d^3\underline{s}\;\left(-\underline{a}\cdot\underline{\nabla}_s\,\delta(\underline{s})\right)\frac{1}{|\underline{r}-\underline{s}|}$$

$$= +\int d^3\underline{s}\;\delta(\underline{s})\;\underline{a}\cdot\underline{\nabla}_s\,\frac{1}{|\underline{r}-\underline{s}|}$$

note: can do integral w/ $\delta(s)$ now if you want:

$$\Rightarrow \quad = \underline{a}\cdot\underline{\nabla}_s\,\frac{1}{|\underline{r}-\underline{s}|}\Big|_{\underline{s}=0}$$

$\nabla_s \frac{1}{r} = 0$ multipole: all vanish @ $s=0$

$$\left[\frac{1}{r} + \frac{\underline{r}\cdot\underline{s}}{r^3} + O(s^2)\right]$$

w/ $\underline{p} = \underline{a}$

$$= \boxed{\frac{\underline{a}\cdot\underline{r}}{r^3}}$$

EXACTLY THE FORM OF A DIPOLE !
EVIDENTLY, ρ IS A 'POINT DIPOLE'
SEE HW #2.4 ↳ $\pm\underline{p}\cdot\underline{\nabla}$ rule

what is a point dipole? like "point" magnetic dipole from Hydrogen atom. OR EFFECTIVE TREATMENT OF DIELECTRIC MEDIA.

[by the way I should tell my $\rho \sim \delta$ story]

(2) WHAT IS THE QUADRUPOLE MOMENT OF A UNIFORMLY CHARGED ELLIPSOID:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$ w/ UNIFORM CHARGE DENSITY ρ.

[LANDAU VOL 2 P. 107]

SYMMETRY: WE'RE ALREADY ALIGNED ALONG THE PRINCIPAL AXES WHICH DIAGONALIZE Q_{ij}. convince yourself!

$$Q_{ij} = \int d^3\underline{s}\, \rho(s) \left(3 s_i s_j - s^2 \delta_{ij}\right)$$

$$Q_{xx} = \int d^3\underline{s}\, \rho(\underline{s}) \left(2x^2 - y^2 - z^2\right)$$ $\}$ sim for y & z.

INTEGRATION REGION IS KIND OF HARD.
SOLUTION: MAKE IT EASY!

def: $x' = x/a \qquad y' = y/b \qquad z' = z/c$

$$d^3\underline{s} = dx\,dy\,dz = abc\, dx'dy'dz'$$

$$Q_{xx} = \int_B abc\, d^3\underline{s}'\, \rho \left(2a^2(x')^2 - b^2(y')^2 - c^2(z')^2\right)$$

B ← UNIT BALL

now we have a bunch of integrals of the form

$$\int_{BALL} \underbrace{z^2}_{r^2\cos^2\theta} r^2 dr\, d(\cos\theta)\, d\phi = \underbrace{\int_{BALL} \underbrace{(r^4\, dr)}_{\frac{1}{5}} \underbrace{\cos^2\theta\, d\cos\theta}_{\frac{2}{3}}\, \underbrace{d\phi}_{2\pi}}_{\frac{1}{5}\cdot\frac{4\pi}{3}}$$

$$\Rightarrow Q_{xx} = \frac{4\pi}{3}\frac{\rho abc}{5}\left(2a^2 - b^2 - c^2\right)$$

$$Q_{yy} = \text{—"—} \left(2b^2 - a^2 - c^2\right)$$

$$Q_{zz} = \text{—"—} \left(2c^2 - a^2 - b^2\right)$$

all other elements $= 0$

Homework Caveats

- A useful skill: calculate along some particular orientation & use symmetry to argue the general case.

- Problem 3 has a few potential pitfalls
  - ↳ keep asking yourself if your results make sense

  Hint: be careful if you try to use spherical coords
  - ↳ I encourage you to try & see why it's hard ... related to viewpoint in last week's section!

  obscure hint: just because a vector's magnitude isn't changing, that doesn't mean ~~#~~ that the vector isn't changing!

The puzzle of the point electron E = m

↳ from my lectures to the LEPP SUMMER STUDENTS '12

the electron has "rest energy" $m_e c^2$
but obtains a correction from the energy of the electric field it generates:

$$\Delta E_{coulomb} = \alpha / r_e$$

← "radius" of electron

$r_e \lesssim 10^{-17}$ cm → $\Delta E \gtrsim 10$ GeV

unobserved

$$[\text{OBSERVED REST ENERGY}] = m_e c^2 + \Delta E$$

$$.5 \text{ MeV} = (-9.\underline{005} + 10) \text{ GeV}$$

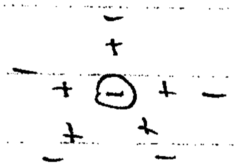
fine tuning

THIS 0.1% TUNING SEEMS SILLY.

TO AVOID THIS TUNING, WOULD NEED THE COULOMB POTENTIAL TO "BREAK DOWN" @

$$\bar{r} = \frac{e^2}{\alpha m_e c^2} \sim \boxed{3 \times 10^{-13} \text{ cm}}$$

INDEED, IT IS. THE COULOMB POTENTIAL IS SINGULAR @ CLASSICAL LIMIT — BUT NOT IN THE QUANTUM LIMIT.



VIRTUAL e^+e^- PAIRS <u>SHIELD</u> THE POTENTIAL

} SMEAR OUT THE POINT CHARGE

THESE VIRTUAL PAIRS OBEY (roughly) $\Delta t \Delta E \sim \hbar$

$\Rightarrow \Delta t \sim \hbar / \Delta E = \hbar / (2 M_e c^2)$

CHARACTERISTIC DISTANCE:

$$d \sim c \Delta t \sim \hbar c / (2 M_e c^2) = \boxed{200 \times 10^{-13} \text{ cm}}$$

cf 3×10^{-13} cm

SO QUANTUM MECHANICS SAVES US @ A LENGTH SCALE 100 TIMES LARGER THAN NEEDED.

↑ wiggle room.

∃ COMPLETELY ANALOGOUS PROBLEM IN PARTICLE PHYSICS → WHY IS THE HIGGS BOSON LIGHT?

proposed solution: double spectrum again → supersymmetry