Integral Control

So far, we only considered proportional control law

Problem : proportional droop

$y_\infty \neq r_\infty$ $\quad$ $y_\infty = r_\infty$ only K_p infinite

Integral control law

$$K_i \int_{-\infty}^{t} e(t')\,dt' \qquad K(s) = \frac{K_i}{\tau s}$$

Frequency dependent gain

infinite for DC $(s=0) \Rightarrow y_\infty = r_\infty$

for $s \neq 0$ gain decreases as $1/s$

Note! With integral feedback, the system goes to desired setpoint, regardless of K_i or G

Also : assume $G = 1/1+\tau s$

$$T(s) = \frac{KG}{1+KG} = \frac{1}{1 + \tau s/K_i + \tau^2 s^2 K_i}$$

Integral control turns 1st order system into effective 2nd order system

Stability

$T = \frac{KG}{1+KG}$ response infinite if $KG = -1$
(this is instability cond. for linear system)

In a FB system instability occurs for
$|KG| = 1$ and a phase lag of $180°$

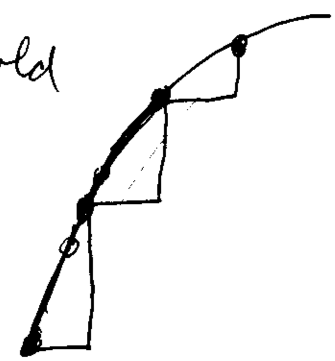
~~Instabilities need at least 3rd order systems~~

$\left(G \sim \frac{1}{s^n} \right.$ Asymptotically frequency response
$\left. \sim (i\omega)^{-n} \rightarrow \text{phase lag} \right)$

Higher order systems tend $n \cdot \frac{\pi}{2}$
to be more unstable

Thus, integral control tends to destabilize

Increasing gain will always lead to $|KG| = 1$
What about phase lag?
In practice phase lags always occur at high frequencies due to delays
e.g. digitization with Zero-Order Hold



PDE's have infinite # of normal modes
Heater – Metal Bar – Thermometer

$D \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$ $\quad T(x \to \infty) = T_\infty$

$-\lambda \left. \frac{\partial T}{\partial x} \right|_{0,t} = J_0 e^{-i\omega t}$ $\quad T_s$

$\frac{T(x,t) - T_\infty}{J_0 e^{-i\omega t}} \propto e^{i(kx + \pi/4)} e^{-kx}$ $\quad k = \sqrt{\frac{\omega}{2D}}$

$\omega^* = \frac{9}{8} \pi^2 \frac{D}{L^2}$ crit. freq. higher for
Heater close to thermometer

Differential Control

$K_d\, s$

much higher 'speed' by anticipating
e.g. if temperature falls, controller starts heating before object has cooled significantly

Problem : sensor noise

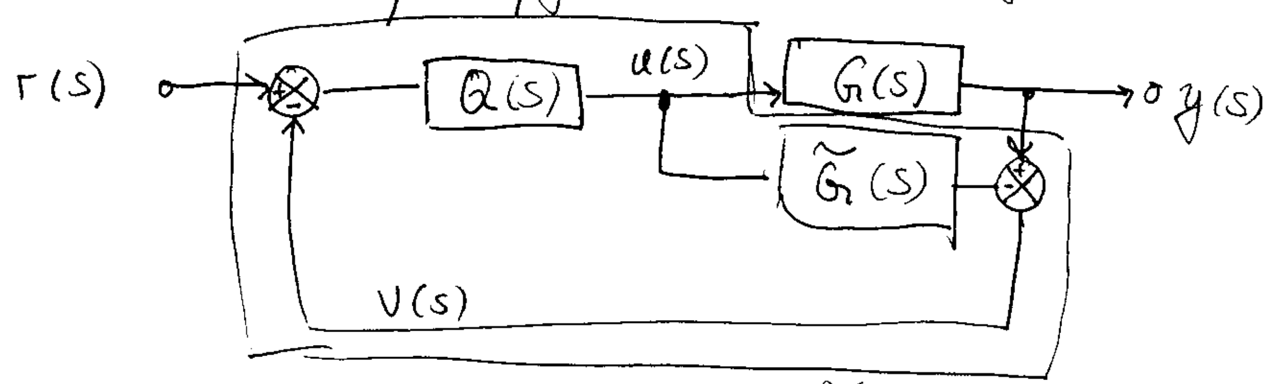
In general, one often uses a PID controller

$$K(s) = K_p + \frac{K_i}{s} + K_d\, s$$

Robust control

Problem : Our system changes and the model we have of our system is inaccurate

How can we quantify this uncertainty?



Open loop system + model $\tilde{G}(s)$

$$y(s) = \frac{G\,Q}{1 + (G - \tilde{G})\,Q}\, r$$

Internal Model Control Parametrization

$$V = (G - \tilde{G})\, u$$

Internal Model Controller $Q(s)$ relates to $K(s)$ as

$$K(s) = \frac{Q}{1-\tilde{G}Q}$$

Stability?

Unstable only if $\quad Q = -\frac{1}{G-\tilde{G}}$

For a reasonable model $G \approx \tilde{G}$, thus Q would have to be very large for instabilities to occur.

Other advantage : can handle nonlinearities for $G \approx \tilde{G}$

Robust Performance

Remember $d(s)$

$$e_0(s) = S(s)\left[r(s) - d(s)\right]$$

$$S = \frac{1}{1+L} \qquad \text{We want } S(i\omega) \ll 1$$

But: for high frequencies loop gain $L = KG$ will go to zero, i.e. $S \to 1$ for $\omega \to \infty$

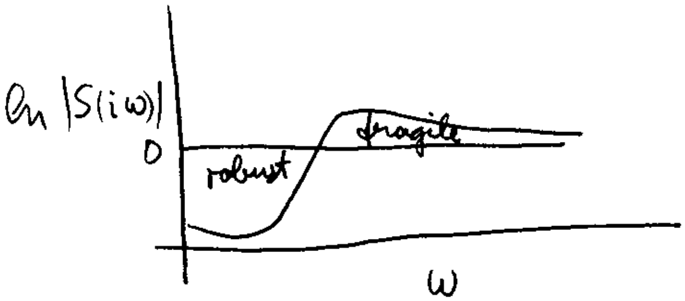
Also the form of L constrains S

E.g. consider L with no poles or zeros in rhplane

assume at least 2nd order, i.e. $\underset{\omega\to\infty}{L \sim \omega^{-n}}$ $n \geq 2$

Then one can show that

$$\int_0^\infty \ln S(i\omega)\, d\omega = 0$$



area below 0 $|S|<1$
must be balanced by area
above 0 $|S|>1$

"conservation of fragility"
"waterbed effect"

higher gain
→ larger ω-range
robust, but higher
fragility for large ω

$d(s) = \Delta(s)\, W_1(s)$

$W_1(s)$ performance weight : largest expected disturbance

$\Delta(s)$: transfer fct with $|\Delta|<1$

largest expected error

$$|e_0(i\omega)| = |d \cdot S| \leq |W_1(i\omega)\, S(i\omega)| = \underbrace{\|W_1 S\|_\infty}$$

desired nominal performance

means $|W_1(i\omega)S(i\omega)| < 1 \quad \forall\omega$

Robust performance

consider uncertainty in S, G

$$\begin{array}{l} S \to S_\Delta \\ T \to T_\Delta \end{array} \qquad \begin{array}{l} G \to G_\Delta = G_0(1+\Delta W_2) \\ L \to L_0(1+\Delta W_2) \end{array}$$

$$|W_1 S_\Delta| = \frac{|W_1|}{|1+L_0+\Delta W_2 L_0|} = \frac{|W_1 S|}{|1+\Delta W_2 T|} \overset{!}{<} 1 \quad \text{for robust}$$

$$|W_1 S| < |1+\Delta W_2 T| \leq 1 - |W_2 T| \quad \forall\omega$$

$$\Rightarrow \boxed{\|\, |W_1 S| + |W_2 T|\, \|_\infty < 1}$$

Condition for robust performance

Nonlinearities

Dissipative chaotic system

- ergodic : comes near to every point in phase space
- has dense set of unstable periodic orbits (UPO's)

e.g. fixed points x^*

$$x_{n+1} = f(\lambda, x_n)$$

linearization around fixed point

$$x_{n+1} - x^* \approx \frac{\partial f}{\partial x}(x_n - x^*) + \frac{\partial f}{\partial \lambda} \Delta \lambda_n$$

By changing the control parameter λ, one can set $x_{n+1} = x^*$ by choosing

$$\Delta \lambda_n = - \frac{\partial f (x_n - x^*)/\partial x}{\partial f/\partial \lambda}$$

HW : $x_{n+1} = \lambda x_n (1 - x_n)$

Find x^* (unstable at $\lambda = 3.8$)

Find $\Delta \lambda_n$ to stabilize response

Very different idea from linear control

Not overpowering the system, but small perturbations

can only control to UPO's

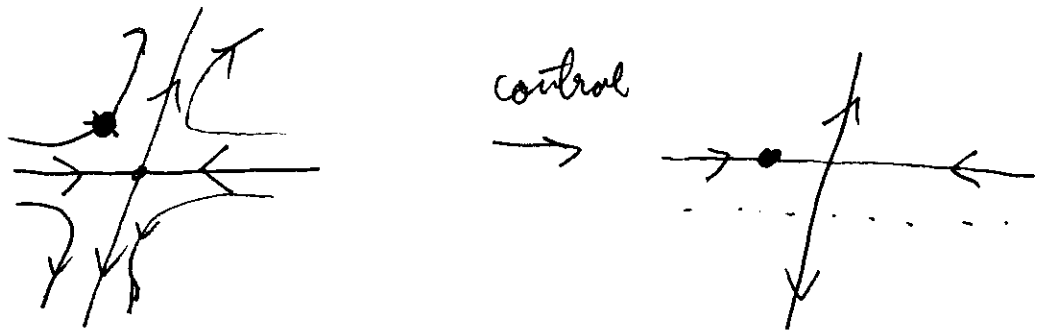
(may not be problem, since many of them present)

wait for control to switch on

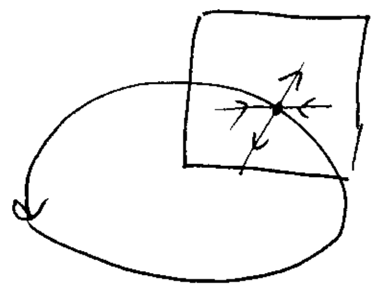
(stick example)

Extensions

One often controls only 1 dimension



Only 1 control step per period
(Poincaré map)



What if several are needed
SV Decomposition
Deal with complex EV
let dissipation do the work

Instabilities may make controls more responsive

Fundamental Mechanisms of Control in Biology

1. Negative Feedback

Expression of Tet R
tetracycline repressor

conc. of Tet R

$$\frac{dR}{dt} = \underbrace{\frac{\alpha}{1+KR}}_{\text{Michaelis Menten type kinetics}} - \lambda R$$

α depends on conc. of RNA-pol. ($\to$ mRNA $\to$ TetR)

λ: degradation rate of TetR

K : binding affinity of TetR to DNA

$K=0$ no feedback

$K>0$ negative FB

Steady-state

$$R^* = \frac{1}{2}\left(-\frac{1}{K} + \sqrt{\frac{1}{K^2} + \frac{4\alpha/\lambda}{K}}\right)$$

Perturbations in cell:

$$R(t) = R^* + r(t)$$

$$\dot{r} = -\lambda^* r \qquad \text{assume decay}$$

$$\lambda^* = \lambda + \frac{\alpha K}{(1+KR^*)^2} \qquad \lambda \to 2\lambda \text{ for } K \text{ from } 0 \to \infty$$

Fluctuations of r obey (assume white noise)

$$\langle r^2 \rangle = \frac{1}{\sqrt{\lambda^*}}$$

$\boxed{\text{Negative FB reduces variability}}$

Positive FB

... leads to switch or oscillations

Well : positive FB occurs at intermediate states
Since Interm. state is between to stable states for topological reasons, we have a switch

Sometimes positive FB <u>creates</u> a second stable state and a switch (think bifurcations)

E: Two proteins that inhibit expression of each other

$$\dot{u} = \frac{\alpha_1}{1+v^n} - u$$

$$\dot{v} = \frac{\alpha_2}{1+u^n} - v$$

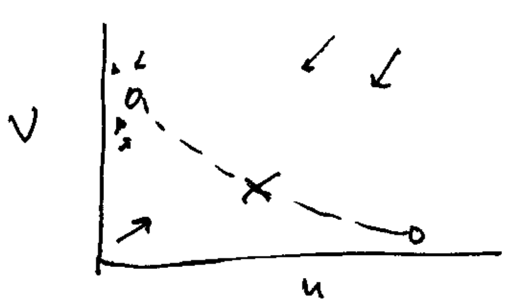
n : coorparitivity parameter ≥ 2

Steady-state solutions

(1) $\alpha_1 = \alpha_2 = \alpha \gg 1$ Three solutions $u \approx \alpha \quad v \approx \frac{1}{\alpha}$

$v \approx \alpha \quad u = \frac{1}{\alpha}$

$u \approx v \approx \alpha^{1/3}$



Example of bistability

Real examples more complex

Positive and negative FB are part of cell regulation

Many of such loops are connected

→ Networks : Loop → Interconnections between nodes

Much emphasis on statist. properties

e.g. distribution of # connections k of a node w. neighbors

Random networks $P(k)$ peaks at $\langle k \rangle$

Scale-free networks $P(k) \sim k^{-\gamma}$

$\gamma : 2-3$

communication

social interactions

Few hubs have many connections

→ small world phenomenon

→ robustness against removal of nodes

metabolic / protein IA networks

Nodes : proteins

chem / phys IA

molecular aspects → network motifs