FLIP TANEDO
http://www.lepp.cornell.edu/~pt267/ ← these notes will be posted

ANNOUNCEMENTS

- EMAIL BIBHUSHAN IF YOU HAVE ANY HW QUESTIONS
- LHC: 3.5 TeV/beam!

The Big Picture

YOU'VE LEARNED ABOUT SOME VERY DEEP & IMPORTANT IDEAS IN QM

- SPIN
- ADDITION OF ANGULAR MOMENTUM (representation theory)
- IDENTICAL PARTICLES
- ATOMS (how to get chemistry from physics)

SPIN-STATISTICS THM:

"force particles"

PARTICLES w/ INTEGER SPIN ARE BOSONS
$\frac{1}{2}$ INTEGER SPIN ARE FERMIONS → "matter particles" BY PAULI EXCLUSION

INTRINSIC ANGULAR MOMENTUM OF AN INDIVIDUAL PARTICLE

BEHAVIOR OF 2 OR MORE PARTICLES

HEURISTIC MOTIVATION (~~not~~ a proof!)

CONSIDER THE STATE $R(\pi)|\Psi(x)\rangle \otimes |\Psi(-x)\rangle$

rotates the spin of the particle about, eg. z-axis

NOW PERFORM AN ADDITIONAL ROTATION ABOUT z-AXIS BY π

$$R(\pi)|\Psi(x)\rangle \otimes |\Psi(-x)\rangle \longrightarrow \underbrace{R(2\pi)}|\Psi(-x)\rangle \otimes R(\pi)|\Psi(x)\rangle$$

WHAT IS THIS OBJECT?

ROTATION OP: $e^{i\frac{\theta}{2}} \longrightarrow \pm 1$ FOR INTEGER/$\frac{1}{2}$ INTEGER SPIN

SO: $R(\pi)|\Psi(x)\rangle \otimes |\Psi(-x)\rangle = \pm |\Psi(-x)\rangle \otimes R(\pi)|\Psi(x)\rangle$

this is a heuristic motivation, not A pf!
REAL PROOF REQUIRES RELATIVISTIC QUANTUM MECHANICS
↳ QFT. = SPECIAL RELATIVITY + QUANTUM MECHANICS

BUT THE TAKE HOME MESSAGE IS THAT SPIN REALLY IS MORE THAN "JUST ANGULAR MOMENTUM"; IT DETERMINES FUNDAMENTAL PROPERTIES OF THE STATE.

REMARK: SUPERSYMMETRY

IN QFT, THERE IS A VERY ELEGANT IDEA CALLED SUPERSYMMETRY → REQUIRES EXISTENCE OF FERMIONS
THAT RELATES BOSONS & FERMIONS.

force particles — matter particles

A NICE INTRODUCTION TO SUSY QM IS Haymaker & Rau, "SUSY IN QM" Am. J. Phys. 54 1986

these "superpartner fields" have funny names:

electron → selectron
quark → squark
photon → photino

[another question: what "is" a fermion (mathematically)]

but have all the same properties except spin.

THIS TURNS OUT TO BE A VERY DEEP IDEA & RELATES TO THE NATURE OF SPACETIME ITSELF ("FERMIONIC" EXTRA DIMENSIONS)

But you could already have guessed this, AS YOU SHOULD ALREADY KNOW: SYMMETRIES ARE VERY IMPORTANT IN PHYSICS.
→ IN FACT, ONE COULD ARGUE THAT SYMMETRY IS EVERYTHING.

eg. ANALYTICAL MECHANICS: Noether's thm: SYMMETRY ↔ CONSERVATION LAW

IN QUANTUM MECHANICS SYMMETRIES ARE GENERATED BY HERMITIAN OPERATORS (that commute w/ H)
YOU ARE ALREADY FAMILIAR WITH SOME: eg. ROTATIONS & TRANSLATIONS

YOU'VE ALSO LEARNED ABOUT THE COMMUTATION RELATIONS OF THESE OBJECTS. THIS IS CALLED THE ALGEBRA OF THE SYMMETRY.

→ IN THE SENSE OF GROUP REPRESENTATION THEORY "the mathematics of symmetry"

VERY IMPORTANT IN PHYSICS.
→ ALL PHYSICISTS (esp. PARTICLE, & really esp. THEORISTS) SHOULD KNOW
- GROUP THEORY, REP. THY., LIE ALGEBRAS
- DIFFERENTIAL GEOMETRY (more = better)
- ℓ ANALYSIS

THIS ISN'T COVERED WELL IN GRIFFITHS.
SEE, eg., SHANKAR (big red book)
SAKURAI (modern QM)

Kind of interesting:

SUSY GENERATORS OBEY ANTICOMMUTATION RELATIONS.
$\{A, B\} = AB + BA$

OPERATORS THAT COMMUTE WITH H ARE INTERESTING.
THEY GIVE US ADDITIONAL QUANTUM NUMBERS
& OFTEN ADDITIONAL RAISING & LOWERING OPERATORS FOR THOSE QUANTUM #'s

$\Rightarrow$ defines <u>DEGENERATE STATES</u> w/ SAME ENERGY

<u>Remark</u>: YOU HAVE RECENTLY SEEN EXAMPLES OF THIS

- SHO in N-DIMENSIONS: N separate ladder operators

  $\hookrightarrow$ can have many states w/ given energy
  eg. $|1,0,0,\dots\rangle$, $|0,1,0,\dots\rangle$, ...

  this came from a <u>symmetry</u>: $\hat{e}_i \leftrightarrow \hat{e}_j$

- DEGENERATE STATES IN THE HYDROGEN ATOM

  STATE SPECIFIED BY (n, ℓ, m)
  $\uparrow$ DETERMINES ENERGY

  $\left.\begin{array}{l} \forall\, n,\ \exists\ n \text{ values of } \ell \\ \forall\, \ell,\ \exists\ (2\ell+1) \text{ values of } m \end{array}\right\}$ degeneracy $= \sum_{l=0}^{n-1} (2\ell+1) = n^2$

  DOES THIS ALSO COME FROM A SYMMETRY?
  <u>Yes</u>: NATURE SELDOM (if EVER) DOES THINGS "BY ACCIDENT"
  $\rightarrow$ especially degeneracies!

  WE KNOW THAT ANGULAR MOMENTUM IS CONSERVED

  $\Rightarrow [L_\pm, H] = 0$

  $\Rightarrow$ WE CAN RAISE & LOWER m VALUES w/o CHANGING E
  EXPLAINS DEGENERACY FOR A FIXED ℓ

  WHY IS THERE A DEGENERACY IN THE VALUES OF ℓ?

  see, eg. Shankar p. 360

  RECALL FROM ELECTRODYNAMICS THAT THE COULOMB POTENTIAL HAS AN ADDITIONAL SYMMETRY: IT PRESERVES THE RUNGE-LENZ VECTOR,

  $$\vec{n} = \frac{\vec{p}\times\vec{\ell}}{m} - \frac{e^2}{r}\vec{r}$$

  $$\Rightarrow \exists \text{ OPERATOR } \vec{N} = \frac{1}{2m}\left[\vec{P}\times\vec{L} - \vec{L}\times\vec{P}\right] - \frac{e^2\vec{R}}{\sqrt{x^2+y^2+z^2}} \quad \text{s.t. } [\vec{N}, H] = 0$$

  & WE CAN BUILD A LADDER OF STATES FOR THIS.

$\Rightarrow$ COULD ANALYZE HYDROGEN w/ "OPERATOR ANALYSIS" OF SHO.
YOU'LL SEE SOON: CAN <u>LIFT</u> THE DEGENERACY BY BREAKING SYMMETRY (fine structure)

NOW WE HAVE SYSTEMS WITH MULTIPLE SOURCES OF ANGULAR MOMENTUM

Q: WHAT IS THE ANGULAR MOMENTUM OF THE WHOLE SYSTEM?

[IN FACT, THIS IS ALSO A SOMEWHAT 'DEEP' QUESTION! WE KNOW PARTICLES HAVE A FUNDAMENTAL SPIN ... BUT WHAT ABOUT COMPOSITE PARTICLES? eg ATOMS OR MESONS. FOR A LONG TIME WE DIDN'T KNOW THESE THINGS HAD SUBSTRUCTURE]

WE NEED A WAY TO ADD ANGULAR MOMENTUM

CAN'T JUST ADD COMPONENTS CLASSICALLY (eg L_x & L_y NOT WELL-DEFINED)

$\rightarrow$ THIS ONLY WORKS FOR THE Z QUANTUM NUMBER

GRIFFITHS WORKS OUT THE SIMPLEST CASE: SPIN $\frac{1}{2} \otimes$ SPIN $\frac{1}{2}$

STRATEGY:

- WRITE OUT A BASIS OF STATES: $\uparrow\uparrow$ ($m=1$), $\uparrow\downarrow$ ($m=0$), $\downarrow\uparrow$ ($m=0$), $\downarrow\downarrow$ ($m=-1$)

  not a good basis but the obvious basis

- START WITH LARGEST m STATE

  THIS IS THE HIGHEST STATE OF A SPIN-m MULTIPLET

  ("multiplet" is group theory lingo, cf "triplet," etc.)

- WORK OUT THE OTHER ELEMENTS OF THE MULTIPLET BY ACTING WITH LOWERING OPERATORS ON THIS HIGHEST STATE.

$$S_- = S_-^1 \oplus S_-^2$$

<u>IMPORTANT</u>: this <u>generally</u> mixes states

$$\text{eg. } S_-^1 \oplus S_-^2 (\uparrow\uparrow) = (S_-^1 \uparrow)\uparrow + \uparrow(S_-^2 \uparrow)$$

$$= \downarrow\uparrow + \uparrow\downarrow \qquad \text{(up to constants)}$$

- END UP W/ A FULL MULTIPLET: "<u>IRREDUCIBLE REP</u> OF THE <u>GROUP</u>"

  IRREDUCIBLE REP vs. <u>REDUCIBLE</u> PRODUCT REP.

  GROUP: SO(3) OR "ANGULAR MOMENTUM" OR "ROTATIONAL SYM."

- STILL HAVE LEFTOVER STATES. START AGAIN W/ THE HIGHEST-SPIN REMAINING STATE & REPEAT.

- REPEAT UNTIL THERE ARE NO LEFTOVER STATES.

$\Rightarrow$ EASY TO GENERALIZE (eg. moon-earth-sun system) ((spin + orbital angular momentum!))

WE CAN KEEP WORKING OUT THESE DECOMPOSITIONS OF REDUCIBLE TENSOR PRODUCTS INTO IRREDUCIBLE REPRESENTATIONS. THIS QUICKLY BECOMES TEDIOUS & IS NOT PARTICULARLY ENLIGHTENING. FORTUNATELY, SOMEONE ELSE HAS ALREADY DONE THIS FOR US ... <u>CLEBSCH-GORDAN TABLES</u>

HOW DO WE USE THIS?

$\boxed{\text{GRIFFITHS 4.36}}$

like a graviton

no orbital angular momentum

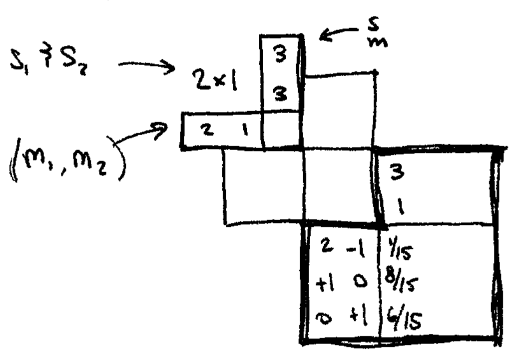
a) SPIN 1 PARTICLE & SPIN 2 PARTICLE @ REST

CONFIGURATION s.t. TOTAL SPIN $s = 3$
Z-COMPONENT $m = 1$ (set $\hbar = 1$)

<u>Q</u>: IF WE MEASURE Z-COMPONENT OF ANGULAR MOMENTUM OF THE SPIN-2 PARTICLE, WHAT ARE THE POSSIBILITIES & PROBABILITIES?

SO WE USE THE CLEBSCH-GORDAN TABLE TO GIVE US THE DECOMPOSITION OF A SPIN 2 × SPIN 1

$|3, 1\rangle$ in s, m BASIS $\longrightarrow$ WANT (s_1, m_1, s_2, m_2) BASIS



$$\Rightarrow |3,1\rangle = \sqrt{\tfrac{1}{15}}\,|2\,2\rangle \otimes |1\,{-1}\rangle + \sqrt{\tfrac{8}{15}}\,|2\,1\rangle \otimes |1\,0\rangle + \sqrt{\tfrac{6}{15}}\,|2\,0\rangle \otimes |1\,1\rangle$$

SO: Z-COMPONENT OF ∡ MOMENTUM OF SPIN-2 PARTICLE CAN BE

$2\hbar$ w/ PROB $1/15$
$\hbar$ w/ PROB $8/15$
0 w/ PROB $6/15$

b) ψ_{510} e^- in HYDROGEN ATOM, SPIN DOWN

IF WE MEASURE (TOTAL ∡ MOMENTUM)^2 OF ELECTRON <u>ALONE</u> (no p^+ spin)

Q. WHAT VALUES ARE POSSIBLE & WHAT PROB?

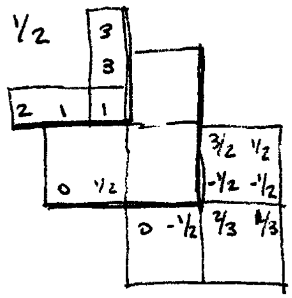
SPIN DOWN e^- IN ψ_{510} STATE

$|s\ m_s\rangle = |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$ $\qquad |l\ m_l\rangle = |1\ 0\rangle$

SO COMBINE SPIN + ORBITAL MOMENTUM OF ELECTRON
LOOK UP TABLE

$\uparrow$ one particle, but combining two sources of angular momentum.

$1 \times 1/2$



$\rightarrow |1\ 0\rangle \otimes |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

$= \sqrt{\tfrac{2}{3}}\, |\tfrac{3}{2}\ \tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\, |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

DON'T FORGET: $L^2\, |l, m_l\rangle = \hbar^2\, l(l+1)\, |l\ m_l\rangle$

$\Rightarrow$ CAN GET $\quad \hbar^2\, \tfrac{3}{2}(\tfrac{3}{2}+1) = \tfrac{15}{4}\hbar^2$ w/ PROB $2/3$

$\qquad \hbar^2\, \tfrac{1}{2}(\tfrac{1}{2}+1) = \tfrac{3}{4}\hbar^2$ w/ PROB $1/3$

REMARKS: CAN PLOT WEIGHT DIAGRAM. (MORE COMPLICATED GROUPS $\rightarrow$ HIGHER DIM DIAGRAM)

$\boxed{\text{GRIFFITHS 5.5}}$ TWO NON-INTERACTING PARTICLES IN THE ∞ SQUARE WELL

a) easy.

b) WHAT ARE THE EXCITED STATES

1 PARTICLE: $\psi_n(x) = \sqrt{\tfrac{2}{a}} \sin\left(\tfrac{n\pi}{a} x\right) \qquad E_n \propto n^2$

2 DISTINGUISHABLE: $\psi_{n_1 n_2}(x_1, x_2) = \psi_{n_1}(x_1)\, \psi_{n_2}(x_2) \qquad E_n \propto n_1^2 + n_2^2$

2 IDENTICAL BOSONS/FERMIONS

$$\psi^{B,F}_{n_1 n_2}(x_1, x_2) = \frac{1}{\sqrt{2}} \left(\psi_{n_1}(x_1)\, \psi_{n_2}(x_2) \pm \psi_{n_2}(x_1)\, \psi_{n_1}(x_2) \right)$$

understand the normalization! $\quad$ * EXCEPT WHEN $n_1 = n_2$!

$\tfrac{1}{\sqrt{2}} \rightarrow \tfrac{1}{2}$

GRIF. 5.4: Why? x-terms nonvanishing!

FERMIONS: $n_1 \neq n_2$

$$H = \frac{1}{2m}(p_1^2 + p_2^2) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$$

LINEAR OPERATOR $\Rightarrow \psi^{B,F}_{n_1 n_2} \propto n_1^2 + n_2^2$ ✓

BUT BECAUSE FERMIONS ARE ANTISYM ($n_1 \neq n_2$), k^{th} STATE HAS DIFF E

	1^{st}	2^{ND}	3^{RD}	4^{th}	5^{th}	6^{th}
B/D	11	12	21	22	13	31
F	[12]	[13]	[23]			

HELIUM & MORE COMPLICATED STUFF

$$H_{He} = \left\{ -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{1}{4\pi\epsilon_0}\frac{2e^2}{r_1} \right\} + \left\{ 1 \to 2 \right\} + \underbrace{\frac{1}{4\pi\epsilon_0}\frac{e^2}{|\vec{r}_1 - \vec{r}_2|}}_{\text{ignore this for simplicity}}$$

$\Rightarrow \psi(\vec{r}_1, \vec{r}_2) = \psi_{n\ell m}(\vec{r}_1)\,\psi_{n'\ell' m'}(\vec{r}_2)$ ← w/ some numerical shifts (eg ½ BOHR RADIUS 4x BOHR E_-)

BECAUSE {…} TERMS DECOUPLE

$\psi_{100}(\vec{r}_1)\psi_{100}(\vec{r}_2)$ is SYMMETRIC

↑ these are wavefunctions for the ELECTRONS → fermions

(★) WE REQUIRE AN ANTISYMMETRIC TOTAL ~~WAVEFUNCTION~~ STATE

state = wavefunction × spin state (× …)

- wavefunction ↑ POSITION
- spin state ↑ SPINOR
- (× …) ↑ internal quantum #'s eg. CHARGE

REMARK: DEEP Q: What is a spinor?
again: rep of a symmetry group
this time: FUNDAMENTAL REP OF THE UNIVERSAL COVER OF THE LORENTZ GROUP.

So: SPINOR STATE MUST BE ANTISYMMETRIC

$\frac{1}{2} \otimes \frac{1}{2} = 1 + 0$ (total spin values)

SYMM. TRIPLET: ↑↑, ↑↓+↓↑, ↓↓

ANTISYM SINGLET: ↑↓−↓↑ → GROUND STATE HELIUM: $\psi_{100}(\vec{r}_1)\psi_{100}(\vec{r}_2)\,(\uparrow\downarrow - \downarrow\uparrow)$

EXCITED STATES: $\psi_{n\ell m}(\vec{r}_1)\,\psi_{100}(\vec{r}_2) \times$ SPIN ? — assume 1 e in ground state

CAN CONSTRUCT:
- $\psi_{n\ell m}(\vec{r}_1)\psi_{100}(\vec{r}_2)$ SYMMETRIC ~~WAVEFUNC~~ & ANTISYM SPIN → PARAHELIUM
- $\psi_{n\ell m}(\vec{r}_1)\psi_{100}(\vec{r}_2)$ ANTISYM WAVEFUNC & SYM. SPIN → ORTHOHELIUM

sym spatial state : attractive exchange force, slightly higher E

$\boxed{\text{GRIFFITHS 5.10}}$

QUALITATIVELY DISCUSS E-LEVEL SCHEME IF He ELECTRONS ARE
1. IDENTICAL BOSONS
2. DISTINGUISHABLE

} w/ spin ½ of Fig 5.2

a) BOSONS: SYMMETRIC SPIN & SYMMETRIC SPATIAL

triplet has 3-fold degeneracy

→ GROUND STATE IS 3-fold degenerate

EXCITED STATES: A-SYM SPIN & ASYM SPATIAL ← singlet = PARAHe
SYM SPIN & SYM SPATIAL ← TRIPLET = ortho He

SPATIAL SYM REQ. ARE SWAPPED W/ FERMIONIC CASE
⇒ ORTHOHELIUM STATES HAVE SLIGHTLY HIGHER E.

b) DISTINGUISHABLE: ↑↑, ↑↓, ↓↑, ↓↓ ARE GOOD EIGENSTATES

⇒ 4-FOLD DEGENERACY.

A MORE INTERESTING EXAMPLE

IN THE 1960s PARTICLE ("HI E") PHYSICS EXPERIMENTS WERE DISCOVERING ALL SORTS OF INTERESTING NEW PARTICLES. (HADRONS, LIKE THE PROTON)

@ AROUND THIS TIME PEOPLE WERE STARTING TO BELIEVE THAT ALL OF THESE ~~PARTICLES~~ HADRONS WERE MADE UP OF MORE FUNDAMENTAL PARTICLES; what we now call QUARKS, SPIN ½ FERMIONS.

PUZZLE: DISCOVERY OF THE Δ^{++} BARYON w/ SPIN 3/2

SPIN 3/2 ⟶ ↑↑↑ SPIN WAVEFUNCTION, symmetric
POSITION WAVEFUNCTION IS ALSO SYMMETRIC
↳ all @ same position

↓

FERMION

QUARKS ARE FERMIONS
→ CANNOT HAVE 3 IDENTICAL FERMIONS IN SAME PLACE!!

WHAT GIVES? ∃ ADDITIONAL QUANTUM #: SU(3) COLOR → r, g, b
UNDER WHICH Q'S ANTISYMMETRIC.