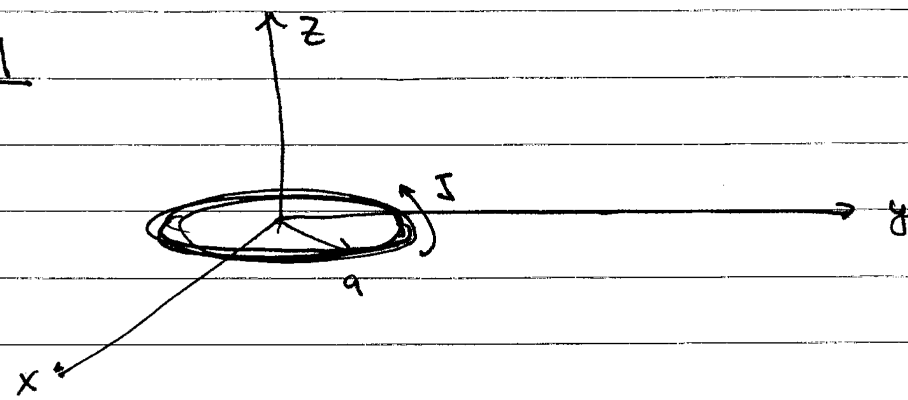
SECTION 7 12 OCT 2012

ANNOUNCEMENTS

- BETHE LECTURES NEXT WEEK
- ADVICE REQ for POSTDOCS: great American frat party

TODAY: BY REQUEST, FOCUS MAINLY ON EXAMPLES.

eg 1



a) WHAT IS THE MAGNETIC FIELD FAR AWAY?
$\Rightarrow$ JUST THE DIPOLE TERM

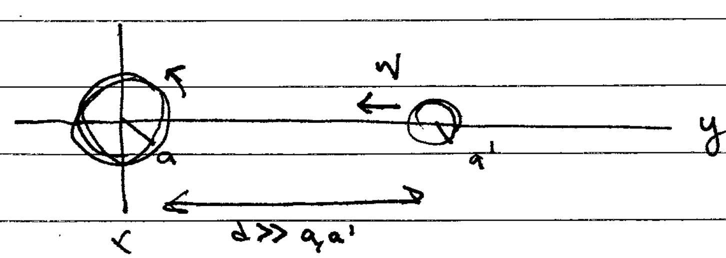
$$m = \frac{I\pi a^2}{c}\hat{z}$$

$$\underline{B} = \frac{1}{r^5}\left[3(\underline{m}\cdot\underline{r})\,\underline{r} - \underline{m}\,r^2\right]$$

$$\text{cf} \quad E^{(2)} = \frac{1}{r^5}\left(3(\underline{p}\cdot\underline{r})\,\underline{r} - \underline{p}\,r^2\right)$$

b) NOW ADD SECOND CONDUCTING LOOP w/ radius a'
MOVING TOWARDS FIRST LOOP w/ VELOCITY $\underline{V} = -v\hat{y}$
& $v \ll c$.

$\boxed{\text{WHAT IS EMF IN MOVING LOOP?}}$



ALONG xy PLANE:

$$\underline{B} = \frac{1}{r^5}\left[3\underbrace{\cancel{(\underline{m}\cdot\underline{r})}}_{0}\,\underline{r} - \underline{m}\,r^2\right]$$

$$= \frac{-m}{r^3} \longleftarrow m = I\pi a^2/c$$

SO IF 2^{ND} LOOP CENTERED @ $s\,\hat{y}$ $(s \gg a')$

THEN THE FLUX IS

$$\Phi_B(s) = \int_S \underline{B}(s')\cdot dA =$$

$$= -m\int_S \frac{dA}{(s')^3} \longleftarrow$$

INTEGRAL OVER SURFACE CENTERED @ s

$\longrightarrow$ to leading order:

$$\Phi_B(s) = -m \cdot \frac{\pi (a')^2}{s^3} = \frac{-I\pi^2 a^2 (a')^2}{c\,s^3}$$

NOW USE $s = \text{const} - vt$

$$\mathcal{E} = \frac{1}{c}\frac{\partial \Phi_B(s)}{\partial t} = \boxed{3I\pi^2 \frac{v}{c^2}\frac{a^2(a')^2}{s^4}}$$

c) SUPPOSE THE SMALLER LOOP HAS RESISTANCE R. WHAT IS THE POWER DISSIPATED IN THE LOOP?

$$P = \frac{\mathcal{E}^2}{R} \qquad \text{(OHM'S LAW)}$$

$$= \frac{9 I^2 \pi^4}{R} \left(\frac{v}{c^2}\right)^2 \left(\frac{qa'}{s\,\blacksquare^2}\right)^4$$

d) <u>WHAT IS THE FORCE ON THE LOOP?</u>

MAGNETIC FIELDS DO NO WORK.

SO POWER MUST BE DISSIPATED AS A <u>CHANGE IN KINETIC ENERGY</u>. (SEE GRIFFITHS)

$$\cancel{dKE}\; d(KE) = \underline{F} \cdot d\underline{x} = \underline{\underline{F\, dy}}$$

$$\underline{F} = q\underline{E} + \underline{\underline{\frac{q}{c}\, \underline{u} \times \underline{B}}}$$

then: $P = \frac{d(KE)}{dt} = F \frac{dy}{dt} = -Fv \leftarrow \underline{v} = -v\hat{y}$

[WE ASSUME LOOP'S VELOCITY IS BEING KEPT CONST BY AN <u>EXTERNAL</u> FORCE]

$$\Rightarrow F = \frac{9 I^2 \pi^4}{R} \frac{v}{c^4} \left(\frac{qa'}{s^2}\right)^4 \qquad \text{in the } \hat{y} \text{ DIR.}$$

H&M 4.14 COULOMB GAUGE

$\nabla \cdot A = 0$ USEFUL WHEN NO NET CHARGE.

(a) CLAIM: SCALAR SATISFIES POISSON EQ.

Gauss' law: $4\pi\rho = \nabla \cdot E$

$= \nabla \cdot \left(-\nabla\phi - \frac{1}{c}\dot{A}\right)$ $\nabla \cdot A = 0$

$= -\nabla^2\phi$ ✓

(b) WHAT is EQN for A?

Ampere $\Rightarrow$ $\nabla \times B = \frac{4\pi}{c} J - \frac{1}{c}\partial_t\left(\nabla\phi - \frac{1}{c}\partial_t A\right) = \nabla \times \nabla \times A$

$= \nabla(\underbrace{\nabla \cdot A}_{0}) - \nabla^2 A$

$$\Rightarrow \underbrace{\nabla^2 A - \frac{1}{c^2}\ddot{A}}_{\text{wave eqn.}} = \frac{1}{c}\nabla\dot{\phi} - \frac{4\pi}{c} J$$

(c) w/o log, WRITE: $J = J_1 + J_2$ w/

$\nabla \times J_1 = 0$ LONGITUDINAL

$\nabla \cdot J_2 = 0$ TRANSVERSE

CLAIM: $\nabla\dot{\phi} = 4\pi J_1$ INDEP of ϕ

Induced electric field from a time-dependent current

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This note is an extension of the problem posed by Shakur,[1] and commented on by Gauthier,[2] which asks what induced electric field exists in the vicinity of a long straight wire carrying a current that increases linearly with time. While the problem appears to be elementary, it is in fact quite subtle—more for what it leaves out than what it includes. Here we pose and discuss a number of queries that might occur to someone not satisfied by Shakur's "solution."

Query 1. Is it not inconsistent to calculate the magnetic field (and thence the induced electric field), inside the wire, on the assumption of uniform current density?

Yes, it is inconsistent. But, as noted by Gauthier,[2] for the given numbers, the *Ohmic* electric field required to drive the prescribed current $I(t)$ exceeds the internal *induced* electric field when $t > 10\ \mu$s. That is, except for the initial microseconds, the Ohmic field (proportional to time, but independent of radius) swamps the induced field (independent of time for $dI/dt =$ constant, but varying with radius), both inside and outside the wire. Thus the total current density is *approximately* independent of radius. But under these conditions, of course, the induced component of the field is only a small perturbation, and the problem becomes a rather pedantic parlor game. A much more complete discussion, explicitly addressing the skin-effect phenomenon, is given by Gauthier *et al.*[3]

Query 2. Where is the "zero" of the induced electric field?

This turns out to be something of a semantic question bearing on what one means by the prescribed current I, and on how one defines "induced" (as opposed to other, "noninduced") electric fields. The simplest answer is to require that the partial current driven by the induced field be zero,[4]

$$I_{\text{ind}} = \int_0^R \sigma E_{\text{ind}}(r)\, 2\pi r\, dr = 0, \tag{1}$$

where $\sigma =$ conductivity and $R =$ radius of wire. This leads to

$$E_{\text{ind}}(r<R) = \frac{\mu_0}{4\pi R^2}\left(r^2 - \frac{R^2}{2}\right)\frac{dI}{dt}, \tag{2}$$

$$E_{\text{ind}}(r>R) = \frac{\mu_0}{4\pi}\left[\frac{1}{2} + 2\ln\left(\frac{r}{R}\right)\right]\frac{dI}{dt}. \tag{3}$$

Note that if one chooses to put the zero at any other radius (e.g., Shakur assumes $E = 0$ at $r = 0$), the effect is simply to add a space-time constant to the "induced" field. This point is discussed further in query 4.

Query 3. Does the magnitude of the induced field really increase without bound as one goes away from the wire?

The solution for the induced field outside the wire varies as $\ln(r/R)$. This suggests a sort of "ultraviolet catastrophe" of arbitrarily large fields at large radii, clearly a nonphysical result. The difficulty lies in the fact that a "long straight wire" is not a complete circuit; we have not defined the global geometry of the circuit.[5] The only fully defined electromagnetic problems are those in which all fields go to zero at large distances. Also at large distances one can no longer neglect retardation and radiation effects.

Query 4. Can we get the induced field directly from Maxwell's equations?

It is reasonable to define the induced field as the *Faraday* field given by

$$\nabla\cdot\mathbf{E}_F = 0, \tag{4}$$

$$\nabla\times\mathbf{E}_F = -\frac{\partial B}{\partial t}. \tag{5}$$

This is in juxtaposition to the *Coulomb* field given by

$$\nabla\cdot\mathbf{E}_C = \rho/\epsilon_0, \tag{6}$$

$$\nabla\times\mathbf{E}_C = 0, \tag{7}$$

The solution of Eqs. (4) and (5) is the dual of the Biot–Savart formula, namely,[6]

$$\mathbf{E}_F = -\frac{1}{4\pi}\int\left(\frac{\partial \mathbf{B}}{\partial t}\right)\times\hat{\mathbf{r}}/r^2\, dv \tag{8}$$

with integration over all space. For the geometry at hand (axial symmetry; azimuthal $\partial\mathbf{B}/\partial t$), the solution is

$$\mathbf{E}_F(r>R) = -\frac{\mu_0}{2\pi}\frac{dI}{dt}\int_r^\infty \frac{1}{r}\, dr, \tag{9}$$

which produces a (negative) logarithmic infinity at any finite distance from the wire! The logarithmic divergence at large radii of query 3 has moved to the vicinity of the wire in query 4. Both are artifacts of incomplete specification of the global geometry.

Note that a spatially constant field (even a logarithmically divergent one!) is consistent with both Eqs. (4) and (5) and Eqs. (6) and (7). So how does one know whether to label such a constant field as "Faraday" or "Coulomb?" This ambiguity can only be resolved by specifying the ultimate energy sources of the global circuit, and is the basis for the ambiguity of the location of zero induced field (query 2).

Query 5. Why not calculate the induced field from the vector potential?

From the standard formula

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}}{r}\, dv \tag{10}$$

we see, without attempting to do the integral, that the vector potential $\mathbf{A}$ must be parallel to the current density $\mathbf{J}$, and hence to the axis of our system, with the sense of the total current I. Then the induced field is given by